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Principles of non-local field theories and their application to polymerized membranes

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Abstract

In these lecture notes, we give an overview about non-local field-theories and their application to polymerized membranes, i.e. membranes with a fixed internal connectivity. The main technical tool is the multi-local operator product expansion (MOPE), generalizing ideas from local field theories to the multi-local situation.

These notes are largely inspired by: Kay Wiese, "Polymerized membranes, a review", Domb and Lebowitz, eds., Academic Press, London (2000).

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1 Field theory for membranes

Let us start by characterizing the different possible types of membranes. One very popular class of membranes are fluid membranes. Biologically most relevant are bilayers of lipid molecules that are composed of a hydrophilic head and two hydrophobic chains. We are now looking for the statistical weight of a membrane configuration. Since the lipid molecules in the membrane are free to move around, the energy, i.e. "Hamiltonian" of the membrane has to be invariant under coordinate transformations. This is achieved by the Canham-Helfrich Hamiltonian [1,2]

$$\mathcal{H}[\vec{r}] = \int \mathrm{d}^2 x \sqrt{g(x)} \left[\tau + \frac{\kappa}{2} \left(H(x) - H_0 \right)^2 \right] . \tag{1.1}$$

$\mathrm{d}^2 x \sqrt{g}$ is the invariant volume-element of the membrane, τ its surface tension, and κ the bending-rigidity, which is coupled to the square of the mean curvature

$$H = \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) , \tag{1.2}$$

where R_1 and R_2 are the two curvature radii. H_0 is a spontaneous curvature, present in the case of symmetry breaking between the two sides of the membrane. Physically, rigidity is explained by the finite thickness of the membrane. RG-calculations indicate that bending-rigidity should be irrelevant at large distances [3,4]; this however has recently been criticized in [5]. For a general review about fluid membranes, see [4,6–8].

Interestingly, the Hamiltonian (1.1) with $\kappa = 0$ also plays a central role in string theory. Here, one of the inner coordinates $\vec{x} = \binom{x_1}{x_2}$ is identified as $i \times$ time, and the other one as length on the string. Eq. (1.1) is then the action generating the motion of the string. Further generalizations use a metric $g_{\alpha\beta}$ independent of the imbedding space [9–13]. Strings are considered as one of the most promising candidates for unifying all fundamental interactions.

Another class of membranes, on which we shall concentrate here, are polymerized tethered (sometimes also called solid) membranes, which have a fixed and constant internal metric. These membranes have not yet found applications in high-energy physics, but are realized in experiments (e.g. the spectrin network of red blood cells [15,16], or sheets of graphite oxide [17,18]).

A microscopic model is given by the so-called "spring and bead model" (see figure 1.1), which consists of balls (beads) which are connected by springs and form a regular lattice. The difficulty is to incororate the self-avoidance of the beads, i.e. the fact that beads cannot intersect.

2 Tethered membranes, a non-local field theory

We start from the continuous model for a D-dimensional flexible polymerized membrane introduced in [19,20]. This model is a simple extension of the well known Edwards' model for continuous chains. The membrane fluctuates in d-dimensional space. Points in the membrane are labeled by coordinates $x \in \mathbb{R}^D$ and the configuration of the membrane in physical space is described by the field $r : x \in \mathbb{R}^D \longrightarrow r(x) \in \mathbb{R}^d$, i.e. from now on we note r instead of $\vec{r}$. At high temperatures the free energy for a configuration is given by the (properly scaled) Hamiltonian

$$\mathcal{H}[r] = \frac{1}{2-D} \int\limits_x \frac{1}{2} (\nabla r(x))^2 + b_0 \iint\limits_{x\ y} \tilde{\delta}^d (r(x) - r(y)) . \tag{2.1}$$

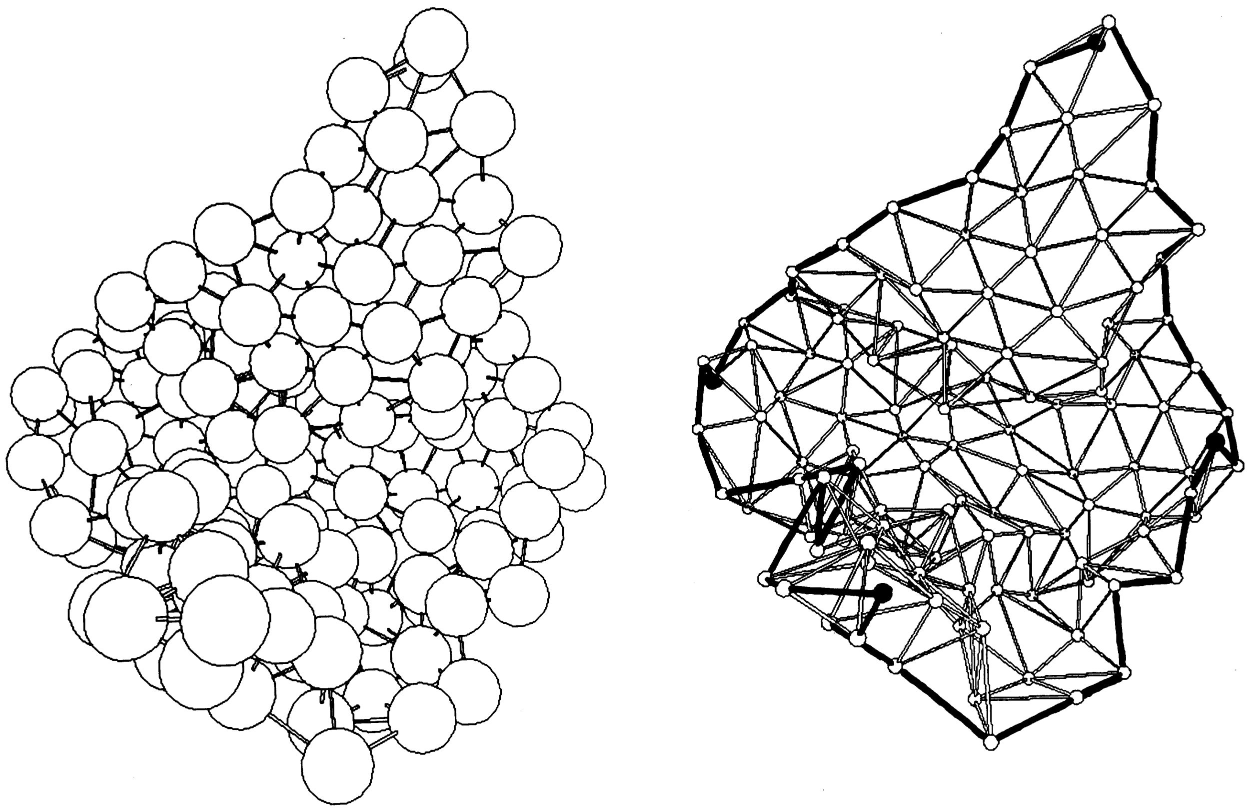


Figure 1.1: A tethered membrane (spring and bead model), from [14].

The integral $\int_x$ runs over D-dimensional space and ∇ is the usual gradient operator. The normalizations are

$$\int\limits_x := \frac{1}{S_D}\int \mathrm{d}^D x\ , \quad S_D = 2\frac{\pi^{D/2}}{\Gamma(D/2)} \tag{2.2}$$

and

$$\tilde{\delta}^d(r(x)-r(y)) = (4\pi)^{d/2}\delta^d(r(x)-r(y))\ . \tag{2.3}$$

The latter term is normally used in Fourier-representation

$$\tilde{\delta}^d(r(x)-r(y)) = \int\limits_p \mathrm{e}^{ip[r(x)-r(y)]}\ , \tag{2.4}$$

where the normalization of $\int_p$ is given by

$$\int\limits_p = \pi^{-d/2}\int \mathrm{d}^d p \tag{2.5}$$

to have

$$\int\limits_p \mathrm{e}^{-p^2 a} = a^{-d/2}\ . \tag{2.6}$$

All normalizations are chosen in order to simplify the calculations, but are unimportant for the general understanding. μ is an internal momentum scale, such that μx is dimensionless. It is

