Large Orders and Strong-Coupling Limit in Functional Renormalization

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We study the large-order behavior of the functional renormalization group (FRG). For a model in dimension zero, we establish Borel-summability for a large class of microscopic couplings. Writing the derivatives of FRG as contour integrals, we express the Borel-transform as well as the original series as integrals. Taking the strong-coupling limit in this representation, we show that all short-ranged microscopic disorders flow to the same universal fixed point. Our results are relevant for FRG in disordered elastic systems.

Introduction. Perturbative expansions are a work horse in theoretical physics. Most of them are not converging, but aysmptotic series [1–5]. The main strategy to obtain a series with a finite radius of convergence is to define its Borel transform by dividing its n-th series coefficient by $n!$. One then continues the latter and reconstructs the original function via an integral over this analytic continuation. The aim is to extend the range of applicability from small expansion parameters, where the series naively converges, to larger ones. Techniques using Padé-Borel resummation or conformal mappings are successful here [3–8], and were employed for the ϵ-expansion of perturbative RG [6, 8, 9]. In simpler examples, as the anharmonic oscillator [10], one can go further, and use *resurgence* [11–13] to reach finite couplings. Resurgence is strong in identifying the closest singularity of the Borel transform, and using this information to extend the domain of convergence. It is less effective to reach strong coupling.

In the renormalization context we wish to take a bare coupling to infinity; we know of few methods to address this beyond postulating the asymptotic behavior [14]. An additional problem arises when the microscopic set of couplings is itself a function, as in the functional renormalization group (FRG) treatment of disordered systems. Here we consider a specific model in dimension $d = 0$, which is later derived from the field theory of disordered elastic manifolds (for a review, see [15]). We wish to answer the following fundamental questions: What is the large-order behavior of FRG? Is it Borel-summable? How can we study its strong-coupling limit? And how does universality arise?

Setting the stage. In order to address these questions, we start with the $O(2)$ model on a single site. This is not only the simplest possible model, but key formulas will prove useful later. Consider the partition function,

$$\mathcal{Z}_{O(2)}(\lambda) := \int_{\tilde\phi,\phi} \mathrm{e}^{-\tilde\phi\phi-\lambda\tilde\phi^2\phi^2}, \quad \mathcal{Z}(0) = 1. \tag{1}$$

Here ϕ and $\tilde\phi$ are complex conjugate fields. Analysis proceeds via Wick's theorem, using the measure induced by $\mathrm{e}^{-\tilde\phi\phi}$,

$$\left\langle \tilde\phi^n f(\phi) \right\rangle = (\partial_\phi)^n f(\phi)\Big|_{\phi=0} \quad \Rightarrow \quad \left\langle \tilde\phi^n \phi^m \right\rangle_0 = n!\,\delta_{n,m}. \tag{2}$$

This implies that

$$\mathcal{Z}_{O(2)}(\lambda) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!}(-\lambda)^n. \tag{3}$$

Stirling's formula shows that this series is divergent. Its *Borel transform*, obtained by dividing the n-th series coefficient by $n!$, has a finite radius of convergence,

$$\mathcal{Z}^{\rm B}_{O(2)}(t) := \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(-t)^n = \frac{1}{\sqrt{1+4t}}. \tag{4}$$

$\mathcal{Z}^{\rm B}_{O(2)}(t)$ has a branch cut starting at $t = -1/4$, and its *analytic continuation* is well defined for $t > 0$. This allows one to obtain $\mathcal{Z}_{O(2)}(\lambda)$ via an *inverse Borel transform*

$$\mathcal{Z}_{O(2)}(\lambda) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \mathcal{Z}^{\rm B}_{O(2)}(t\lambda) = \sqrt{\frac{\pi}{4\lambda}}\, \mathrm{e}^{\frac{1}{4\lambda}} \mathrm{erfc}\Big(\frac{1}{2\sqrt{\lambda}}\Big). \tag{5}$$

The crucial step in this resummation is our ability to analytically continue the Borel-transform $\mathcal{Z}^{\rm B}_{O(2)}(t)$ beyond its radius of convergence of $1/4$, to the positive real axis.

