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Vorlesung: Ungeordnete Systeme

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Contents

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1 Introduction

1.1 Motivation

1.2 The Mean Field Approximation (MF)

Take the Ising-Model

$$\mathcal{H}[\sigma] := -\frac{1}{2}\sum_{r,r'} J(r-r')\sigma(r)\sigma(r') - h\sum_r \sigma(r) \ . \tag{1}$$

<u>Lemma:</u>

$$\mathcal{Z} = \sum_{\{\sigma\}} \mathrm{e}^{-\beta\mathcal{H}[\sigma]} \geq \mathrm{e}^{-\beta\langle\mathcal{H}-\mathcal{H}_0\rangle_{\mathcal{H}_0}} \sum_{\{\sigma\}} \mathrm{e}^{-\beta\mathcal{H}_0[\sigma]} \ . \tag{2}$$ variational ansatz

The mean-field solution is obtained by optimizing the r.h.s. of Eq. (2) w.r.t. $\mathcal{H}_0$.

For the Ising-model, the variational Ansatz is

$$\mathcal{H}_0 = \sum_r \tilde{h}\sigma(r) \ . \tag{3}$$

The best $\tilde{h}$ is for

$$\tilde{h} = h + \sum_{r\neq 0} J(r)\tanh(\beta\tilde{h}) \ . \tag{4}$$

2 Spin-Glasses

2.1 General Considerations

2.1.1 The Model

$$\mathcal{H} = -\frac{1}{2}\sum_{i\neq j} J_{ij}\sigma_i\sigma_j \ , \tag{5}$$

where J_{ij} is a random variable.

2.1.2 Frustration

2.1.3 Ergodicity Breaking

2.1.4 Order-parameter for Disordered Systems

$$q_{\alpha\beta} := \frac{1}{N}\sum_{i=1}^{N} \overline{\langle\sigma_\alpha\rangle\langle\sigma_\beta\rangle} \ . \tag{6}$$

2.2 Mean-Field Theory for Spin-Glasses

2.2.1 The Sherrington Kirkpatrick (SK) Model

For N spins, the Sherrington Kirkpatrick (SK) Model is defined as

$$\boxed{\mathcal{H} = -\frac{1}{2}\sum_{i\neq j}^{N} J_{ij}\sigma_i\sigma_j \ , \quad \mathcal{P}[J_{ij}] = \prod_{i<j}\sqrt{\frac{N}{2\pi}}\mathrm{e}^{-\frac{N}{2}J_{ij}^2}} \ . \tag{7}$$

Attention: The scaling with N is chosen such that thermodynamic quantities like the free energy are extensive.

2.2.2 Saddle Point Equations

We want to calculate

$$Z_n := \overline{Z[J]^n} \ . \tag{8}$$

This can be written as

$$\begin{aligned}
&\sum_{\{\sigma\}}\int\mathcal{D}[J]\exp\left[\beta\sum_{i<j}\sum_{\alpha=1}^{n} J_{ij}\sigma_i^\alpha\sigma_j^\alpha - \frac{N}{2}\sum_{i<j}J_{ij}^2\right] \\
&= \sum_{\{\sigma\}}\exp\left[\frac{\beta^2}{2N}\sum_{i<j}\left(\sum_{\alpha=1}^{n}\sigma_i^\alpha s_j^\alpha\right)^2\right]
\end{aligned}$$

$$= \sum_{\{\sigma\}} \exp\left[-\frac{\beta^2 n^2}{4} + \frac{\beta^2 nN}{4} + \frac{\beta^2}{2N} \sum_{\alpha<\beta} \left(\sum_{i=1}^{N} \sigma_i^\alpha \sigma_i^\beta \right)^2 \right] . \tag{9}$$

The last square is disentangled using

$$\exp\left[\frac{\beta^2}{2N} \left(\sum_{i=1}^{N} \sigma_i^\alpha \sigma_i^\beta \right)^2 \right] = \beta \int_{-\infty}^{\infty} \frac{\mathrm{d}Q_{\alpha\beta}}{\sqrt{2\pi N}} \exp\left[\beta^2 Q_{\alpha\beta} \left(\sum_{i=1}^{N} \sigma_i^\alpha \sigma_i^\beta \right) - \frac{\beta^2 N}{2} Q_{\alpha\beta}^2 \right] . \tag{10}$$