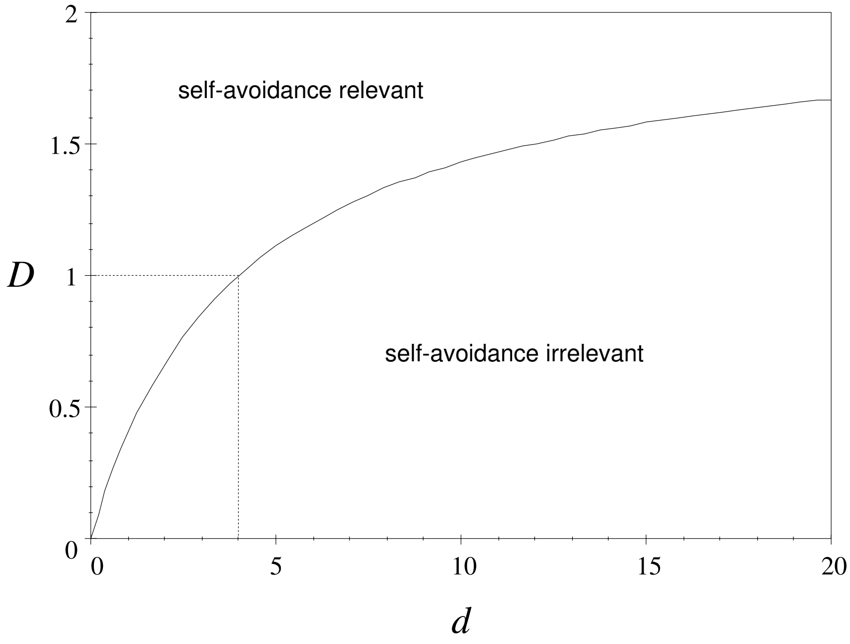


Figure 2.1: The critical curve $\varepsilon(D, d) = 0$. The dashed line corresponds to the standard polymer perturbation theory, critical in $d = 4$.

introduced to render the coupling b dimensionless. The first term in the Hamiltonian is a Gaussian elastic energy which is known to describe the free "phantom" surface. The interaction term corresponds (for $b > 0$) to a weak repulsive interaction upon contact. The expectation values of physical observables are obtained by performing the average over all field-configurations $r(x)$ with the Boltzmann weight $e^{-\mathcal{H}[r]}$. This average can not be calculated exactly, but one can expand about the configurations of a phantom, i.e. non-interacting surface.

Such a perturbation theory is constructed by performing the series expansion in powers of the coupling constant b. This expansion suffers from ultraviolet (UV) divergences which have to be removed by renormalization and which are treated by dimensional regularization, i.e. analytical continuation in D and d. A physical UV-cutoff could be introduced instead, but would render the calculations more complicated. Long-range infrared (IR) divergences also appear. They can be eliminated by using a finite membrane, or by studying translationally invariant observables, whose perturbative expansion is also IR-finite in the thermodynamic limit (infinite membrane). Such observables are "neutral" products of vertex operators

$$\mathcal{O} = \prod_{a=1}^{N} \mathrm{e}^{ik_a r(x_a)} , \qquad \sum_{a=1}^{N} k_a = 0 . \tag{2.7}$$

An example is given at the end of section 3.

Let us now analyze the theory by power-counting. We use internal units $\mu \sim 1/x$, and note $[x]_x = 1$, and $[\mu]_x = -[\mu]_\mu = -1$. The dimension of the field and of the coupling-constant are:

$$\nu := [r]_x = \frac{2-D}{2} , \quad \varepsilon := [b_0]_\mu = 2D - \nu d . \tag{2.8}$$

In the sense of Wilson [21] the interaction is relevant for $\varepsilon > 0$, see figure 2.1. Perturbation theory is then expected to be UV-finite except for subtractions associated to relevant operators. We shall come back to this point later.

For clarity, we represent graphically the different interaction terms which have to be considered. The local operators are

$$1 = \mathbf{1} \tag{2.9}$$

$$\frac{1}{2}(\nabla r(x))^2 = \;+\!\!\!\!\bullet\; . \tag{2.10}$$

The bi-local operator, the dipole, is

$$\tilde{\delta}^d(r(x) - r(y)) = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \; . \tag{2.11}$$

The expectation-value of an observable is

$$\left\langle \mathcal{O}[r] \right\rangle_b = \frac{\displaystyle\int \mathcal{D}[r]\, \mathcal{O}[r]\, \mathrm{e}^{-\mathcal{H}[r]}}{\displaystyle\int \mathcal{D}[r]\, \mathrm{e}^{-\mathcal{H}[r]}} \; . \tag{2.12}$$

Perturbatively, all expectation-values are taken with respect to the free theory:

$$\left\langle \mathcal{O}[r] \right\rangle_0 = \frac{\displaystyle\int \mathcal{D}[r]\, \mathcal{O}[r]\, \mathrm{e}^{-\frac{1}{2-D}\int_x \frac{1}{2}(\nabla r(x))^2}}{\displaystyle\int \mathcal{D}[r]\, \mathrm{e}^{-\frac{1}{2-D}\int_x \frac{1}{2}(\nabla r(x))^2}} \; . \tag{2.13}$$

A typical term in the expansion of (2.12) is

$$(-bZ_b\mu^\varepsilon)^n \iint \ldots \iint \left\langle \mathcal{O}\, \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \ldots \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \right\rangle_0^{\mathrm{c}} \; , \tag{2.14}$$

where the integral runs over the positions of all dipole-endpoints.

3 Locality of divergences

In this section, we show that all divergences are short distance divergences. Note that even for massless theories and in the absence of IR-divergences, this is not trivial. Divergences could as well appear, when some of the distances involved become equal, or multiple of each other. A simple counter-example is the integral of $\Big||a| - |b|\Big|^{-\nu d}$, where a and b are two of the distances involved.

That divergences only occur at short distances (i.e. when at least one of the distances involved tends to 0), is a consequence of Schoenbergs theorem [22]. Here, we present a proof, based on the equivalence with electrostatics.

We first state that with our choice of normalizations, the free correlation-function $C(x_1, x_2)$

$$\begin{aligned} C(x_1, x_2) := \frac{1}{d} \left\langle \frac{1}{2} \left[r(x_1) - r(x_2) \right]^2 \right\rangle_0 &= |x_1 - x_2|^{2-D} \\ &\equiv (2-D) S_D \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2} \left(1 - \mathrm{e}^{ip(x_1 - x_2)} \right) \end{aligned} \tag{3.1}$$

is the Coulomb potential in D dimensions. Furthermore, the interaction part of the Hamiltonian $\mathcal{H}$ is reminiscent of a dipole, and can be written as

$$\begin{aligned}\mathcal{H}_{\text{int}} &= b_0 \int_{x_1}\int_{x_2} \tilde{\delta}^d(r(x_1)-r(x_2)) \\ &= b_0 \int_{x_1}\int_{x_2}\int_{k} \mathrm{e}^{ik[r(x_1)-r(x_2)]} \ ,\end{aligned} \tag{3.2}$$

where k may be seen as a d-component (vector-) charge.

The next step is to analyze the divergences appearing in the perturbative calculation of expectation values of observables. To simplify the calculations, we focus on the normalized partition function

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \frac{1}{\mathcal{Z}_0} \sum_{\text{all configurations}} \mathrm{e}^{-\mathcal{H}} = \left\langle \mathrm{e}^{-\mathcal{H}_{\text{int}}} \right\rangle_0 \ . \tag{3.3}$$

To exhibit the similarity to Coulomb systems, consider the second order term

$$\begin{aligned}\frac{1}{2}\left\langle \mathcal{H}_{\text{int}}^2 \right\rangle_0 &= \frac{b_0^2}{2} \int_{x_1}\int_{x_2}\int_{y_1}\int_{y_2}\int_{k}\int_{p} \left\langle \mathrm{e}^{ik[r(x_1)-r(x_2)]}\, \mathrm{e}^{ip[r(y_1)-r(y_2)]} \right\rangle_0 \\ &= \frac{b_0^2}{2} \int_{x_1}\int_{x_2}\int_{y_1}\int_{y_2}\int_{k}\int_{p} \mathrm{e}^{-E_{\mathrm{c}}} \\ E_{\mathrm{c}} &= k^2 C(x_1-x_2) + p^2 C(y_1-y_2) \\ &\quad + kp\left[C(x_1-y_2)+C(x_2-y_1)-C(x_1-y_1)-C(x_2-y_2)\right] \ ,\end{aligned} \tag{3.4}$$

where E_{c} is the Coulomb-energy of a configuration of dipoles with charges $\pm k$, and $\pm p$, respectively. More generally, for any number of dipoles (and even for any Gaussian measure) we have

$$\left\langle \mathrm{e}^{i\sum_i k_i r(x_i)} \right\rangle_0 = \mathrm{e}^{-E_{\mathrm{c}}} \ , \qquad E_{\mathrm{c}} = \frac{1}{2}\sum_{i,j} \left\langle k_i r(x_i)\, k_j r(x_j) \right\rangle_0 \ . \tag{3.5}$$

