Modern Physics Letters A, Vol. 8, No. 40 (1993) 3845–3852

ANOMALOUS DIMENSIONS OF SOFT OPERATORS IN SUPERSYMMETRIC NONLINEAR SIGMA-MODELS

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Received 27 September 1993

The renormalization ζ-function for supersymmetric nonlinear sigma-models is calculated up to three-loop order. For a wide class of models, which includes the N-vector model and matrix models, the result can be summarized as follows: If the ζ-function for the bosonic model is $\zeta_{\text{Bos}}(t_R) = at_R + O(t_R^2)$, then the ζ-function for the supersymmetric model takes the form $\zeta_{\text{SUSY}}(t_R) = at_R + O(t_R^4)$. This is the case for arbitrary harmonic polynomials of the field variables (so called "soft operators").

1. Introduction

Bosonic nonlinear σ-models have widely been discussed in the past years. The analysis started with the N-vector model[3,4] and was then generalized for other symmetric spaces[2,9,15] and general Riemannian manifolds.[11] The form of the manifold no longer remains fixed. Therefore new consistency-relations arise.[5] It is claimed but not proved that there are enough relations to derive all renormalization group functions from the metric β-function.[12]

For the supersymmetric case results from direct loop calculations exist only for the metric β-function.[1,7] These calculations are performed using a massless propagator. Other renormalization functions (e.g. ζ_l) were neither derived from the metric β-function nor by direct loop expansions.

In this letter the renormalization ζ-function, which describes the renormalization of polynomial operators in the fields, is actually calculated up to three-loop order. In contrast to the existing calculations for the metric β-function[1,7] they are performed using a massive supersymmetric propagator. The results for the bosonic case are known from Refs. 8, 9 and 15 but are recalculated here in order to enable a comparison with the supersymmetric functions. In the supersymmetric case our result is in agreement with Ref. 6, where for the special case of the N-vector model and the simplest mass-term the ζ-function was derived from a $1/N^2$-expansion.

This letter is organized as follows: First we outline some basic ideas and terminology of space-time supersymmetry and discuss the relation to the supersymmetry arising in the context of disorder. Then a definition of the nonlinear

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σ-model is given. It is discussed how, and whether at all, the model can be renormalized. The explicit renormalization is presented in Sec. 5. The results hold for a wide class of operators and models. This is discussed in the last section.

2. Supersymmetry

In this approach supersymmetry is induced by an additional generator Q_α, which extends the two-dimensional Poincaré-algebra[a]

$$[P_\tau, M_{\mu\nu}] = \frac{1}{i}(g_{\mu\tau}\, g_{\nu\lambda} - g_{\mu\lambda}\, g_{\nu\tau})P_\lambda\,, \tag{1}$$

$$[P_\mu, P_\nu] = 0 \tag{2}$$

to a superalgebra[14,16,13]

$$[Q_\alpha, P_\mu] = 0\,, \tag{3}$$

$$\begin{aligned}[Q_\alpha, M_{\mu\nu}] &= \mathcal{D}_S(M_{\mu\nu})_{\alpha\beta} Q_\beta \\ &= \frac{i}{4}\,[\gamma_\mu, \gamma_\nu]_{\alpha\beta}\, Q_\beta\,,\end{aligned} \tag{4}$$

$$\{Q_\alpha, Q_\beta\} = 2(\not{P}\gamma_0)_{\alpha\beta}\,. \tag{5}$$

It is easy to show that $\mathcal{D}_S(M_{\mu\nu})$ has to be a representation of $M_{\mu\nu}$. Here it is chosen to be the two-dimensional spin representation in order to incorporate spinorial quantities. With this choice $\{Q_\alpha, Q_\beta\}$ is fixed up to a prefactor.

As translations by P_μ are parametrized by a vector x_μ, translations by Q_α are parametrized by a Grassmannian spinor θ_α. So one is led to extend scalar fields $\varphi(x)$ to superfields[a] $\pi(x,\theta)$:

$$\pi(x,\theta) := \varphi(x) + \bar{\theta}\psi(x) + \frac{1}{2}\bar{\theta}\theta F(x)\,. \tag{6}$$

As there are two independent Grassmannian (anticommuting) variables θ_1 and θ_2, the Taylor expansion terminates after the second term. Since $\pi(x,\theta)$ is supposed to be real, $\varphi(x)$ and $F(x)$ are also real scalar fields. $\psi(x)$ is a real (Majorana-) bi-spinor.