Defining

$$\int \mathcal{D}[Q_{\alpha\beta}] := \prod_{\alpha<\beta} \left[\beta\sqrt{N} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q_{\alpha\beta}}{\sqrt{2\pi}} \right] \tag{11}$$

this is written as

$$Z_n = \sum_{\{\sigma\}} \int \mathcal{D}[Q_{\alpha\beta}] \, \exp\left[\frac{\beta^2 n(N-n)}{4} + \beta^2 \sum_{\alpha<\beta} Q_{\alpha\beta} \sum_{i=1}^{N} \sigma_i^\alpha \sigma_i^\beta - \frac{\beta^2 N}{2} \sum_{\alpha<\beta} Q_{\alpha\beta}^2 \right] . \tag{12}$$

The integral remains invariant under the substitution $Q_{\alpha\beta} \to Q_{\alpha\beta} + \delta Q_{\alpha\beta}$, implying that ($\langle \ldots \rangle_Q$ means the expectation value w.r.t. Q)

$$\langle Q_{\alpha\beta} \rangle_Q = \frac{1}{N} \sum_i \overline{\left\langle \sigma_i^\alpha \sigma_i^\beta \right\rangle} . \tag{13}$$

The sum over σ_i^α factorizes for different sites i. We finally obtain (for large N)

$$Z_n = \int \mathcal{D}[Q_{\alpha\beta}] \, \exp\left[-nN\beta f_n(Q_{\alpha\beta}) \right] \tag{14}$$

with

$$f_n(Q_{\alpha\beta}) = -\frac{\beta}{4} + \frac{\beta}{2n} \sum_{\alpha<\beta} Q_{\alpha\beta}^2 - \frac{1}{\beta n} \ln\left(\sum_{\{\sigma^\alpha\}} \exp\left[\sum_{\alpha<\beta} \beta^2 Q_{\alpha\beta} \sigma^\alpha \sigma^\beta \right] \right) . \tag{15}$$

In the thermodynamic limit of $N \to \infty$, one can approximate Z_n by $f_n(Q_{\alpha\beta})$ evaluated at the saddlepoint w.r.t. $Q_{\alpha\beta}$. For general $Q_{\alpha\beta}$ this is difficult.

2.2.3 Replica symmetric solution

Since the general problem is far from beeing tractable, we start with the simmplist approximation, called the replica symmetric (RS) solution:

$$Q_{\alpha\beta} = q \quad \forall \alpha \neq \beta \, , \qquad Q_{\alpha\alpha} = 0 \, . \tag{16}$$

The free enery f as a function of q is

$$f(q) = -\frac{\beta}{4}(1-q)^2 - \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2} \ln(2\cosh(\beta z \sqrt{q})) \, . \tag{17}$$

The saddle-point equation $\frac{\mathrm{d}}{\mathrm{d}q} f(q) = 0$ reads

$$q = \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2} \tanh^2(\beta z \sqrt{q}) \tag{18}$$

SKrepsymSP

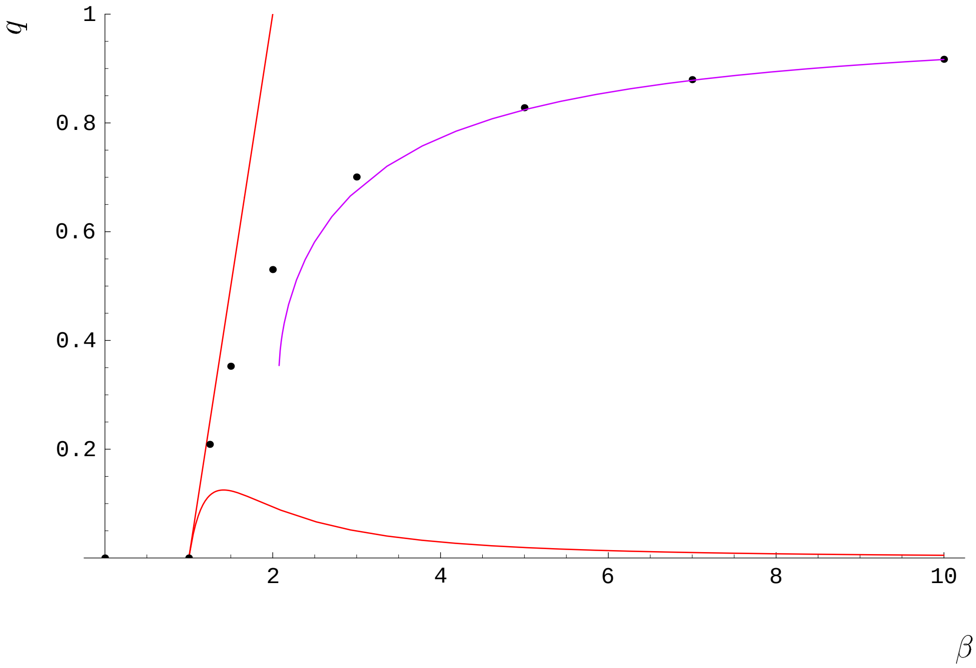

Figure 1: The solution of the saddle-point equation (18), as well as the limiting analytic solutions for $\beta \to 1$ and $\beta \to \infty$. f:order paramter