Since $\sum_i k_i = 0$, the latter can be rewritten with the help of the usual correlation function $C(x-y) = \frac{1}{2d}\left\langle [r(x)-r(y)]^2 \right\rangle_0$ as

$$E_{\mathrm{c}} = -\frac{1}{4d}\sum_{i,j} k_i k_j \left\langle [r(x_i)-r(x_j)]^2 \right\rangle_0 \ . \tag{3.6}$$

As for any configuration of dipoles, specified by their coordinates and charges, the total charge is zero, the Coulomb-energy is bounded from below, i.e.

$$E_{\mathrm{c}} \geq 0 \ . \tag{3.7}$$

Formally, this is proven by the following line of equalities (remember that $D < 2$)

$$\begin{aligned}E_{\mathrm{c}} &= \frac{1}{2}\sum_{i,j} \left\langle k_i r(x_i)\, k_j r(x_j) \right\rangle_0 \\ &= \frac{(2-D)S_D}{2} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \sum_{i,j} k_i k_j \frac{1}{p^2} \mathrm{e}^{ip(x_i-x_j)} \\ &= \frac{(2-D)S_D}{2} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2} \left| \sum_i k_i \mathrm{e}^{ipx_i} \right|^2 \geq 0 \ .\end{aligned} \tag{3.8}$$

The last inequality is again due to the global charge neutrality, which ensures convergence of the integral for small p. Hence, E_c vanishes, if and only if the charge density vanishes everywhere. This implies that

$$\mathrm{e}^{-E_\mathrm{c}} \leq 1 \, , \tag{3.9}$$

and the equality is obtained for vanishing charge density. Noting $E_\mathrm{c} = \sum_{i,j} k_i k_j Q_{ij}$, Eq. (3.8) even states that as long as $x_i \neq x_j$ for all $i \neq j$, Q_{ij} is a non-degenerate form on the space of k_i with $\sum_i k_i = 0$. This implies that integrating $\mathrm{e}^{-E_\mathrm{c}}$ as in Eq. (3.4) over all k_i with $\sum_i k_i = 0$ gives a finite result, as long as not some of the x_i coalesce. Consequently, divergences in the integration over x_i can only appear when at least some of the distances vanish, as stated above.

This does of course not rule out IR-divergences. However, they are absent in translationally invariant observables; for a proof see [23]. As an example, use in Eq. (2.7) the observable $\mathcal{O} = \mathrm{e}^{ik[r(s)-r(t)]}$, which is the generating function for the moments of $[r(s) - r(t)]$; the series up to first order in b reads

$$\begin{aligned}\left\langle \mathcal{O} \right\rangle_b = \mathrm{e}^{-k^2 C(s-t)} \times \\ \Bigg\{ 1 + b_0 \iint\limits_{x\ y} \left[1 - \exp\left(\frac{1}{4} k^2 \frac{\left[C(s-x) + C(t-y) - C(s-y) - C(t-x) \right]^2}{C(x-y)} \right) \right] C(x-y)^{-d/2} \\ + O(b_0^2) \Bigg\} \, . \end{aligned} \tag{3.10}$$

Note that the integral over x and y is IR-convergent, but UV-divergent at $\varepsilon \leq 0$: There is a singularity for $|x - y| \to 0$. This is a general feature of such expectation values. The purpose of the rest of this section is to introduce the basic tools to handle these divergences. On the example of Eq. (3.10), this is verified in appendix B, see page 20.

4 Multilocal operator product expansion (MOPE)

In section 3, we showed that for self-avoiding membranes divergences only occur at short distances. The situation is thus similar to local field-theories for which we discussed in the last section how the techniques of operator product expansion can be used to analyze the divergences. Our aim is now, to generalize these techniques to the multilocal case [24,23]. Intuitively, in the context of multilocal theories – by which we mean that the interaction depends on more than one point – we also expect multilocal operators to appear in such an operator product expansion, which therefore will be called “multi-local operator product expansion” (MOPE). Its precise definition is the aim of this section, whereas we shall calculate some examples in the following one.

We start our analysis by recalling the general form of a (local) operator product expansion of two scaling-operators $\Phi_A(z + \lambda x)$ and $\Phi_B(z + \lambda y)$ in a massless theory in the limit of $\lambda \to 0$:

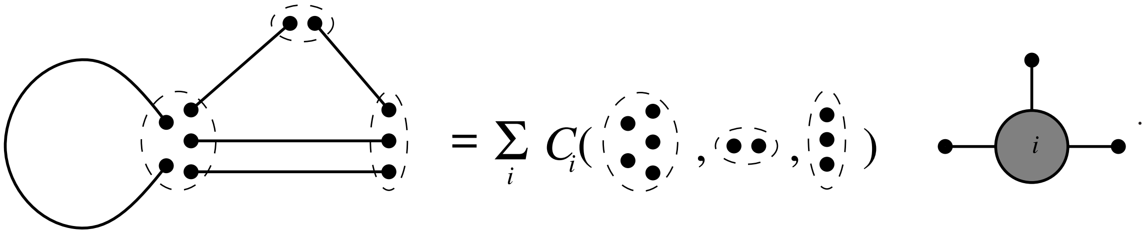
$$\Phi_A(z + \lambda x) \Phi_B(z + \lambda y) = \sum_i C_i(z, \lambda x, \lambda y)\, \Phi_i(z) \, , \tag{4.1}$$

where $C_i(z, \lambda x, \lambda y)$ are homogeneous functions of λ

$$C_i(z, \lambda x, \lambda y) = \lambda^{[\Phi_A]_x + [\Phi_B]_x - [\Phi_i]_x} C_i(z, x, y) \, . \tag{4.2}$$

Here $[\Phi]_x$ is the canonical dimension of the operator Φ in space-units such that $[x]_x = 1$, as obtained by naive power-counting. If the theory is translationally invariant, $C_i(z, x, y)$ is also independent of z, and we will suppose that this is the case, if not stated otherwise†. Also recall that this relation is to be understood as an operator identity, i.e. it holds inserted into *any* expectation value, as long as none of the other operators sits at the point z, to which the contraction is performed.

An example for the multilocal theory is



(4.3)

Let us explain the formula. We consider n dipoles (here $n = 5$) and we separate the $2n$ end-points into m subsets (here $m = 3$) delimited by the dashed lines. The MOPE describes how the product of these n dipoles behaves when the points inside each of the m subsets are contracted towards a single point z_j. The result is a sum over *multilocal* operators $\Phi_i(z_1, \ldots, z_m)$, depending on the m points $z_1, \ldots, z_m$, of the form

$$\sum_i C_i(x_1 - z_1, \ldots)\, \Phi_i\,(z_1, z_2, \ldots, z_m) \ , \tag{4.4}$$

where the MOPE-coefficients $C_i(x_1 - z_1, \ldots)$ depend only on the distances $x_l - z_j$ *inside each subset*. This expansion is again valid as an operator-identity, i.e. inserted into any expectation value and in the limit of small distances between contracted points. Again, no other operator should appear at the points $z_1, \ldots, z_m$, towards which the operators are contracted. As the Hamiltonian (2.1) does not contain a mass-scale, the MOPE-coefficients are as in Eq. (4.2) homogeneous functions of the relative positions between the contracted points, with the degree of homogeneity given by simple dimensional analysis. In the case considered here, where n dipoles are contracted to an operator Φ_i, this degree is simply $-n\nu d - [\Phi_i]_x$. This means that

$$C_i(\lambda(x_1 - z_1), \ldots) = \lambda^{-n\frac{2-D}{2}d - [\Phi_i]_x} C_i(x_1 - z_1, \ldots) \ , \tag{4.5}$$

where $[\Phi_i]_x$ is the canonical dimension of the operator Φ_i and $-d(2 - D)/2$ is simply the canonical dimension of the dipole.

In order to evaluate the associated singularity, one finally has to integrate over all relative distances inside each subset. This gives an additional scale factor with degree $D(2n - m)$. A singular configuration, such as in Eq. (4.3), will be UV-divergent if this degree of divergence

$$D(2n - m) - n\frac{2 - D}{2}d - [\Phi_i]_x \ , \tag{4.6}$$

is negative. It is superficially divergent if the degree is zero and convergent otherwise. The idea of renormalization, is *to remove exactly these superficially divergent contributions recursively.*

†Translation invariance is e.g. broken when regarding systems with boundaries or initial time problems. It is also broken when the underlying metric is not constant, see [23,25].