Given the algebra (1)–(5) it is straightforward to construct a representation on $C^\omega(\mathbb{R}^2 \times \mathcal{G}^2)$:

$$\mathcal{D}_\omega(P_\sigma) = \frac{1}{i}\frac{\partial}{\partial x_\sigma}\,, \tag{7}$$

$$\mathcal{D}_\omega(M_{\mu\nu}) = \frac{1}{i}(g_{\mu\tau}\, g_{\nu\lambda} - g_{\mu\lambda}\, g_{\nu\tau})x_\tau \frac{\partial}{\partial x_\lambda}\,, \tag{8}$$

$$\mathcal{D}_\omega(Q_\alpha) = (\not{\partial}\,\theta)_\alpha - i\frac{\partial}{\partial \bar{\theta}_\alpha}\,. \tag{9}$$

[a]For the conventions see the Appendix.

It is easy to see that the Berezin-integral ($\int d^2\theta\, \pi(x,\theta) := -F(x)$) of any power of the superfield $\pi(x,\theta)$ is, up to a total divergence, invariant under translations Q_α. Thus ($d = 2$)

$$\int d^d x\, d^2\theta\, \pi^n(x,\theta) \tag{10}$$

is supercovariant and may be used as a contribution to an invariant Hamiltonian. This is not the case for terms like

$$\int d^d x\, d^2\theta\, \bar{Q}_\alpha \pi^n(x,\theta)\, Q_\alpha \pi^m(x,\theta)\,, \tag{11}$$

because Q_α does not anticommute with itself. However one can define a so-called supercovariant derivative D_β which anticommutes with Q_α:

$$\{D_\alpha, Q_\beta\} = 0\,, \tag{12}$$

$$D_\alpha := \frac{\partial}{\partial\bar{\theta}_\alpha} - i(\partial\!\!\!/\,\theta)_\alpha\,. \tag{13}$$

Then

$$\int d^2\theta\, d^d x\, \bar{D}_\alpha \pi^n(x,\theta)\, D_\alpha \pi^m(x,\theta) \tag{14}$$

is invariant under the transformations Q_α.

The construction given above however should not be confused with internal supersymmetries, which arise in the treatment of disorder (p. 683 of Ref. 10) and characteristically leave invariant terms like

$$x^2 + \bar{\theta}\theta\,. \tag{15}$$

This term is affected by Q_α as can be seen from Eq. (5). Moreover Q_α does not preserve any scalar quantity like (15).

3. The Nonlinear Sigma-Model

The Hamiltonian of the $O(N)$-symmetric bosonic model in Cartesian coordinates $\{\pi^1, \ldots, \pi^n, \sigma\}$ with $d = 2$ and $n = N - 1$ is[3,4]:

$$\mathcal{H}_{\text{Bos}} = \int d^d x\, \frac{1}{2}(\nabla\pi)^2 + \frac{1}{2}(\nabla\sigma)^2 - H\sigma\,, \tag{16}$$

where σ is expressed in terms of π as

$$\sigma = \sqrt{1 - \pi^2}\,. \tag{17}$$

The supersymmetric generalization is[1,7]:

$$\mathcal{H}_{\text{SUSY}} = \int d^d x\, d^2\theta\, \frac{1}{4}\bar{D}_\alpha \pi D_\alpha \pi + \frac{1}{4}\bar{D}_\alpha \sigma D_\alpha \sigma - H\sigma \tag{18}$$

π and σ are both superfields. Again (17) is used to eliminate σ.

4. Renormalization

The renormalization of the model is achieved by replacing

$$\frac{1}{t_B}\mathcal{H}_{\mathrm{SUSY},B} = \frac{1}{t_B}\int d^d x\, d^2\theta\, \frac{1}{4}\bar{D}_\alpha \pi_B D_\alpha \pi_B + \frac{1}{4}\bar{D}_\alpha \sigma_B D_\alpha \sigma_B - H_B \sigma_B \tag{19}$$