2.2.4 Thermodynamcis

This solution however is ill, as can be seen from the following calculation. The entropy-density is

$$s = -\frac{\mathrm{d}f}{\mathrm{d}T} = \beta^2 \frac{\mathrm{d}f}{\mathrm{d}\beta} = \beta^2 \frac{\partial f}{\partial \beta} + \beta^2 \frac{\partial q}{\partial \beta} \frac{\partial f}{\partial q} \ . \tag{19}$$

Since at the saddle-point $\frac{\partial f}{\partial q} = 0$, we can drop the last term. After some lines of calculation, one gets

$$s = -\frac{\beta^2}{4}(1-q)(3q+1) + \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2} \ln(2\cosh(\beta z \sqrt{q})) \ . \tag{20}$$ bad entropy

After a long and nasty calculation, we obtain the entropy in the limit of $\beta \to \infty$

$$s(T=0) = -\frac{1}{2\pi} \ , \tag{21}$$

in contradiction to elementary principles of statistical mechancis.
Exercise: Show Eq. (20)
Hint: You have to use the following large γ-expansion

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} \frac{\mathrm{e}^{-\frac{1}{2}z^2}}{\cosh^2(\gamma z)} &= \frac{1}{\gamma\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-\frac{1}{2\gamma^2}z^2} \frac{1}{\cosh^2(z)} \\ &= \frac{1}{\gamma\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}z \left[1 - \frac{1}{2\gamma^2}z^2 + \frac{1}{16\gamma^4}z^4 - \dots\right] \frac{1}{\cosh^2(z)} \end{aligned} \tag{22}$$

which shows that corrections are of order $1/\gamma^2$. (The first correction in $1/\gamma^2$ is easily done analytically.) There is a second similar formula. One thus obtains the violett curve in figure 1.

2.3 Replica-Symmetry Breaking (RSB)

2.3.1 Motivation

2.3.2 1-Step Replica-Symmetry Breaking

$$
\mathbb{Q} = \begin{pmatrix}
0 & q_1 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\
q_1 & 0 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\
q_1 & q_1 & 0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 & 0 & q_1 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 & q_1 & 0 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 & q_1 & q_1 & 0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & 0 & q_1 & q_1 & q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_1 & 0 & q_1 & q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_1 & q_1 & 0 & q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & 0 & q_1 & q_1 \\
q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_1 & 0 & q_1 \\
q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_1 & q_1 & 0
\end{pmatrix} \tag{23}
$$

2.3.3 Multiple Step Replica-Symmetry Breaking

2.3.4 The Parisi RSB-Algebra

Be $\mathbb{Q}$ a Parisi-Matrix

$$
\mathbb{1}_i \equiv \mathbb{1}_{m_i} = \begin{pmatrix}
1&1&1&0&0&0&0&0&0&0&0&0 \\
1&1&1&0&0&0&0&0&0&0&0&0 \\
1&1&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&1&1&0&0&0&0&0&0 \\
0&0&0&1&1&1&0&0&0&0&0&0 \\
0&0&0&1&1&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&1&1&0&0&0 \\
0&0&0&0&0&0&1&1&1&0&0&0 \\
0&0&0&0&0&0&1&1&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&1&1 \\
0&0&0&0&0&0&0&0&0&1&1&1 \\
0&0&0&0&0&0&0&0&0&1&1&1
\end{pmatrix} , \tag{24}
$$

where here $m_i = 3$. The matrices are constructed such that $i < j$ implies $m_i > m_j$ and $\mathrm{ggt}(m_i, m_j) = \min\{m_i, m_j\}$. The set $\mathcal{P}$ of all Parisi-matrices forms a closed algebra. This is explicitly seen from the following rules: One has (denoting by $\cdot$ the Hadamard-product $(\mathbb{P} \cdot \mathbb{Q})_{\alpha\beta} = \mathbb{P}_{\alpha\beta}\mathbb{Q}_{\alpha\beta}$)