When no analytic result is available, the standard procedure is to do a saddle-point (instanton) analysis of the integral [1–5], and then use resummation techniques or resurgence. In practice one is often constrained to either approximate $\mathcal{Z}_{\rm B}(t)$ via a Padé approximant [3], Meijer G-function [16], or use a conformal mapping [3, 6, 8]. While this allows one to extend the range of convergence, say by a factor of five, the question of the strong-coupling behavior remains elusive.

Resummation of a functional expansion. Let us proceed to a 0-dimensional model for functional RG,

$$\mathcal{Z}_{\rm FRG}(w,\lambda) = \int_{\tilde\phi,\phi} \mathrm{e}^{-\tilde\phi(\phi-w)+\lambda\tilde\phi^2[\Delta(0)-\Delta(\phi)]}. \tag{6}$$

We consider this a mathematical problem. We assume that $\Delta(\phi)$ is an analytic function, fast and monotonously decaying for $\phi \geq 0$, and that $\Delta(0) = 1$. A good example is $\Delta(\phi) = \mathrm{e}^{-\phi}$. The field ϕ has an expectation w. Wick's theorem (2) allows us to write the perturbative expansion for $w > 0$,

$$\mathcal{Z}_{\rm FRG}(w,\lambda) := \sum_{n=0}^{\infty} \lambda^n \mathcal{Z}^{(n)}_{\rm FRG}(w), \tag{7}$$

$$\mathcal{Z}^{(n)}_{\rm FRG}(w) := \frac{1}{n!}(\partial_w)^{2n}\big[1-\Delta(w)\big]^n. \tag{8}$$

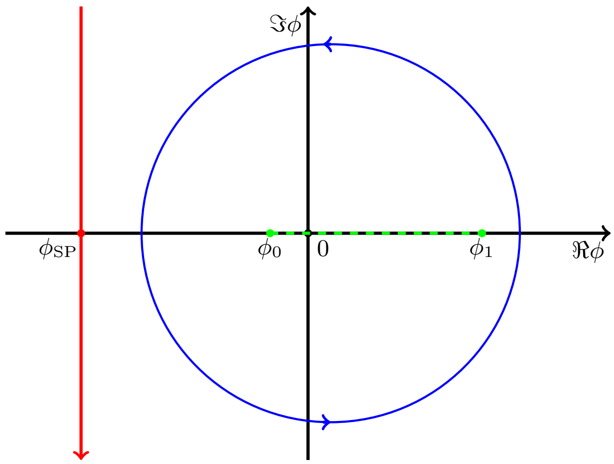

FIG. 1. The different paths and contour integrals. In blue the one used for Eqs. (15) and (16), encircling the cut in Eq. (17) (green/dashed). In red the path used for the derivation of Eq. (14) which passes through $\phi_{\rm SP}$.

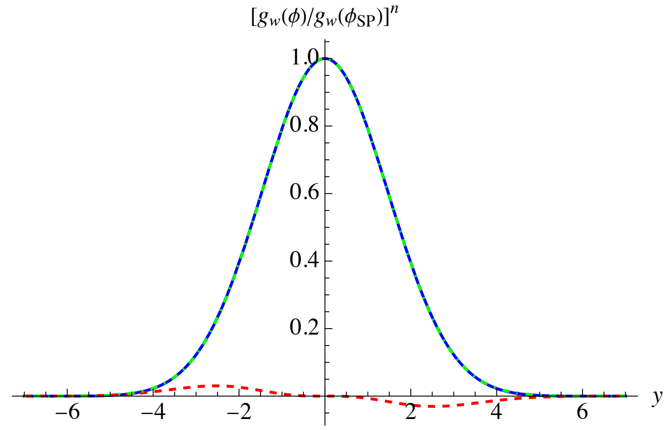

FIG. 2. Plot of $[g_w(\phi)/g_w(\phi_{\rm SP})]^n$ for $w=0$, $n=100$, with real part in blue (solid) and imaginary part in red (dashed); $\phi=\phi_{\rm SP}+iy/\sqrt{n}$, as indicated by the red curve on Fig. 1. In green (dot-dashed) $\exp(-\frac{g_w(\phi)}{\phi^2 g_w''(\phi)}y^2/2)|_{\phi=\phi_{\rm SP}}$, whose integration leads to Eq. (14).