through

$$\frac{1}{t_R}\mathcal{H}_{\mathrm{SUSY},R} = \frac{\mu^\varepsilon}{t_R}\int d^d x\, d^2\theta\, \frac{Z}{Z_1}\left(\frac{1}{4}\bar{D}_\alpha \pi_R D_\alpha \pi_R + \frac{1}{4}\bar{D}_\alpha \sigma_R D_\alpha \sigma_R\right) - H_R \sigma_R \tag{20}$$

and

$$\pi_R = \frac{1}{\sqrt{Z}}\pi_B\,, \tag{21}$$

$$\sigma_R = \frac{1}{\sqrt{Z}}\sigma_B\,, \tag{22}$$

$$t_R = \frac{1}{Z_1}\mu^\varepsilon t_B\,, \tag{23}$$

$$H_R = \frac{1}{Z_H}H_B\,. \tag{24}$$

The renormalizability of the bosonic model has been proved by Brézin *et al.* [4] It is possible to generalize this result for the supersymmetric case, for details see Ref. 17. The proof is straightforward if one replaces every field by a superfield and takes care of the anticommuting property of the supercovariant derivative. Additionally one finds that like in the bosonic case

$$Z_H = \frac{Z_1}{\sqrt{Z}}\,, \tag{25}$$

i.e. the renormalization of the external field H is not independent.

Since the model is renormalizable one is free to renormalize any quantity. The simplest one is the free energy[15]: In one-loop order there is only one graph to be calculated, whereas for the renormalization of the two-point function $\Gamma^{(2)}$ there are six graphs. The well known problem which arises in the supersymmetric case is that due to supersymmetry the free energy always vanishes. Nevertheless it is possible to define an equivalent $\mathcal{F}_{\mathrm{SUSY}} \neq 0$ by explicit breaking of supersymmetry.[17] This is also done if one calculates the two-point function: It depends on x *and* θ.

5. Results for the Standard Model

The calculations are performed employing the superpropagator technique as described in Ref. 7. The minimal subtraction scheme is used combined with dimensional reduction. Dimensional reduction means that one does the calculation of the supersymmetric algebra in two dimensions until there are only left mere scalars or scalar products of (space-) vectors. The integrals arising in that way are analytically continued to $d = 2+\varepsilon$ dimensions. This prescription is well-defined since for the spinorial part the representation as bi-spinors is valid also for three-dimensional space-time. This is simply the Pauli representation of quantum mechanics.

Up to three-loop order the divergent part of the free energy is

$$\mathcal{F}^{(3)}_{\text{Bos}}(t_B, H_B) = \frac{H_B}{t_B} + \frac{nH_B^{1+\varepsilon/2}}{\varepsilon(2+\varepsilon)} + \frac{t_B H_B^{1+\varepsilon}(2n-n^2)}{8\varepsilon^2} + \frac{H_B^{1+3\varepsilon/2} t_B^2 n(20\varepsilon - 20n\varepsilon + 3n^2\varepsilon + 16 - 20n + 12n\varepsilon^2 - 12\varepsilon^2 + 6n^2)}{96\varepsilon^3}, \tag{26}$$

$$\mathcal{F}^{(3)}_{\text{SUSY}}(t_B, H_B) = \frac{H_B}{t_B} + \frac{nH_B^{1+\varepsilon}}{2\varepsilon(1+\varepsilon)} + \frac{t_B H_B^{1+2\varepsilon}(2n-n^2)}{8\varepsilon^2} + \frac{H_B^{1+3\varepsilon} t_B^2 n(12\varepsilon - 12n\varepsilon + 3n^2\varepsilon + 8 - 10n + 3n^2)}{48\varepsilon^3}. \tag{27}$$

One can extract the Z-factors:

$$Z_{\text{Bos}}(t_R) = 1 + \frac{n}{\varepsilon} t_R + \frac{n(2n-1)}{2\varepsilon} t_R^2 + \left(\frac{n(n-2)}{4\varepsilon} + \frac{n(n-1)}{3\varepsilon^2} + \frac{n(2n-1)(3n-2)}{6\varepsilon^3}\right) t_R^3 + O(t_R^4), \tag{28}$$

$$Z_{\text{SUSY}}(t_R) = 1 + \frac{n}{\varepsilon} t_R + \frac{n(2n-1)}{2\varepsilon} t_R^2 + \frac{n(2n-1)(3n-2)}{6\varepsilon^3} t_R^3 + O(t_R^4), \tag{29}$$

$$Z_{1,\text{Bos}}(t_R) = 1 + \frac{n-1}{\varepsilon} t_R + \left(\frac{n-1}{2\varepsilon} + \frac{(n-1)^2}{\varepsilon^2}\right) t_R^2 + O(t_R^3), \tag{30}$$

$$Z_{1,\text{SUSY}}(t_R) = 1 + \frac{n-1}{\varepsilon} t_R + \frac{(n-1)^2}{\varepsilon^2} t_R^2 + O(t_R^3). \tag{31}$$