$$
\mathbb{1}_i \cdot \mathbb{1}_j = \mathbb{1}_{\max\{i,j\}} \tag{25}
$$

$$
\mathbb{1}_i \mathbb{1}_j = m_{\max\{i,j\}} \mathbb{1}_{\min\{i,j\}} \tag{26}
$$

$$
\overline{\mathbb{1}}_i := \mathbb{1}_i - \mathbb{1}_{i+1} \tag{27}
$$

$$
\overline{\mathbb{1}}_i \overline{\mathbb{1}}_j = \begin{cases} (m_i - m_{i+1})\mathbb{1}_i - m_{i+1}\overline{\mathbb{1}}_i & \forall i = j \\ \overline{\mathbb{1}}_i (m_j - m_{j+1}) & \forall i < j \end{cases} . \tag{28}
$$

Setting

$$
\mathbb{P} = \tilde{p}\mathbb{1} + \sum_i p_i \overline{\mathbb{1}}_i . \tag{29}
$$

Then

$$\mathbb{R} = \mathbb{P}\mathbb{Q} \tag{30}$$

is equivalent to

$$\tilde{r} = \tilde{p}\tilde{q} + \sum_i p_i q_i (m_i - m_{i+1}) \tag{31}$$

$$r_i = \tilde{p}q_i + \tilde{q}p_i + \sum_{j\le i} p_j q_j (m_j - m_{j+1}) - p_i q_i m_{i+1} + \sum_{j>i} (p_i q_j + p_j q_i)(m_j - m_{j+1})\,. \tag{32}$$

Taking $n < 1$ gives

$$\tilde{r} = \tilde{p}\tilde{q} - \int_n^1 \mathrm{d}x\, p(x)q(x) \tag{33}$$

$$r(x) = (\tilde{p} - \langle p\rangle)q(x) + (\tilde{q} - \langle q\rangle)p(x) - np(x)q(x) - \int_n^x \mathrm{d}y\ \left(p(y) - p(x)\right)\left(q(y) - q(x)\right) \tag{34}$$

We will mostly use $n = 0$:

$$\boxed{\begin{aligned} \tilde{r} &= \tilde{p}\tilde{q} - \int_0^1 \mathrm{d}x\, p(x)q(x) \\ r(x) &= (\tilde{p} - \langle p\rangle)q(x) + (\tilde{q} - \langle q\rangle)p(x) - \int_0^x \mathrm{d}y\ \left(p(y) - p(x)\right)\left(q(y) - q(x)\right) \end{aligned}} \tag{35}$$ parisi-mul

Inverse Matrix $\mathbb{Q}^{-1}$ We want to solve $\mathbb{P}\mathbb{Q} = \mathbb{1}$ for $\mathbb{P}$. This is obtained from (35) for $\tilde{r} = 1$ and $r(x) = 0$. We define

$$\boxed{[q](y) := \int_n^y \mathrm{d}x\, [q(y) - q(x)]}\,, \tag{36}$$ [q]

which will be mostly used at $n = 0$. We do that now. We have

$$[q]'(y) = yq'(y)\,. \tag{37}$$ [q]'

We state that

$$\left(\tilde{q} - \langle q\rangle - [q](x)\right)\left(\tilde{p} - \langle p\rangle - [p](x)\right) = 1\,. \tag{38}$$ MPAII4

This is proven by observing that deriving the second equation in (35) by x gives x times the derivative of (38). The constant (w.r.t. x) is obtained by evaluating (35) (both equations!) and (38) at $x = 1$.

From (38) we obtain

$$[p](x) = -\frac{1}{\tilde{q} - \langle q\rangle}\frac{[q](x)}{\tilde{q} - \langle q\rangle - [q](x)}\,. \tag{39}$$ K1

Deriving this gives

$$p'(x) = -\frac{1}{\tilde{q} - \langle q\rangle}\frac{1}{x}\frac{\partial}{\partial x}\frac{[q](x)}{\tilde{q} - \langle q\rangle - [q](x)} \tag{40}$$ K2

Using (35) and (38) at $x = 0$, we obtain

$$p(0) = -\frac{q(0)}{(\tilde{q} - \langle q \rangle)^2} \tag{41}$$ K3

(40) and (41) together give after partial integration

$$p(x) = -\frac{1}{\tilde{q} - \langle q \rangle}\left(\frac{[q](x)}{x(\tilde{q} - \langle q \rangle - [q](x))} + \frac{q(0)}{\tilde{q} - \langle q \rangle} + \int_0^x \frac{\mathrm{d}y}{y^2}\frac{[q](y)}{\tilde{q} - \langle q \rangle - [q](y)}\right) \tag{42}$$ K4