To evaluate Eq. (6) non-perturbatively, integration contours need to be specified. As we show later, this is not an obvious task. Therefore we *define* our model by Eqs. (7)–(8). The latter are motivated by perturbative results for the renormalization of disordered elastic manifolds in dimension $d=0$ [17–22], for which $\Delta(\phi)$ is the microscopic disorder correlator.

Let us start with the large-order behavior of $\mathcal{Z}^{(n)}_{\rm FRG}(w)$. This is given by the saddle point of Eq. (6) over both ϕ and $\tilde\phi$. It implies two saddle-point equations, is quite formal, and difficult to control. A more powerful approach is to evaluate Eq. (8) via the residue theorem,

$$\mathcal{Z}^{(n)}_{\rm FRG}(w) = \frac{(2n)!}{n!}\frac{1}{2\pi i}\oint\frac{{\rm d}\phi}{\phi}g_w(\phi)^n, \tag{9}$$

$$g_w(\phi) := \frac{1-\Delta(w+\phi)}{\phi^2}. \tag{10}$$

The contour goes counter clockwise around the origin, see Fig. 1. It picks out the coefficient of order ϕ^0 in the Laurent series at $\phi=0$. Since $\Delta(\phi)$ is bounded for $\Re\phi>0$, we can push the path in that domain to ∞. We expect a saddle point (SP) elsewhere, given by

$$\frac{\rm d}{{\rm d}\phi}g_w(\phi)=0. \tag{11}$$

To make our analysis concrete, set $\Delta(\phi):={\rm e}^{-\phi}$. For $w=0$, the saddle point is at

$$\phi_{\rm SP} = -W(-2\,{\rm e}^{-2})-2=-1.59362, \tag{12}$$

$$g_0(\phi_{\rm SP}) = -1.54414, \tag{13}$$

where W is the Lambert W function. Fig. 2 shows that the large-order behavior of Eq. (8) is captured by the integral running over $\phi=\phi_{\rm SP}+i\mathbb{R}$ (see Fig. 1 for the path). This gives the leading order of the large-n behavior,

$$\mathcal{Z}^{(n)}_{\rm FRG}(w)\simeq\frac{\Gamma(2n{+}1/2)}{\Gamma(n{+}1)\sqrt{\pi}}[g_w(\phi)]^n\sqrt{\frac{g_w(\phi)}{\phi^2 g_w''(\phi)}}\Bigg|_{\phi=\phi_{\rm SP}}. \tag{14}$$

Several remarks are in order: First of all, the large-order behavior is asymptotic and its Borel transform exists, as $\Gamma(2n{+}1/2)/\Gamma(n{+}1)\simeq n!$. Second, for disordered systems $\Delta(\phi)=\Delta(|\phi|)$ thus the branch for $\phi<0$ is given by the branch for $\phi>0$. In contrast, the saddle point is at negative ϕ, on the analytic continuation of the branch for positive ϕ, outside its physically relevant domain. A numerical check for $n=100$ is shown in Fig. 2. The relative error for $\mathcal{Z}^{(n)}_{\rm FRG}(0)$ is 10^{-4}, which can systematically be improved by further $1/n$ corrections. The latter are relevant for resurgence [23].

There are surprises in this procedure: when changing the microscopic disorder from $\Delta(\phi)={\rm e}^{-\phi}$ to $\Delta(\phi)={\rm e}^{-\phi-a\phi^2}$, there is a critical value $a_{\rm c}\approx0.0649$ s.t. if $a>a_{\rm c}$ the saddle point disappears. We will see later that while the large-order behavior changes, the large-λ limit can still be taken, and is independent of a, as long as $a>0$. We retain that, despite its widespread use, the information contained in the large-order behavior may be quite limited [23].

Borel transform. Define the Borel transform of Eqs. (7)-(9) as

$$\mathcal{Z}^{\rm B}_{\rm FRG}(w,t):=\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}\frac{t^n}{2\pi i}\oint\frac{{\rm d}\phi}{\phi}g_w(\phi)^n. \tag{15}$$

Exchanging sum and integration, Eq. (4) yields

$$\mathcal{Z}^{\rm B}_{\rm FRG}(w,t)=\oint\frac{{\rm d}\phi}{2\pi i\phi}\frac{1}{\sqrt{1-4tg_w(\phi)}}. \tag{16}$$