5 Evaluation of the MOPE-coefficients

The MOPE therefore gives a convenient and powerful tool to calculate the dominant and all subdominant contributions from singular configurations. In this section, we explain how to calculate the MOPE-coefficients on some explicit examples. These examples will turn out to be the necessary diagrams at 1-loop order.

In the following we shall use the notion of normal-ordering. The first thing, which we use, is that

$$:\mathrm{e}^{ikr(x)}: = \mathrm{e}^{ikr(x)} \ . \tag{5.1}$$

Explicitly, tadpole-like contributions which are powers of

$$\int \mathrm{d}^D p \, \frac{1}{p^2} \tag{5.2}$$

are omitted. This is done via a finite part prescription (analytic continuation, dimensional regularization), valid for infinite membranes, for which the normal-order prescription is defined. Let us stress that this is a pure technical trick, which is not really necessary. However, adopting this notation, the derivation of the MOPE-coefficients is much simplified, and we will henceforth stick to this convention. The suspicious reader may always check that the same results are obtained without this procedure. This is clear from the uniqueness of the finite-part prescription.

The key-formula for all further manipulations is

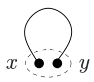
$$:\mathrm{e}^{ikr(x)}::\mathrm{e}^{ipr(y)}: = \ \mathrm{e}^{kpC(x-y)} \ :\mathrm{e}^{ikr(x)}\mathrm{e}^{ipr(y)}: \ . \tag{5.3}$$

This can be proven as follows: Consider the (free) expectation value of any observable $\mathcal{O}$ times the operators of Eq. (5.3). Then the the left- and right-hand sides of the above equation read

$$\begin{aligned}
\mathcal{L} &= \left\langle \mathcal{O} :\mathrm{e}^{ikr(x)}::\mathrm{e}^{ipr(y)}: \right\rangle_0 \\
\mathcal{R} &= \mathrm{e}^{kpC(x-y)} \left\langle \mathcal{O} :\mathrm{e}^{ikr(x)}\mathrm{e}^{ipr(y)}: \right\rangle_0 \ .
\end{aligned}$$

First of all, for $\mathcal{O}=1$, the desired equality of $\mathcal{L}=\mathcal{R}$ holds, because $\left\langle :\mathrm{e}^{ikr(x)}\mathrm{e}^{ipr(y)}: \right\rangle_0 = 1$ and $\left\langle :\mathrm{e}^{ikr(x)}::\mathrm{e}^{ipr(y)}: \right\rangle_0 = \mathrm{e}^{kpC(x-y)}$. Now consider a non-trivial observable $\mathcal{O}$, and contract all its fields r with $\mathrm{e}^{ikr(x)}$ or $\mathrm{e}^{ipr(y)}$, before contracting any of the fields $r(x)$ with $r(y)$. The result is a product of correlation-functions between the points in $\mathcal{O}$ and x or y, and these are equivalent for both $\mathcal{L}$ and $\mathcal{R}$. However, contracting an arbitrary number of times $\mathrm{e}^{ikr(x)}$, leaves the exponential $\mathrm{e}^{ikr(x)}$ invariant. Completing the contractions for $\mathcal{L}$ therefore yields a factor of $\mathrm{e}^{kpC(x-y)}$, and the latter one also appears in $\mathcal{R}$. Thus, the equality of $\mathcal{L}$ and $\mathcal{R}$ holds for all $\mathcal{O}$ and this proves Eq. (5.3).

Now proceed to the first explicit example, the contraction of a single dipole with endpoints x and y.



$$= \int_k :\mathrm{e}^{ikr(x)}::\mathrm{e}^{-ikr(y)}: \ . \tag{5.4}$$

This configuration may have divergences when x and y come close together. Let us stress that in contrast to ϕ^4-theory, these divergences are not obtained as a finite sum of products of correlators: Since $C(x-y) = |x-y|^{2-D}$, the latter is always well-behaved at $x = y$. The singularity only appears when summing an infinite series of diagrams as we will do now. To this purpose, we first normal-order the two exponentials using Eq. (5.3)

$$\int_k :\mathrm{e}^{ik[r(x)-r(y)]}:\, \mathrm{e}^{-k^2|x-y|^{2\nu}} \ . \tag{5.5}$$

Note that the operators $\mathrm{e}^{ikr(x)}$ and $\mathrm{e}^{-ikr(y)}$ are free of divergences upon approaching each other, since no more contractions can be made. The divergence is captured in the factor $\mathrm{e}^{-k^2|x-y|^{2\nu}}$. Therefore, we can expand the exponential $:\mathrm{e}^{ik[r(x)-r(y)]}:$ for small $x-y$ and consequently in powers of $[r(x)-r(y)]$. This expansion is

$$\int_k :\left\{\mathbf{1} + ik\,[r(x)-r(y)] - \frac{1}{2}\left(k\,[r(x)-r(y)]\right)^2 + \dots\right\}:\, \mathrm{e}^{-k^2|x-y|^{2\nu}} \ . \tag{5.6}$$

We truncated the expansion after the third term. It will turn out later that this is sufficient, since subsequent terms in the expansion are proportional to irrelevant operators for which the integral over the MOPE-coefficient is UV-convergent.

Due to the symmetry of the integration over k the term linear in k vanishes. Also due to symmetry, the next term can be simplified with the result

$$\int_k \left[\mathbf{1} - \frac{k^2}{2d}\ :[r(x)-r(y)]^2:\ + \dots\right] \mathrm{e}^{-k^2|x-y|^{2\nu}} \ . \tag{5.7}$$

Finally, the integration over k can be performed. Recall that normalizations were chosen such that $\int_k \mathrm{e}^{-sk^2} = s^{-d/2}$ to obtain

$$|x-y|^{-\nu d}\,\mathbf{1} - \frac{1}{4}\ :\left[(x-y)\nabla r\left(\frac{x+y}{2}\right)\right]^2:\ |x-y|^{-\nu(d+2)} + \dots \ . \tag{5.8}$$

The second operator has a tensorial structure, which has to be taken into account in order to construct the subtraction operator. Using the short-hand notation ${}_\alpha{+}_\beta = \frac{1}{2}(\partial_\alpha r)(\partial_\beta r)$, we can write this symbolically as

 $= \Big($  $\Big|\mathbf{1}\Big)\mathbf{1} + \Big($  $\Big|{}_\alpha{+}_\beta\Big)\ {}_\alpha{+}_\beta\ + \dots \ ,$ (5.9)

with the MOPE-coefficients (reminding Feynman's bra and ket notation)

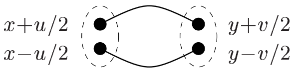
$\Big($  $\Big|\mathbf{1}\Big) = |x-y|^{-\nu d}$ (5.10)

$\Big($  $\Big|{}_\alpha{+}_\beta\Big) = -\frac{1}{2}\,(x-y)_\alpha\,(x-y)_\beta\,|x-y|^{-\nu(d+2)} \ .$ (5.11)

As long as the angular average is taken (and this will be the case when integrating the MOPE-coefficient to obtain the divergence), we can replace in Eq. (5.9) ${}_\alpha{+}_\beta$ by $+ := \frac{1}{2}(\nabla r)^2$ and Eq. (5.11) by

$\Big($  $\Big|+\Big) = -\frac{1}{2D}|x-y|^{D-\nu d} \ .$ (5.12)

Next consider a real multi-local example of an operator-product expansion, namely the contraction of two dipoles towards a single dipole:



$$= \int_k \mathrm{e}^{ik[r(x+u/2)-r(y+v/2)]} \int_p \mathrm{e}^{ip[r(x-u/2)-r(y-v/2)]} \ . \tag{5.13}$$

This has to be analyzed for small u and v, in order to control the divergences in the latter distances. As above, we normal-order operators which are approached, yielding

$$\mathrm{e}^{ikr(x+u/2)}\mathrm{e}^{ipr(x-u/2)} = :\mathrm{e}^{ikr(x+u/2)}::\mathrm{e}^{ipr(x-u/2)}: = :\mathrm{e}^{ikr(x+u/2)}\mathrm{e}^{ipr(x-u/2)}:\ \mathrm{e}^{kpC(u)} \ . \tag{5.14}$$

A similar formula holds when approaching $\mathrm{e}^{-ikr(y+v/2)}$ and $\mathrm{e}^{-ipr(y-v/2)}$

$$\mathrm{e}^{-ikr(y+v/2)}\mathrm{e}^{-ipr(y-v/2)} = :\mathrm{e}^{-ikr(y+v/2)}::\mathrm{e}^{-ipr(y-v/2)}: = :\mathrm{e}^{-ikr(y+v/2)}\mathrm{e}^{-ipr(y-v/2)}:\ \mathrm{e}^{kpC(v)} \ . \tag{5.15}$$