$\tilde{q}$ can be infered from (38) at $x = 0$ and (42) as

$$\tilde{q} = \frac{1}{\tilde{q} - \langle q \rangle}\left(1 - \frac{q(0)}{\tilde{q} - \langle q \rangle} - \int_0^1 \frac{\mathrm{d}y}{y^2}\frac{[q](y)}{\tilde{q} - \langle q \rangle - [q](y)}\right) \tag{43}$$ K5

Other useful formulas are

$$\begin{aligned} \tilde{q} - q(x) &= \frac{1}{x(\tilde{q} - \langle q \rangle - [q](x))} - \int_x^1 \frac{\mathrm{d}y}{y^2}\frac{1}{\tilde{q} - \langle q \rangle - [q](y)} && (44)\ \text{K6} \\ &= \frac{1}{\tilde{q} - q(1)} - \int_x^1 \mathrm{d}y \frac{q'(y)}{(\tilde{q} - \langle q \rangle - [q](y))^2}\ . && (45) \end{aligned}$$

<u>Eigenvalues and Multiplicities:</u> Look at matrix $\mathbb{Q}$. Eigenvector $\vec{e}_0 = (1, 1, \ldots, 1)$ with Eigenvalue

$$\lambda_0 = \tilde{q} + \sum_i q_i(m_i - m_{i+1}) \longrightarrow \tilde{q} - \langle q \rangle \quad \text{with multiplicity} = 1\ . \tag{46}$$ EM1

Now define the projectors (we also give their algebra):

$$\begin{aligned} \mathbb{P}_i &:= \frac{1}{m_i}\mathbb{1}_i && (47)\ \text{EM4} \\ \mathbb{P}_i\mathbb{P}_j &= \mathbb{P}_{\min\{i,j\}} && (48) \\ \overline{\mathbb{P}}_i &:= \mathbb{P}_i - \mathbb{P}_{i-1}\ , \qquad \mathbb{P}_0 = 0 && (49) \\ \overline{\mathbb{P}}_i\overline{\mathbb{P}}_j &= \delta_{ij}\overline{\mathbb{P}}_i && (50) \\ \mathbb{1} &= \sum_{i=1}^k \overline{\mathbb{P}}_i && (51) \end{aligned}$$

We note that

$$\begin{aligned} \dim\ker \mathbb{P}_1 &= 1 && (52)\ \text{EM5} \\ \dim\ker \mathbb{P}_i &= \frac{n}{m_i} && (53)\ \text{EM5a} \\ \dim\ker(\overline{\mathbb{P}}_i) &= \frac{n}{m_i} - \frac{n}{m_{i-1}} = n\frac{m_{i-1} - m_i}{m_i m_{i-1}} && (54)\ \text{EM5b} \end{aligned}$$

For any vector v, define

$$v_i := \overline{\mathbb{P}}_i v\ . \tag{55}$$ EM6

This gives the decomposition

$$v = \sum_i v_i\ . \tag{56}$$ EM7

We state that v_i is Eigenvector of $\mathbb{Q}$. To see this, write $\mathbb{Q}$ as

$$\mathbb{Q} = \tilde{q}\mathbb{1} + \sum_i \left(m_i \mathbb{P}_i - m_{i+1}\mathbb{P}_{i+1}\right) \ . \qquad (57)\ \text{EM8}$$

This gives

$$\begin{aligned} \mathbb{Q}v_j &= \left(\tilde{q}\mathbb{1} + \sum_i \left(m_i \mathbb{P}_i - m_{i+1}\mathbb{P}_{i+1}\right)\right)\left(\mathbb{P}_j - \mathbb{P}_{j-1}\right) v \\ &= \tilde{q}v_j + \sum_{i\geq j} q_i \left(m_i - m_{i+1}\right) v_j - q_{j-1}m_{j-1}v_j \end{aligned} \qquad (58)\ \text{EM9}$$

This gives the Eigenvector

$$\hat{\mathbb{Q}}_i = \sum_{j>i}(m_j - m_{j+1})q_j - m_{i+1}q_i + \tilde{q} \qquad (59)\ \text{EM2}$$