While Eq. (9) is valid for any contour circling the origin, in order to avoid the branch cut induced by the denominator in Eq. (16), one needs to make the contour in Eq. (16) large enough, see Fig. 1. One can then shrink the contour until it hugs the branch cut. Evaluating the discontinuity across the cut, we can rewrite Eq. (16) as

$$\mathcal{Z}^{\rm B}_{\rm FRG}(w,t) = \frac{1}{\pi}\int_{\phi_0}^{\phi_1} \mathrm{d}\phi \frac{1}{\sqrt{4t[1-\Delta(w+\phi)] - \phi^2}}, \qquad (17)$$

where $\phi_0 \le 0 < \phi_1$ are the two zeros of the denominator, and the sign inside the square root is reversed between Eqs. (16) and (17). For $w = 0$, $\phi_0 = 0$. One could extend this integral from $-\infty$ to ∞, if one keeps only the real part of the integrand. A numerical check of Eqs. (16) and (17) is presented in Fig. 4 of the appendix.

Inverse Borel transform. Using Eq. (5), the inverse Borel transform (from t to λ) of the integrand in Eq. (16) is (noting $g := g_w(\phi)$)

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{-t}}{\sqrt{1-4\lambda g t}}\,\mathrm{d}t &= \frac{\sqrt{\pi}\mathrm{e}^{-\frac{1}{4\lambda g}}}{2} \frac{\mathrm{erfc}\Big(\frac{1}{2\sqrt{-\lambda g}}\Big)}{\sqrt{-\lambda g}} \\ &= \frac{\sqrt{\pi}\mathrm{e}^{-\frac{1}{4\lambda g}}}{2}\left[\frac{1}{\sqrt{-\lambda g}} + \sum_{n\in\mathbb{N}} \frac{a_n}{(g\lambda)^n}\right]. \qquad (18)\end{aligned}$$

On the second line is the large-λ expansion. The key observation is that the terms $\sim a_n$ are analytic in ϕ around the origin, and thus do not contribute to the integral (16). As a consequence, the latter can be simplified to

$$\mathcal{Z}_{\rm FRG}(w,\lambda) = \oint \frac{\mathrm{d}\phi}{2\pi i\phi} \frac{\sqrt{\pi}\mathrm{e}^{-\frac{1}{4\lambda g_w(\phi)}}}{2\sqrt{-\lambda g_w(\phi)}}. \qquad (19)$$

In order for this equality to be valid, the contour is not allowed to cross the cut which now extends to $\phi = \infty$, and starts at $\phi = -w$. As in the derivation of Eq. (17), we can simplify Eq. (19) by retaining only the discontinuity across the cut,

$$\mathcal{Z}_{\rm FRG}(w,\lambda) = \frac{1}{\sqrt{4\pi\lambda}}\int_0^\infty \mathrm{d}\phi\, \frac{\mathrm{e}^{\frac{-(\phi-w)^2}{4\lambda[1-\Delta(\phi)]}}}{\sqrt{1-\Delta(\phi)}}. \qquad (20)$$

To arrive here, we moved the factor of $1/\phi$ inside the square root, evaluated its discontinuity, and finally shifted $\phi \to \phi + w$. This result is checked on Fig. 5 of the appendix. Finally, Eq. (20) can be derived from Eq. (6), if one choses for the integration contours $\tilde\phi \in i\mathbb{R}$, and $\phi \ge 0$.

Strong-coupling behavior. Eq. (20) allows us to extract the large-λ behavior. The key observation is that due to the factor of $1/\lambda$ in the exponent, larger and larger values for ϕ contribute. On these scales, $\Delta(\phi)$ is negligible and can be dropped, leading to

$$\begin{aligned}\mathcal{Z}_{\rm FRG}(w,\lambda) &\simeq \frac{1}{\sqrt{4\pi\lambda}}\int_0^\infty \mathrm{d}\phi\, \mathrm{e}^{-\frac{(\phi-w)^2}{4\lambda}} \\ &= \frac{1}{\sqrt{4\pi}}\int_0^\infty \mathrm{d}\phi\, \mathrm{e}^{-\frac{(\phi-w/\sqrt{\lambda})^2}{4}}. \qquad (21)\end{aligned}$$