Eq. (5.13) then becomes

$$\int_k \int_p :\mathrm{e}^{ikr(x+u/2)+ipr(x-u/2)}::\mathrm{e}^{-ikr(y+v/2)-ipr(y-v/2)}:\ \mathrm{e}^{kp[C(u)+C(v)]} \ . \tag{5.16}$$

In order to keep things as simple as possible, let us first extract the leading contribution before analyzing subleading corrections. This leading contribution is obtained when expanding the exponential operators (here exemplified for the second one) as

$$:\mathrm{e}^{-ikr(y+v/2)}\mathrm{e}^{-ipr(y-v/2)}: = :\mathrm{e}^{-i(k+p)r(y)}\left(1+O(\nabla r)\right): \tag{5.17}$$

and dropping terms of order ∇r. This simplifies Eq. (5.16) to

$$\int_k \int_p :\mathrm{e}^{i(k+p)r(x)}::\mathrm{e}^{-i(k+p)r(y)}:\ \mathrm{e}^{kp[C(u)+C(v)]} \ . \tag{5.18}$$

In the next step, first k and second p are shifted

$$k \longrightarrow k-p \ , \quad \text{then} \qquad p \longrightarrow p+\frac{k}{2} \ . \tag{5.19}$$

The result is (dropping the normal-ordering according to Eq. (5.1))

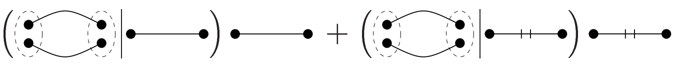
$$\int_k \mathrm{e}^{ik[r(x)-r(y)]} \int_p \mathrm{e}^{(\frac{1}{4}k^2-p^2)[C(u)+C(v)]} \ . \tag{5.20}$$

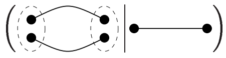
The factor of $\int_k \mathrm{e}^{ik[r(x)-r(y)]}$ is again a δ-distribution, and the leading term of the short distance expansion of Eq. (5.20). Derivatives of the δ-distribution appear when expanding

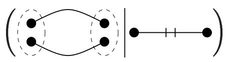
$$\mathrm{e}^{(\frac{1}{4}k^2-p^2)[C(u)+C(v)]}$$

in k^2; these are less relevant and only the first sub-leading term will be displayed for illustration:

$$\int_k \mathrm{e}^{ik[r(x)-r(y)]} \int_p \mathrm{e}^{-p^2[C(u)+C(v)]}\left(1+\frac{k^2}{4}\left[C(u)+C(v)\right]+\ldots\right)$$

$=$  $+\ldots \ ,$ (5.21) where in analogy to Eqs. (5.9) and (5.11)

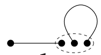
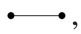
 $= [C(u) + C(v)]^{-d/2}$,

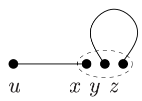
 $= \frac{1}{4} [C(u) + C(v)]^{1-d/2}$ (5.22)

and

 $= \tilde{\delta}^d(r(x) - r(y))$, $= (-\Delta_r)\tilde{\delta}^d(r(x) - r(y))$. (5.23)

Let us already mention that the leading contribution proportional to the δ-distribution will renormalize the coupling-constant, and that the next-to-leading term is irrelevant and can be neglected. The same holds true for the additional term proportional to (∇r) which was dropped in Eq. (5.17).

There is one more possible divergent contribution at the 1-loop level, namely . We now show that the leading term of its expansion, which is expected to be proportional to , is trivial. To this aim consider



$$= \int_{k,p} :\mathrm{e}^{ikr(u)}: :\mathrm{e}^{-ikr(x)}: :\mathrm{e}^{ipr(y)}: :\mathrm{e}^{-ipr(z)}:$$

$$= \int_{k,p} :\mathrm{e}^{ikr(u)}: :\mathrm{e}^{-ikr(x)}\, \mathrm{e}^{ipr(y)}\, \mathrm{e}^{-ipr(z)}: \; \mathrm{e}^{-p^2 C(y-z)}\, \mathrm{e}^{kp[C(x-z)-C(x-y)]} \ . \qquad (5.24)$$

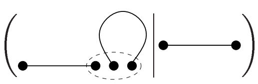
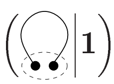
We want to study the contraction of x, y, and z, and look for all contributions which are proportional to

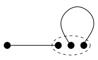
 $= \int_k :\mathrm{e}^{ikr(u)}: :\mathrm{e}^{-ikr((x+y+z)/3)}: \ .$ (5.25)

The key-observation is that in Eq. (5.24) the leading term is obtained by approximating

$$\mathrm{e}^{kp[C(x-z)-C(x-y)]} \approx 1 \ .$$

All subsequent terms yield factors of k, which after integration over k give derivatives of the $\tilde{\delta}^d$-distribution. The result is that

 $-$  $= 0$. (5.26)

This means that divergences of  are already taken into account by a proper treatment of the divergences in , analyzed in Eq. (5.9).

6 Strategy of renormalization

In the last two sections, we discussed how divergences occur, how their general structure is obtained by the MOPE, and how the MOPE-coefficients are calculated. In the next step, the theory shall be renormalized. The basic idea is to identify the divergences through the MOPE, and then to introduce counter-terms which subtract these divergences. These counter-terms are nothing else than integrals over the MOPE-coefficients, properly regularized, i.e. cut off.

In order to properly understand this point, let us recall the two main strategies employed in renormalization: The first one subtracts divergences in correlation-functions or equivalently vertex-functions. This amounts to adding counter-terms to the Hamiltonian which can be interpreted as a change of the parameters in this Hamiltonian. Calculating observables with this modified Hamiltonian leads to finite physical expectation values, but it is not evident that the integrals appearing in these calculations are convergent.

The other procedure is inspired by ideas employed in a formal proof of renormalizability, or more precisely when applying the $\mathbf{R}$-operation to the perturbation expansion. Details of this procedure can not be given here, but are discussed in [26,23]. It consists in adding to the Hamiltonian counter-terms which are *integrals*, such that each *integrand* which appears in the perturbative expansion becomes an integrable function, and *as a consequence* the integrals and thus the perturbation expansion are finite. Of course, to finally obtain the critical exponents, the integral counter-terms have to be reduced to numbers. However, we really want to think of them as integrals in the intermediate steps. The reason is the following: It is extremely difficult to calculate observables. However, this is not really necessary as long as one is only interested in renormalization. The above-mentioned procedure is then sufficient to ensure finiteness of any observable as long as there is no additional divergence when the dipole is contracted towards this observable. The latter situation would require a new counter-term, which is a proper renormalization of the observable itself. The procedure of considering whole integrals as counter-terms is in the heart of our renormalization procedure, and the reader should bear this idea in mind.

7 Renormalization at 1-loop order

Let us continue on the concrete example of the 1-loop divergences, from which are obtained the scaling exponents to first order in the dimensional regularization parameter ε. Explicitly, the model shall be renormalized through two renormalization group factors Z (renormalizing the field r) and Z_b (renormalizing the coupling b). Recalling Eq. (2.1), this is

$$\mathcal{H}[r] = \frac{Z}{2-D} \int_x \frac{1}{2} (\nabla r(x))^2 + b Z_b \mu^{\varepsilon} \int_x \int_y \tilde{\delta}^d(r(x) - r(y)) \,, \tag{7.1}$$

where r and b are the renormalized field and renormalized dimensionless coupling constant, and $\mu = L^{-1}$ is the renormalization momentum scale.