In the continuum limit this yields the famous combination

$$\hat{q}(x) = \tilde{q} - xq(x) - \int_x^1 \mathrm{d}y\, q(y) = \tilde{q} - \langle q \rangle - [q]\,(x) \ . \qquad (60)\ \text{K7}$$

The multiplicity is obtained from (54) as

$$n\frac{m_{i-1} - m_i}{m_i m_{i-1}} \ \longrightarrow -n\frac{\mathrm{d}x}{x^2} \ . \qquad (61)\ \text{Dot1}$$

<u>Replica-Fourier-Transform:</u>
We have

$$\mathbb{Q}\mathbb{R} = \mathbb{Q}\mathbb{1}\mathbb{R} = \mathbb{Q}(\sum_i \overline{\mathbb{P}}_i)\mathbb{R} = \ldots = \sum_i \hat{q}_i \hat{r}_i \overline{\mathbb{P}}_i \ , \qquad (62)\ \text{rf1}$$

thus multiplication is diagonal in the space of *replica-Fourier-transformed* fields $\hat{q}_i$.

Let us give some application: The inversion becomes very simple. Suppose $\mathbb{1} = \mathbb{Q}\mathbb{P}$, then $\hat{q}(x)\hat{p}(x) = 1$, which (this time without suffering) reproduces (38).

<u>Trace Log:</u> Using (46) – (61) we obtain

$$\begin{aligned} \frac{1}{n}\mathrm{tr}\ln\mathbb{Q} &= \frac{1}{n}\ln(\tilde{q} - \langle q \rangle) - \int_n^1 \frac{\mathrm{d}y}{y^2}\ln\left(\tilde{q} - \langle q \rangle - [q](y)\right) \\ &= \ln(\tilde{q} - \langle q \rangle) - \int_n^1 \frac{\mathrm{d}y}{y^2}\ln\left(\frac{\tilde{q} - \langle q \rangle - [q](y)}{\tilde{q} - \langle q \rangle}\right) \end{aligned} \qquad (63)\ \text{TrLog1}$$

Taking the (well-defined) limit of $n \to 0$, this gives nearly the Mezard-Parisi [14] result

$$\lim_{n\to 0}\frac{1}{n}\mathrm{tr}\ln\mathbb{Q} = \ln(\tilde{q} - \langle q \rangle) + \frac{q(0)}{\tilde{q} - \langle q \rangle} - \int_0^1 \frac{\mathrm{d}y}{y^2}\ln\left(\frac{\tilde{q} - \langle q \rangle - [q](y)}{\tilde{q} - \langle q \rangle}\right) \qquad (64)\ \text{TrLog2}$$

which could e.g. be obtained from

$$\frac{1}{n}\ln(\tilde{q} - \langle q \rangle + nq(0)) - \int_n^1 \frac{\mathrm{d}y}{y^2}\ln\left(\tilde{q} - \langle q \rangle - [q](y)\right) \ . \qquad (65)\ \text{TrLog3}$$

This means that this formula is quite subtel. For clarification consider

$$\mathbb{Q} = \tilde{q}\mathbb{1} + q_0(\mathbb{1}_1 - \mathbb{1}) \; , \tag{66}$$ TrLog4

where $\mathbb{1}_1$ is the matrix with 1 everywhere. Then it is easy to see that

$$\begin{aligned} f(\mathbb{Q}) &= f(\tilde{q}-q_0)\mathbb{1} + f'(\tilde{q}-q_0)q_0\mathbb{1}_1 + \frac{1}{2}f''(\tilde{q}-q_0)q_0^2(\mathbb{1}_1)^2 + \ldots \\ &= f(\tilde{q}-q_0)\mathbb{1} + f'(\tilde{q}-q_0)q_0\mathbb{1}_1 + \frac{n}{2}f''(\tilde{q}-q_0)q_0^2\mathbb{1}_1 + \ldots \end{aligned} \tag{67}$$ TrLog5

and taking the trace gives

$$\begin{aligned} \frac{1}{n}\mathrm{tr} f(\mathbb{Q}) &= f(\tilde{q}-q_0) + f'(\tilde{q}-q_0)q_1 + \frac{n}{2}f''(\tilde{q}-q_0)q_0^2 + \ldots \\ &= \frac{1}{n}f(\tilde{q}-q_0+nq_0) + \left(1-\frac{1}{n}\right)f(\tilde{q}-q_0) \end{aligned} \tag{68}$$ TrLog6

which gives us the desired term. *Give a general and satisfying derivation.*