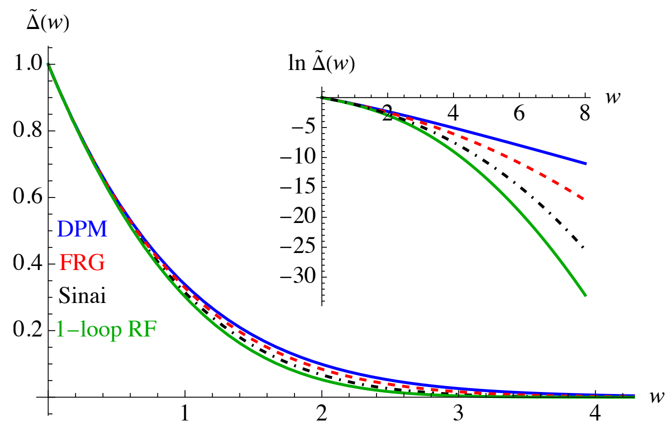

FIG. 3. Different solutions for $\tilde\Delta(w)$, all rescaled to $\tilde\Delta(0) = |\tilde\Delta'(0)| = 1$. From top to bottom: driven particle (DPM) in Gaussian disorder (blue), Eq. (85) of [24], Eq. (24) (red, dashed), Sinai model, Eq. (202) of [15] (black, dot-dashed), and the 1-loop random-field fixed point, Eq. (88) of [15] (green, solid).

The second line shows that the limit $\mathcal{Z}^\infty_{\rm FRG}(w) := \lim_{\lambda\to\infty} \mathcal{Z}_{\rm FRG}(w\sqrt{\lambda},\lambda)$ exists, and is given by

$$\mathcal{Z}^\infty_{\rm FRG}(w) = \frac{1}{2}\left[1 + \mathrm{erf}\Big(\frac{w}{2}\Big)\right]. \qquad (22)$$

To derive this it is essential that the singularity in the denominator of Eq. (20) is integrable. Numerically we checked the passage from Eq. (20) to Eq. (21) for λ up to 10^{20}.

Finally, Eqs. (7) and (8) imply that $\mathcal{Z}_{\rm FRG}(w,\lambda) = 1 - \lambda\Delta''_{\rm FRG}(w,\lambda)$. Therefore the dimensionless rescaled limit for $\Delta''_{\rm FRG}$ reads

$$\tilde\Delta''_{\rm FRG}(w) := \lim_{\lambda\to\infty} \lambda^{-1}\Delta''_{\rm FRG}(w\sqrt{\lambda},\lambda) = \frac{1}{2}\mathrm{erfc}\Big(\frac{w}{2}\Big). \qquad (23)$$

Integrating twice yields

$$\tilde\Delta_{\rm FRG}(w) = \frac{w^2+2}{4}\mathrm{erfc}\left(\frac{w}{2}\right) - \frac{\mathrm{e}^{-\frac{w^2}{4}}w}{2\sqrt{\pi}}. \qquad (24)$$

What is remarkable about Eq. (20) is that the final result, given in Eq. (22), is largely independent of the microscopic $\Delta(\phi)$. What we used is that $\Delta(\phi)$ is analytic, has a linear cusp at the origin, and decays quickly. The cusp is a technical requirement, necessary to transform the contour integral into a cut integral. We believe that this is more a technical constraint than a physical one: we could regularize the microscopic disorder in order to obtain a linear cusp, and then remove the regularization. We have studied this for $\Delta(\phi) = \mathrm{e}^{-\phi^2}$. While we clearly see that convergence is non-uniform and slow, we have no indication that the process does not converge, or converges against a different fixed point. On the practical side, when applied to disordered systems, as the disorder usually lives on a grid, we can well approximate it by a function with a linear cusp.

What is reassuring about our findings is that while it is believed that all microscopic disorders converge to the same

FRG fixed point, this has only be seen perturbatively [17–22], in simulations [25–27] and in experiments [28–30]. The mechanism by which this happens here is non-perturbative, and apparently robust.

Finally, let us compare the shape of $\tilde{\Delta}(w)$ as derived in Eq. (24) to other analytical solutions (Fig. 3): A $d = 0$ solution for depinning, the $d = 0$ solution in equilibrium with random-field (RF) disorder (Sinai model), and the 1-loop solution in the RF universality class. While these solutions are similar, they are distinct and allow one to determine the universality class, as was done for magnetic domain walls [29].