The β- and ζ-functions are defined as[b]:

$$\beta(t_R) := \mu \frac{\partial}{\partial \mu}\bigg|_B t_R \equiv \frac{\varepsilon t_R}{1 + t_R \dfrac{\partial}{\partial t_R} \ln Z_1(t_R)}, \tag{32}$$

$$\zeta(t_R) := \mu \frac{\partial}{\partial \mu}\bigg|_B \ln Z(t_R) \equiv \beta(t_R) \frac{\partial}{\partial t_R} \ln Z(t_R). \tag{33}$$

[b]The subscript "B" indicates that the differentiations are performed for fixed bare quantities.

So one arrives at

$$\beta_{\rm Bos}(t_R) = \varepsilon t_R + (1-n)t_R^2 + (1-n)t_R^3 + O(t_R^4)\,, \tag{34}$$

$$\beta_{\rm SUSY}(t_R) = \varepsilon t_R + (1-n)t_R^2 + O(t_R^4)\,, \tag{35}$$

$$\zeta_{\rm Bos}(t_R) = n t_R + \frac{3}{4}n(n-1)t_R^3 + O(t_R^4)\,, \tag{36}$$

$$\zeta_{\rm SUSY}(t_R) = n t_R + O(t_R^4)\,. \tag{37}$$

The results for the bosonic case has already been given in Ref. 8. The β-function for the supersymmetric case has been obtained in Ref. 1. The result for the supersymmetric ζ-function is in agreement with $1/N^2$-calculations.[6]

As there are 51 graphs to be calculated, it is useful to have some consistency conditions in order to ensure that the calculations are correct. To extract the β- and ζ-functions from the free energy in three-loop order one has to fulfill 10 equations, but five equations will already be sufficient. Then the β- and ζ-functions have to be finite. Together these tests are very sensitive.

6. Generalizations

Two generalizations can be given:

6.1. *Renormalization of arbitrary polynomials in π and σ (soft operators)*

In order to avoid operator-mixing one has to choose eigenfunctions of the renormalization group equation. One finds that these are harmonic polynomials in the fields:

$$\Delta\Big|_{\rm sphere} h_l = l(l+n-1)\, h_l\,. \tag{38}$$

If one adds a term

$$\int d^d x\, d^2\theta\, c_l h_l \tag{39}$$

to the Hamiltonian, its renormalization is performed via

$$c_{l,B} = Z_l\, c_{l,R}\,. \tag{40}$$

The calculation is essentially the same as in Sec. 5 with the same diagrams but different prefactors. The ζ-function

$$\zeta_l(t_R) := \mu\frac{\partial}{\partial\mu}\Big|_B \ln Z_l(t_R) \equiv \beta(t_R)\frac{\partial}{\partial t_R}\ln Z_l(t_R) \tag{41}$$

becomes

$$\zeta_{l,\mathrm{Bos}} = -\frac{l(l+n-1)}{2}t_R - \frac{3l(n-1)(l+n-1)}{8}t_R^3 + O(t_R^4)\,, \tag{42}$$

$$\zeta_{l,\mathrm{SUSY}} = -\frac{l(l+n-1)}{2}t_R + O(t_R^4)\,. \tag{43}$$

The result for the bosonic case is consistent with Refs. 9, 11 and 15.