Let us start to eliminate the divergences in the case, where the end-points (x, y) of a single dipole are contracted towards a point (taken here to be the center-of-mass $z = (x+y)/2$). The MOPE is

$$x \quad y = \left(x \quad y \,\middle|\, \mathbf{1} \right) \mathbf{1} \; + \; \left(x \quad y \,\middle|\, \alpha \quad \beta \right) \alpha \quad \beta \; + \dots \,. \tag{7.2}$$

The MOPE-coefficients were obtained in the last section as

$$\left(x \quad y \,\middle|\, \mathbf{1} \right) = |x-y|^{-\nu d} \,, \tag{7.3}$$

$$\left(x \quad y \,\middle|\, \alpha \quad \beta \right) = -\frac{1}{2} \, (x-y)_\alpha \, (x-y)_\beta \, |x-y|^{-\nu(d+2)} \,. \tag{7.4}$$

We now have to distinguish between counter-terms for relevant operators and those for marginal operators. The former can be defined by analytic continuation, while the latter require a subtraction scale. Indeed, the divergence proportional to $\mathbf{1}$ is given by the integral

$$\int\limits_{\Lambda^{-1}<|x-y|<L} \left({}_x \bigcirc_{\bullet\bullet} {}_y \middle| \mathbf{1} \right) = \int\limits_{\Lambda^{-1}}^{L} \frac{\mathrm{d}x}{x} x^{D-\nu d} = \frac{1}{D-\varepsilon} \left(\Lambda^{D-\varepsilon} - L^{\varepsilon - D} \right) \,, \tag{7.5}$$

where Λ is a high-momentum UV-regulator and L a large distance regulator. For $\varepsilon \approx 0$ this is UV-divergent but IR-convergent. The simplest way to subtract this divergence is therefore to replace the dipole operator by

$$ {}_x\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet_y \longrightarrow {}_x\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet_y - {}_x\bullet\cdots\cdots\bullet_y \,, \tag{7.6}$$

where ${}_x\bullet\cdots\cdots\bullet_y = |x-y|^{-\nu d}$. This amounts to adding to the bare Hamiltonian (2.1) the UV-divergent counter-term

$$\Delta\mathcal{H}_{\mathbf{1}} = -b Z_b \mu^{\varepsilon} \int\limits_x \int\limits_y |x-y|^{-\nu d} \,, \tag{7.7}$$

which is a pure number and thus does not change the expectation-value of any physical observable.

We next consider marginal operators: In the MOPE of Eq. (7.2), the integral over the relative distance of $\int_{x-y} \left({}_x \bigcirc_{\bullet\bullet} {}_y \middle| {}_\alpha\!+\!{}_\beta \right) {}_\alpha\!+\!{}_\beta$ is logarithmically divergent at $\varepsilon = 0$. In order to find the appropriate counter-term, we use dimensional regularization, i.e. set $\varepsilon > 0$. An IR-cutoff L, or equivalently a subtraction momentum scale $\mu = L^{-1}$, has to be introduced in order to define the subtraction operation. *As a general rule, let us integrate over all distances appearing in the MOPE-coefficient, bounded by the subtraction scale $L = \mu^{-1}$.* Defining

$$\left\langle \bigcirc_{\bullet\bullet} \middle| {}_\alpha\!+\!{}_\beta \right\rangle_L := \int\limits_{|x-y|<L} \left({}_x \bigcirc_{\bullet\bullet} {}_y \middle| {}_\alpha\!+\!{}_\beta \right) \tag{7.8}$$

we need the following counter-term in the Hamiltonian

$$\Delta\mathcal{H}_{+} = -b\mu^{\varepsilon} \left\langle \bigcirc_{\bullet\bullet} \middle| {}_\alpha\!+\!{}_\beta \right\rangle_L \int\limits_x {}_\alpha\!+\!{}_\beta\, {}_x \,, \tag{7.9}$$

subtracting explicitly the divergence in the integrals, as discussed in the last section. The reader is invited to verify this explicitly on the example of the expectation value of $\mathcal{O} = \mathrm{e}^{ik[r(s)-r(t)]}$, as given in Eq. (3.10). The solution is given in appendix B.

Since the angular integration in Eq. (7.8) reduces ${}_\alpha\!+\!{}_\beta$ to $+$, we can replace Eq. (7.9) by the equivalent expression

$$\Delta\mathcal{H}_{+} = -b\mu^{\varepsilon} \left\langle \bigcirc_{\bullet\bullet} \middle| + \right\rangle_L \int\limits_x +_x \,, \tag{7.10}$$

for which the numerical value of the diagram is calculated as

$$\left\langle \bigcirc_{\bullet\bullet} \middle| + \right\rangle_L = \int\limits_{|x-y|<L} \left({}_x \bigcirc_{\bullet\bullet} {}_y \middle| + \right) = -\frac{1}{2D} \int\limits_0^L \frac{\mathrm{d}x}{x} x^{2D-\nu d} = -\frac{1}{2D} \frac{L^{\varepsilon}}{\varepsilon} \,. \tag{7.11}$$

We can now subtract this term in a minimal subtraction scheme (MS). The internal dimension of the membrane D is kept fixed and (7.11) is expanded as a Laurent series in ε, which here starts at ε^{-1}. Denoting by $\langle\ |\ \rangle_{\varepsilon^p}$ the term of order ε^p of the Laurent expansion of $\langle\ |\ \rangle_L$ for $L = 1$, the residue of the pole in Eq. (7.11) is found to be

$$\Big\langle \quad \Big| \quad \Big\rangle_{\varepsilon^{-1}} = -\frac{1}{2D}\frac{1}{\varepsilon}\,. \tag{7.12}$$

We shall also employ the notation for the residue

$$\Big\langle \quad \Big| \quad \Big\rangle_{\varepsilon} = -\frac{1}{2D}\,. \tag{7.13}$$

It is this pole that is subtracted in the MS-scheme by adding to the Hamiltonian a counter-term

$$\Delta\mathcal{H}_{\ } = -\,b\Big\langle \quad \Big| \quad \Big\rangle_{\varepsilon^{-1}} \int_x \ {}_x\,. \tag{7.14}$$

Note that by going from Eq. (7.9) to Eq. (7.14), we have reduced the integral counter-term to a number. We recall our initial remark that if one wants to check that this counter-term renders the theory finite, one should think of it as its defining integral (7.9), and verify that in the resulting perturbation theory, the first-order divergence is absent.

Similarly, the divergence arising from the contraction of two dipoles to a single dipole is subtracted by a counter-term

$$\Delta\mathcal{H}_{\ } = b^2\mu^{2\varepsilon}\Big\langle \quad \Big| \quad \Big\rangle_L \int_x\int_y {}_x \quad {}_y\ , \tag{7.15}$$

with

$$\Big\langle \quad \Big| \quad \Big\rangle_L = \int_{|x|<L}\int_{|y|<L} \Big(\quad \Big| \quad \Big)\,. \tag{7.16}$$

Reducing this integral counter-term to a number, we subtract the residue of the single pole of

$$\Big\langle \quad \Big| \quad \Big\rangle_L = \int_{|x|<L}\int_{|y|<L} \Big(\quad \Big| \quad \Big) = \int_{|x|<L}\int_{|y|<L} \left(|x|^{2\nu}+|y|^{2\nu}\right)^{-d/2}\,. \tag{7.17}$$

Note that the regulator L cuts off both integrations. One can now either utilize some simple algebra or show by the methods of conformal mapping (see [26]) that the residue is obtained by fixing one distance to equal 1 and by freely integrating over the remaining one

$$\Big\langle \quad \Big| \quad \Big\rangle_\varepsilon = \int_0^\infty \frac{\mathrm{d}x}{x}x^D\left(1+x^{2-D}\right)^{-2D/(2-D)}\,. \tag{7.18}$$

(Recall that $d/2 = 2D/(2-D) + O(\varepsilon)$.) The above is easily related to Euler's B-function and reads

$$\Big\langle \quad \Big| \quad \Big\rangle_\varepsilon = \frac{1}{2-D}\frac{\Gamma\left(\frac{D}{2-D}\right)^2}{\Gamma\left(\frac{2D}{2-D}\right)}\,. \tag{7.19}$$

As a result, the model is UV-finite at 1-loop order, if we use in the renormalized Hamiltonian (7.1) the renormalization factors Z and Z_b

$$Z = 1 - (2-D)\left\langle \quad \middle| \quad \right\rangle_{\varepsilon} \frac{b}{\varepsilon} + O(b^2) \tag{7.20}$$

$$Z_b = 1 + \left\langle \quad \middle| \quad \right\rangle_{\varepsilon} \frac{b}{\varepsilon} + O(b^2)\ . \tag{7.21}$$

Note that due to Eq. (5.26) no counter-term for is necessary.