Field theory for disordered elastic systems. Let us connect our findings to the field theory of disordered elastic systems. This is best done by comparing to the formulation of Ref. [31, 32] which uses Grassmanian variables ("*supersymmetry*") [33–36] to average over disorder. The relevant action contains two physical replicas located at positions u_1 and u_2. Denoting their center of mass by u, and their difference by ϕ, only ϕ appears inside the disorder correlator Δ, and u decouples. The corresponding action becomes (see the appendix)

$$\begin{aligned}\mathcal{S} = \int_x & \tilde{\phi}(x)(m^2 - \nabla^2)[\phi(x) - w] \\ & + \sum_{a=1}^{2} \bar{\psi}_a(x)(m^2 - \nabla^2)\psi_a(x) \\ & + \tilde{\phi}(x)^2\Big[\Delta\big(\phi(x)\big) - \Delta(0)\Big] \\ & + \tilde{\phi}(x)\Delta'\big(\phi(x)\big)\Big[\bar{\psi}_2(x)\psi_2(x) + \bar{\psi}_1(x)\psi_1(x)\Big] \\ & + \bar{\psi}_2(x)\psi_2(x)\bar{\psi}_1(x)\psi_1(x)\Delta''\big(\phi(x)\big)\ . \end{aligned} \tag{25}$$

Here $\tilde{\phi}$ and ϕ are bosonic fields (complex numbers), while $\bar{\psi}_i$ and ψ_i are Grassmann variables [37]. Going to dimension $d = 0$, dropping the Grassmann fields, and rescaling $\tilde{\phi} \to \tilde{\phi}/m^2$ yields the integral (6), with

$$\lambda \equiv m^{-4}. \tag{26}$$

(Again, $\Delta(0) = 1$). Eq. (23) implies that $w \sim \sqrt{\lambda} = m^{-2}$, thus the scaling exponent of the field ϕ, known as the roughness exponent ζ, is

$$\zeta = 2. \tag{27}$$

Let us rewrite these findings in terms of the action (25). Retaining only the bosonic fields, we get for the model (6)-(8),

$$\mathcal{Z}_{\rm FRG}(w,\lambda) \equiv \mathcal{Z}^{\mathcal{S}}_{\rm bos}(w,\lambda) := \int_{\phi,\tilde{\phi}} \mathrm{e}^{-\mathcal{S}|_{\psi_i \to 0}}. \tag{28}$$

By construction, the partition function of action (25) over all bosonic and Grassmann fields is 1,

$$\mathcal{Z}_{\mathcal{S}}(w,\lambda) := \langle 1 \rangle_{\mathcal{S}} = 1, \quad \langle \mathcal{O} \rangle_{\mathcal{S}} := \int_{\phi,\tilde{\phi},\bar{\psi}_1,\psi_1,\bar{\psi}_2,\psi_2} \mathrm{e}^{-\mathcal{S}}\mathcal{O}. \tag{29}$$

The renormalized $\Delta(w)$ is given [15] by the connected expectation of $m^4(\phi - w)^2/2$,

$$\Delta_{\rm Susy}(0,\lambda) - \Delta_{\rm Susy}(w,\lambda) = \frac{m^4}{2}\left\langle (\phi - w)^2 \right\rangle^{\rm c}_{\mathcal{S}}. \tag{30}$$

This function has a limit (we use Eq. (26))

$$\tilde{\Delta}_{\rm Susy}(w) := \lim_{\lambda\to\infty} \Delta_{\rm Susy}(w\sqrt{\lambda},\lambda). \tag{31}$$

Surprisingly, the functions defined in Eqs. (31) and (24) agree,

$$\tilde{\Delta}_{\rm Susy}(w) = \tilde{\Delta}_{\rm FRG}(w). \tag{32}$$

Thus what we obtained for the simple model (6) also applies to the disordered system defined by the action (25), order by order in perturbation theory.

Applications. Our results agree up to 1-loop order with that for disordered elastic manifolds in equilibrium and at depinning [17–22]. Beyond that, amplitudes are different in the ϵ-expansion, and there are additional *anomalous terms* which are hard to recuperate [23]. Though our model can formally be derived from a field theory in equilibrium, we do not believe $\Delta_{\rm FRG}(w)$ to be relevant for a specific physical situation, even though the predicted roughness exponent is equal to that of depinning, and the shape of $\Delta_{\rm FRG}(w)$ on Fig. 3 is between a driven particle and Sinai's model, both relevant in $d = 0$.