6.2. *Renormalization of generalized nonlinear sigma-models*

Let the expansion of the Hamiltonian $\mathcal{H}_{\mathrm{SUSY}}$ of a generalized nonlinear σ-model[2] in the fields π be

$$\begin{aligned}\mathcal{H}_{\mathrm{SUSY}} = \int d^dx\, d^2\theta\, a + b_{ij}^{(2)}\pi^i\pi^j + p_{ijkl}^{(4)}\pi^i\pi^j\pi^k\pi^l + \cdots \\ + c_{ij}^{(2)}\bar{D}_\alpha\pi^i D_\alpha\pi^j + c_{ijkl}^{(4)}\pi^i\pi^j\bar{D}_\alpha\pi^k D_\alpha\pi^l + \cdots\end{aligned} \tag{44}$$

with arbitrary constants a, $b_{\dots}^{(i)}$ and $c_{\dots}^{(i)}$. This property holds for example for matrix models. In order to perform the renormalization one again has to calculate the same diagrams. Let the ζ-function for the bosonic model be

$$\zeta_{\mathrm{Bos}}(t_R) = dt_R + O(t_R^2) \tag{45}$$

then

$$\zeta_{\mathrm{SUSY}}(t_R) = dt_R + O(t_R^4)\,. \tag{46}$$

This property is essentially due to the fact that in two- and three-loop orders there is no contribution to the free energy proportional to $1/\varepsilon$, which is the only term that contributes to the ζ-function. This is definitely not the case if another measure is taken in momentum space (see the Appendix).

By the same method one can conclude that the term $\sim t_R^2$ in (45) actually vanishes. This is generally true for models fulfilling the bosonic equivalent of Eq. (44).

Appendix. Conventions

Euclidean notation is used. The metric is

$$g_{\mu\nu} = \delta_{\mu\nu} \tag{A.1}$$

and so upper and lower indices are treated equally. In order to construct Majorana-(real) spinors, which are needed to make the superfield real, we use a Majorana-representation for the Dirac matrices:

$$C = \gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{A.2}$$

As usual

$$\not{P} = \sum_{\mu=1}^{2} P_\mu \gamma^\mu \tag{A.3}$$

and the adjoint spinor

$$\bar{\chi}_\beta = \chi^*_\alpha \gamma^0_{\alpha\beta} \tag{A.4}$$

so that

$$\bar{\chi}\psi = \bar{\chi}_\alpha \psi_\alpha \,. \tag{A.5}$$

The measure of momentum space is normalized according to ($d = 2 + \varepsilon$)

$$\int \frac{d^d x}{(2\pi)^d} \frac{1}{1+x^2} = -\frac{1}{\varepsilon} \,. \tag{A.6}$$

Acknowledgments

It is a great pleasure for me to thank F. Wegner for the possibility to write my diploma thesis on this subject as well as for many valuable discussions. I would also like to thank S. Kehrein for helpful discussions.

References

1. L. Alvarez-Gaumé, D. Z. Freedman and S. Mukhi, *Ann. Phys.* **134**, 85 (1981).
2. E. Brézin, S. Hikami and J. Zinn-Justin, *Nucl. Phys.* **B165**, 528 (1980).
3. E. Brézin and J. Zinn-Justin, *Phys. Rev.* **B14**, 3110 (1976).
4. E. Brézin, J. Zinn-Justin and J. C. Le Guillou, *Phys. Rev.* **D14**, 2615 (1976).
5. G. Curci and G. Paffuti, *Nucl. Phys.* **B286**, 399 (1987).
6. J. A. Gracey, *Nucl. Phys.* **B348**, 737 (1991).
7. M. T. Grisaru, A. E. M. van de Ven and D. Zanon, *Nucl. Phys.* **B277**, 409 (1986).
8. S. Hikami and E. Brézin, *J. Phys.* **A11**, 1141 (1978).
9. D. Höf and F. Wegner, *Nucl. Phys.* **B275**, 561 (1986).
10. C. Itzykson and J.-M. Drouffe, *Statistical Field Theory* (Cambridge Univ. Press, 1989), Vol. 2.
11. J. Jack, D. R. T. Jones and D. A. Ross, *Nucl. Phys.* **B307**, 531 (1988).
12. J. Jack, D. R. T. Jones and N. Mohammedi, *Nucl. Phys.* **B332**, 333 (1990).
13. M. Sohnius, *Phys. Rep.* **128**, 39 (1985).
14. D. V. Volkov and V. P. Akulov, *Phys. Lett.* **B46**, 109 (1973).
15. F. Wegner, *Nucl. Phys.* **B280**, 193 (1987).
16. J. Wess and B. Zumino, *Nucl. Phys.* **B70**, 39 (1974).
17. K. J. Wiese, "Die Zeta-Funktion für ein supersymmetrisches nichtlineares sigma-model', Diplomarbeit, Universität Heidelberg, 1992.