The renormalized field and coupling are re-expressed in terms of their bare counterparts through

$$r_0(x) = Z^{1/2}\, r(x)\ , \qquad b_0 = b\, Z_b\, Z^{d/2}\, \mu^{\varepsilon}\ . \tag{7.22}$$

Finally, the renormalization group functions are obtained from the variation of the coupling constant and the field with respect to the renormalization scale μ, keeping the bare coupling fixed. (For a derivation, see appendix A). The flow of the coupling is written in terms of Z and Z_b as

$$\begin{aligned}
\beta(b) := \mu \frac{\partial}{\partial\mu}\bigg|_{b_0} b &= \frac{-\varepsilon b}{1 + b\frac{\partial}{\partial b}\ln Z_b + \frac{d}{2} b \frac{\partial}{\partial b}\ln Z} \\
&= -\varepsilon b + \left(\left\langle \quad \middle| \quad \right\rangle_{\varepsilon} - \nu d \left\langle \quad \middle| \quad \right\rangle_{\varepsilon} \right) b^2 + O(b^3) \\
&= -\varepsilon b + \left(\frac{1}{2-D} \frac{\Gamma\left(\frac{D}{2-D}\right)^2}{\Gamma\left(\frac{2D}{2-D}\right)} + \frac{(2-D)d}{4D} \right) b^2 + O(b^3)\ .
\end{aligned} \tag{7.23}$$

Similarly, the full dimension of the field (the exponent entering into the correlation function) is obtained as

$$\begin{aligned}
\nu(b) := \frac{2-D}{2} - \frac{1}{2}\mu \frac{\partial}{\partial\mu}\bigg|_{b_0} \ln Z &= \frac{2-D}{2} - \frac{1}{2}\beta(b)\frac{\partial}{\partial b}\ln Z \\
&= \frac{2-D}{2}\left[1 - b\left\langle \quad \middle| \quad \right\rangle_{\varepsilon}\right] + O(b^2) \\
&= \frac{2-D}{2}\left[1 + b\frac{1}{2D}\right] + O(b^2)\ .
\end{aligned} \tag{7.24}$$

Note that minimal subtraction is used on the level of counter-terms or equivalently Z-factors. Since Z enters as Z^d into the β-function, the latter also contains a factor of d in the 1-loop approximation, i.e. Z^d is not minimally renormalized. In order to calculate the leading order in ε, the factor of d can be replaced by $d_c = \frac{4D}{2-D}$.

The β-function has a non-trivial fixed-point with $\beta(b^*) = 0$, which has positive slope and thus describes the behavior of the model at large distances:

$$b^* = \frac{\varepsilon}{\frac{1}{2-D}\frac{\Gamma\left(\frac{D}{2-D}\right)^2}{\Gamma\left(\frac{2D}{2-D}\right)} + 1} + O(\varepsilon^2)\ . \tag{7.25}$$

The anomalous dimension $\nu^* := \nu(b^*)$ becomes to first order in ε

$$\nu^* = \frac{2-D}{2}\left[1 + \frac{\varepsilon}{2D}\frac{1}{\frac{1}{2-D}\frac{\Gamma\left(\frac{D}{2-D}\right)^2}{\Gamma\left(\frac{2D}{2-D}\right)} + 1}\right] + O(\varepsilon^2)\ . \tag{7.26}$$

For polymers, this result reduces to the well-known formula

$$\nu^*(D=1) = \frac{1}{2} + \frac{4-d}{16} + O((4-d)^2)\ . \tag{7.27}$$

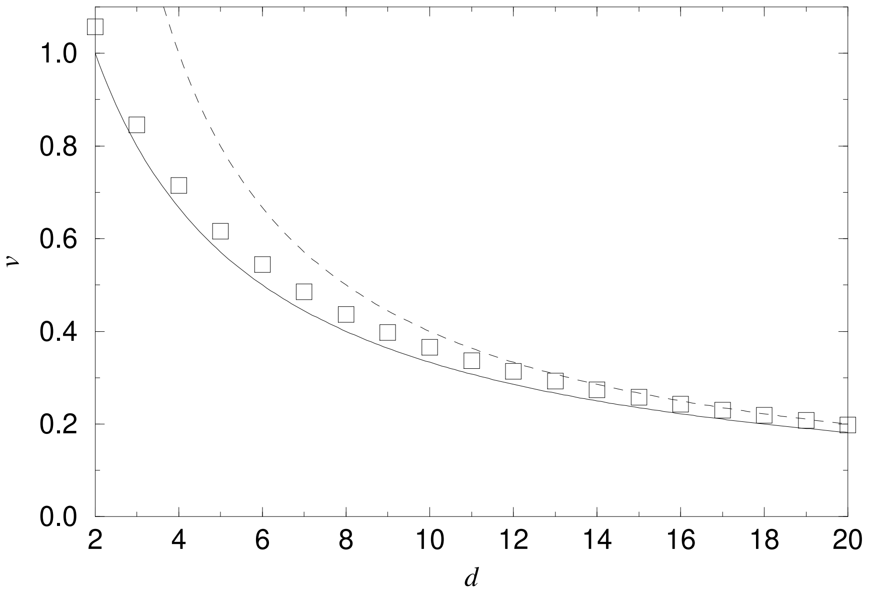
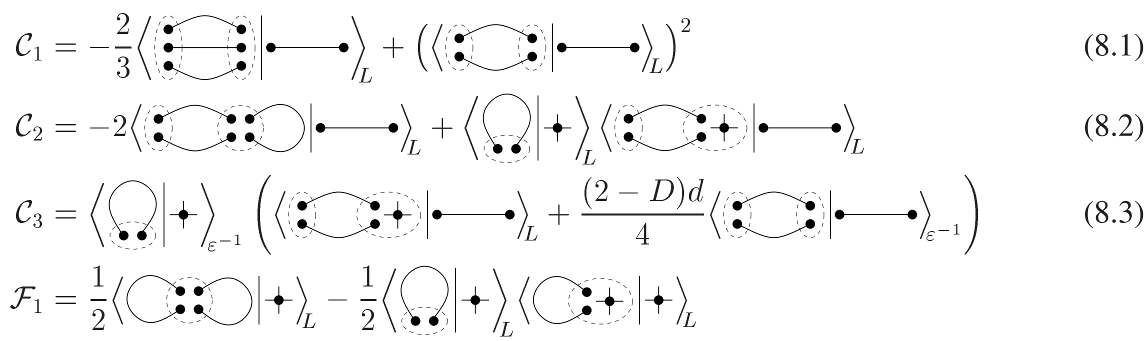
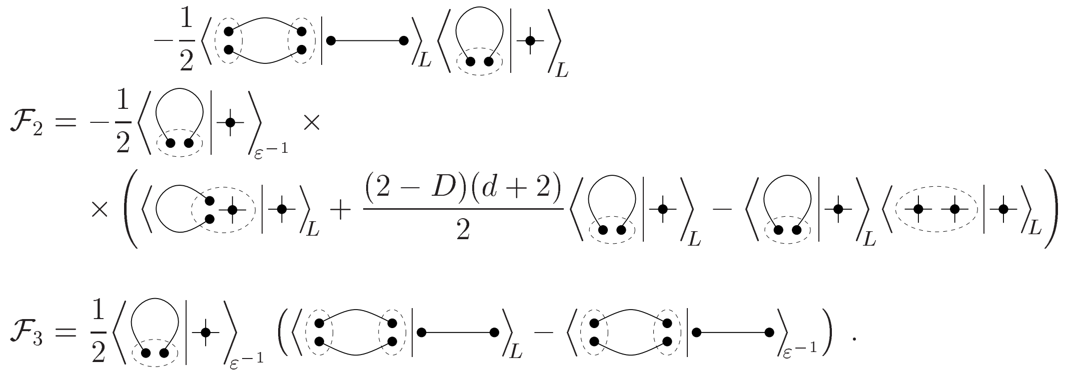


Figure 7.1: Extrapolation of the 2-loop results in d and ε for membranes $D = 2$ in d dimensions, using the expansion of $\nu^*(d+2)$ (squares). The solid line is the prediction made by Flory's approximation, the dashed line by the variational ansatz.

8 Results for self-avoiding membranes from 2-loop calculations

Two loop results are obtained by an explicit analytic and numeric integration of the following combination of diagrams (which are all of order $1/\varepsilon$)





(8.4)

(8.5)

(8.6)

One then has the freedom to extrapolate about any point on the critical curve $\varepsilon(D,d)=0$. This freedom – or ambiguity – can be used to optimize the results. The results of such a procedure for the 2-loop cacluations of ν^* are given on figure 7.1 for membranes ($D=2$) in d dimensions ($2 \le d \le 20$). We see that for $d \to \infty$ the prediction of a Gaussian variational method

$$\nu_{\mathrm{var}} = \frac{2D}{d} \tag{8.7}$$

becomes exact. For small d, the prediction made by Flory's argument

$$\nu_{\mathrm{Flory}} = \frac{2+D}{2+d} \tag{8.8}$$

is close to our results. This is a non-trivial result, since the membrane case corresponds to $\varepsilon = 4$ and in comparison with polymers in $d=3$, where $\varepsilon = 1/2$, the 2-loop corrections were expected to be large. In fact, they are small when one expands around the critical curve $\varepsilon = 0$ for an adequate range of $D \sim 1.5$ (depending slightly on d and on the choice of variables) and a suitable choice of extrapolation variables. In this case the 2-loop corrections are even smaller than the 1-loop corrections and allow for more reliable extrapolations to $\varepsilon = 4$. This can be understood from the large order behavior [27].