Given these caveats, we turn to the strength of our approach. Our model contains all ingredients of functional renormalization: it shows that the perturbative series is Borel summable, how the limit of strong coupling is reached, that it cannot be inferred from the large-order behavior, and how universality emerges. This poses a solid framework for the strong-coupling behavior in functional renormalization. We also saw that to define the path integral non-perturbatively, one needs to specify the integration contours, and possibly restrict variables to part of their physically allowed domains.

Appendices

Field Theory. Building on the Susy formulation of [38], Ref. [31] introduces two physical copies located at u_1 and u_2, which are subject to confining potentials displaced by w, s.t. their difference $\phi := u_1 - u_2$ has expectation $\langle\phi\rangle = w$; its center of mass is $u := (u_1 + u_2)/2$. The field theory, given in Eq. (36) of [31] reads

$$\begin{aligned}\mathcal{S} = \int_x & \tilde{\phi}(x)(m^2 - \nabla^2)[\phi(x) - w] + \tilde{u}(x)(m^2 - \nabla^2)u(x) \\ & + \sum_{a=1}^{2} \bar{\psi}_a(x)(m^2 - \nabla^2)\psi_a(x) \\ & + \tilde{\phi}(x)^2\Big[\Delta\big(\phi(x)\big) - \Delta(0)\Big] \\ & - \frac{1}{4}\tilde{u}(x)^2\Big[\Delta\big(\phi(x)\big) + \Delta(0)\Big] \\ & + \frac{1}{2}\tilde{u}(x)\Delta'\big(\phi(x)\big)\Big[\bar{\psi}_2(x)\psi_2(x) - \bar{\psi}_1(x)\psi_1(x)\Big] \\ & + \tilde{\phi}(x)\Delta'\big(\phi(x)\big)\Big[\bar{\psi}_2(x)\psi_2(x) + \bar{\psi}_1(x)\psi_1(x)\Big] \\ & + \bar{\psi}_2(x)\psi_2(x)\bar{\psi}_1(x)\psi_1(x)\Delta''\big(\phi(x)\big)\ . \end{aligned} \tag{33}$$

Here $\tilde{u}$ and $\tilde{\phi}$ are the response fields for u and ϕ, while ψ_i and $\bar{\psi}_i$ are Grassmannian variables introduced to ensure that the partition function equals 1. Integrating over u forces $\tilde{u} \to 0$, resulting in Eq. (25) of the main text.

Let us next take dimension $d=0$ in action (25), and integrate over the Grassmann variables. This gives the partition function

$$\mathcal{Z}=\frac{1}{m^4}\int_\phi\int_{\tilde\phi}\left\{\left[\tilde\phi\Delta'(\phi)+m^2\right]^2-\Delta''(\phi)\right\}\exp\Big(-\left[\tilde\phi^2\big(\Delta(\phi)-\Delta(0)\big)\right]-m^2\tilde\phi(\phi-w)\Big). \tag{34}$$

Integrating $\tilde\phi$ over the imaginary axis yields

$$\mathcal{Z}=\frac{1}{2m^2\sqrt{\pi}}\int_0^\infty \mathrm{d}\phi\left\{m^4-\Delta''(\phi)+\frac{m^4(w-\phi)^2\Delta'(\phi)^2}{4[\Delta(0)-\Delta(\phi)]^2}-\frac{\Delta'(\phi)\left[\Delta'(\phi)+2m^4(w-\phi)\right]}{2[\Delta(0)-\Delta(\phi)]}\right\}\frac{\mathrm{e}^{-\frac{m^4(w-\phi)^2}{4[\Delta(0)-\Delta(\phi)]}}}{\sqrt{\Delta(0)-\Delta(\phi)}}. \tag{35}$$

After Eq. (20) we stated that the latter can be derived from Eq. (6) if $\tilde\phi\in i\mathbb{R}$ and $\phi>0$. We use the same prescription to pass from Eq. (34) to Eq. (35). This allows us to evaluate Eqs. (28)-(30), both perturbatively and non-peturbatively.