Let us now turn to the physically relevant case of membranes in three dimensions ($D=2$, $d=3$). Our calculations predict an exponent $\nu^* \approx 0.85$ or equivalently a fractal dimension of $d_{\mathrm{f}} \approx 2.4$, which is in agreement with those experiments and simulations which find a fractal phase. However, this is still under debate, and a lot of evidence has been collected that the flat phase is generic. (For a more detailed discussion of this problem, see the review [26].)

9 Outlook

In these lecture notes, we have only been able to discuss the simplest applications of non-local field theories. Much more has been achieved during the last years.

First of all, it has been demonstrated that the dynamics, described by a Langevin-equation

$$\frac{\mathrm{d}}{\mathrm{d}t} r(x,t) = -\frac{\partial \mathcal{H}[r]}{\partial r(x,t)} + \zeta(x,t) \ , \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = \lambda \delta^D(x-x')\delta(t-t') \tag{9.1}$$

also leads to a renormalizable field theory [28], where the dynamic exponent z, defined by the decay of the auto-correlation function $\langle [r(x,t)-r(x,t')]^2\rangle \sim |t-t'|^{2/z}$ is given by $z =$

$2+D/\nu^* = 2+d_{\rm f}$, where $d_f = D/\nu^*$ is the fractal dimension of the membrane. This result had been suggested long-time ago by DeGennes [29] for polymers and by Kardar et al. for membranes [14], but only with the methods discussed above, a proof of that conjectures could be given [28]. When hydrodynamics is included, the dynamical exponent z changes to $z = d$ [30].

Interestingly, also disorder can be treated via the same methods, since averaging over disorder leads to interactions very similar to self-avoidance. The above methods have been applied to the motion of a polymer or membrane in a static disordered force field with both potential and non-conserved parts – leading to new universal physics [31,32].

Also the advection of a polymer in a turbulent flow has been analyzed [33], paralling the discusion of the passive advection of particles, which in the turbulence community is known as the passive scalar problem.

Another interesting generalization is to anisotropically tethered membranes [34–36]. In these models, the membrane is more rigid in one direction, forming a tubular phase.

One of the most useful tricks for self-avoiding polymers is the mapping onto a massive scalar field-theory, i.e. a ϕ^4-theory [37]. In the limit, where the number of compoents $n \to 0$, results for self-avoiding polymers are obtained. Stated differently, ϕ^4-theory is a generalization of self-avoiding polymers. Another generalization has been discussed above: The generalization to membranes with internal dimension $D \neq 1$. The question arizes, whether a common generalization of both the ϕ^4-theory and polymerized membranes is possible. Such a model has indeed been constructed [38,39], and leads to interesting new physics.

Another still open question is the analysis of the spectrum of subdominant operators [40]. This may give a hint of why polymerized membranes are generically seen flat in simulations.

10 Acknowledgements

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A Derivation of the RG-equations

In this section, we give a derivation of the renormalization group functions. Starting from

$$b = b_0\, Z_b^{-1}\, Z^{-d/2}\, \mu^{-\varepsilon} \;, \tag{A.1}$$

where b is the dimensionless renormalized coupling and b_0 the dimensionfull bare one, the β-function is given through the variation of the renormalized coupling, at fixed bare quantities, as

$$\beta(b) := \mu \frac{\partial}{\partial \mu}\bigg|_0 b \;. \tag{A.2}$$

From the derivative of Eq. (A.1) with respect to μ, we obtain

$$\beta(b)\left(1 + b\frac{\partial}{\partial b}\ln(Z_b Z^{d/2})\right) = -\varepsilon b \;, \tag{A.3}$$

leading to

$$\beta(b) = \frac{-\varepsilon b}{1 + b\frac{\partial}{\partial b}\ln(Z_b Z^{d/2})} \ . \tag{A.4}$$

For infinite membranes, the scaling exponent ν^* is obtained from the large-distance behavior of the correlation-function

$$C(|x-y|, b, \mu) := \frac{1}{2d}\left\langle (r(x) - r(y))^2 \right\rangle_{b,\mu} \ , \tag{A.5}$$

as

$$C(\ell, b, \mu) \sim \ell^{2\nu^*} \ . \tag{A.6}$$

To calculate ν^*, we first observe that the bare correlator $C_0(\ell, b_0) = ZC(\ell, b, \mu)$ is independent of the renormalization-scale μ. Therefore

$$\begin{aligned} 0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}C_0(\ell, b_0) &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left[Z(b)C(\ell, b, \mu)\right] \\ &= Z(b)\left[\beta(b)\frac{\partial}{\partial b}\ln Z(b) + \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\right]C(\ell, b, \mu) \ . \end{aligned} \tag{A.7}$$

Since $C(\ell, b, \mu)$ has canonical dimension $D-2$ in units of μ, it can be written as

$$C(\ell, b, \mu) = \mu^{D-2} f(b, \mu\ell) \ . \tag{A.8}$$

We thus obtain

$$\begin{aligned} 0 &= \left[\beta(b)\frac{\partial}{\partial b}\ln Z(b) + \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\right]\mu^{D-2} f(b, \mu\ell) \\ &= \left[\beta(b)\frac{\partial}{\partial b}\ln Z(b) + D - 2 + \ell\frac{\partial}{\partial \ell} + \beta(b)\frac{\partial}{\partial b}\right]\mu^{D-2} f(b, \mu\ell) \ . \end{aligned} \tag{A.9}$$

Denoting

$$\nu(b) = \frac{2-D}{2} - \frac{1}{2}\beta(b)\frac{\partial}{\partial b}\ln Z(b) \ , \tag{A.10}$$

and reinserting Eq. (A.8), Eq. (A.9) is written as

$$\ell\frac{\partial}{\partial \ell}C(\ell, b, \mu) = \left[2\nu(b) - \beta(b)\frac{\partial}{\partial b}\right]C(\ell, b, \mu) \ . \tag{A.11}$$

At the IR-fixed point b^* with $\beta(b^*) = 0$, the dimension of the field is thus

$$\nu^* = \nu(b^*) \ . \tag{A.12}$$

B Finiteness of observables within the renormalized model

Show that within the renormalized model (7.1) and to first order in b, the expectation value of $\mathcal{O} = \mathrm{e}^{ik[r(s)-r(t)]}$ as given for the bare model in Eq. (3.10) is UV- and IR-finite.
<u>Solution:</u>

Two counter-terms are relevant at first order in the renormalized coupling b, namely $\Delta\mathcal{H}_{\mathbf{1}}$, Eq. (7.7) and $\Delta\mathcal{H}_{+}$, Eq. (7.9). The third 1-loop term, $\Delta\mathcal{H}_{\bullet\!-\!-\!\bullet}$, Eq. (7.9), will only show up at order b^2. Also note, that the first one, $\Delta\mathcal{H}_{\mathbf{1}}$, has already been taken into account in Eq. (3.10), due to the normalization introduced in Eq. (2.12).

At first order in the renormalized coupling b, we thus only have to take care of the counter-term (7.9), with the full tensorial structure, resulting in

$$
\begin{aligned}
\langle \mathcal{O} \rangle_b = \mathrm{e}^{-k^2 C(s-t)} \times & \\
\Bigg\{ 1 + b\mu^{\varepsilon} \iint\limits_{x\ y} \Bigg[& 1 - \exp\left(\frac{1}{4} k^2 \frac{\left[C(s-x) + C(t-y) - C(s-y) - C(t-x) \right]^2}{C(x-y)} \right) \\
& + \frac{1}{4} k^2 \frac{\left[(x-y)\nabla_z \left(C(s-z) - C(t-z) \right) \right]^2}{C(x-y)} \, \Theta(|x-y| < L) \Bigg] C(x-y)^{-d/2} \\
& + O(b^2) \Bigg\} \ ,
\end{aligned}
\tag{B.1}
$$

where z may either be chosen as x or y, or the symmetrized version may be used. The terms in the big square brackets exactly cancel the divergence for small $x - y$. Note that there is a possible IR-divergence for $|x - y| \to \infty$. In the original term, this is regularized by the observable-positions s and t, which effectively cut off the integral at $|x - y| \approx |s - t|$. In the counter-term, the IR-cutoff is explicit.

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