Additional numerical checks. Fig. 4 shows that for $w=0$,

$$\mathcal{Z}^{\mathrm{B}}_{\mathrm{FRG}}(w):=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathcal{Z}^{(n)}_{\mathrm{FRG}}(w), \tag{36}$$

with $\mathcal{Z}^{(n)}_{\mathrm{FRG}}(w)$ defined in Eq. (8), agrees with both Eq. (16) and Eq. (17) inside its radius of convergence, at least for $\lambda>0$. For $\lambda>0$ and outside the radius of convergence, the latter two agree with each other and a Padé resummation of Eq. (36).

Fig. 5 shows the rescaled $\tilde\Delta''_{\mathrm{FRG}}(w)$ for $\lambda=10$, i.e. well outside the range of convergence of the Borel transform. We tested the integral (19) against a Padé-Borel approximation of the original series. Deviations for some values of w are visible due to the large value of λ, but are absent for smaller λ (not shown). We also tested that there is no difference when keeping the erfc in Eq. (18), instead of replacing it by 1, as was done in the derivation of Eq. (19). Finally, the solution approaches the asymptotic form (23). We checked this convergence for λ up to 10^{20} using the cut integral (20) (not shown).

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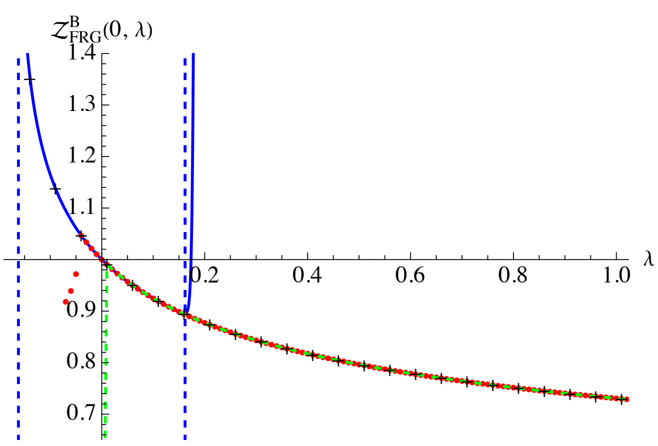

FIG. 4. $\mathcal{Z}^{\mathrm{B}}_{\mathrm{FRG}}(w=0,\lambda)$, evaluated in four different ways: (i) explicit sum from derivatives as given in Eqs. (7) and (8) (blue solid line). The vertical blue-dashed lines indicate its radius of convergence estimated from Eq. (14). (ii) the contour integral (16) (red dots), (iii) the cut integral (17) (green dashed), and (iv) a diagonal Padé resummation of the original series (black crosses). Both integral representations work for λ larger than the radius of convergence of the series (but are as expected problematic for negative λ).

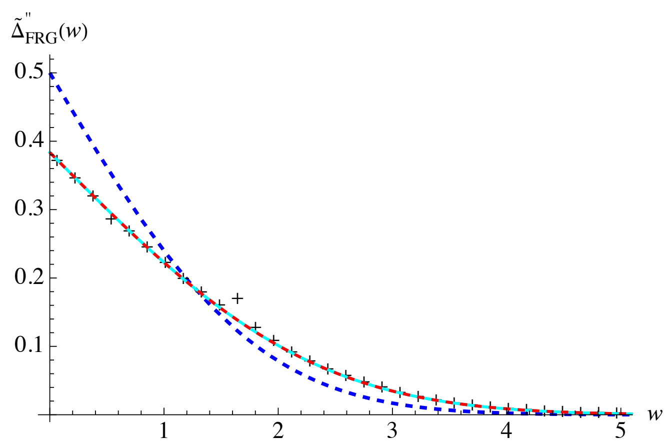

FIG. 5. The function $\tilde\Delta''_{\mathrm{FRG}}(w,\lambda):=1-\mathcal{Z}_{\mathrm{FRG}}(w\sqrt{\lambda},\lambda)$ for $\lambda=10$, evaluated via: Padé-Borel (black crosses) on the combinatorial series at order 100. (Some glitches appear, and Padé-Borel breaks down for larger λ; each Padé is constructed at fixed w.) Evaluation of the integral (20) (cyan, solid), indistinguishable form an implementation which keeps the erfc of Eq. (18) (red, dashed). In blue dashed the asymptotic form (23).

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