How to measure Functional RG fixed-point functions for dynamics and at depinning

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Abstract. – We show how the renormalized force correlator $\Delta(u)$, the function computed in the functional RG (FRG) field theory, can be measured directly in numerics and experiments on the *dynamics* of elastic manifolds in presence of pinning disorder. For equilibrium dynamics we recover the relation obtained recently in the statics between $\Delta(u)$ and a physical observable. Its extension to depinning reveals interesting relations to stick-slip models of avalanches used in dry friction and earthquake dynamics. The particle limit ($d = 0$) is solved for illustration: $\Delta(u)$ exhibits a cusp and differs from the statics. We propose that the FRG functions be measured in wetting and magnetic interfaces experiments.

Models involving elastic objects driven through random media are important for numerous physical systems and phenomena including magnets [1], superconductors [2], density waves [3], wetting [4], dry friction [5], dislocation and crack propagation [6], and earthquake dynamics [7]. There has been progress in qualitative understanding of, e.g. the existence of a depinning treshold for persistent motion at zero temperature $T = 0$, scale invariance at the threshold and the analogy with critical phenomena, collective pinning and roughness exponents, avalanche motion at $T = 0$, and ultra-slow thermally activated creep motion over diverging barriers. These phenomena are predicted by theory, i.e. phenomenological arguments [2], mean field models [8], functional renormalisation group [9–12], and were seen in numerical studies. Experimental evidence for creep motion was found in vortex lattices, in ferroelectrics, and in magnetic interfaces in ferromagnets [1, 13]. Some cases exhibit clear discrepancies with the simplest theories, e.g. the depinning of the contact line of a fluid [4, 14]. Even when agreement exists, much remains to be done for a precise comparison.

Recent theoretical progress makes these quantitative tests possible. For interfaces, powerful algorithms now allow to find the exact depinning threshold and critical configuration on a cylinder [15] and to study creep dynamics [16]. The functional RG has been extended beyond the lowest order (one loop), and it was shown that differences between statics and depinning become manifest only at two loops [11, 12]. The FRG is the candidate for a field theoretic description of statics and depinning, beyond mean field. It captures the complex glassy physics of numerous metastable states at the expense of introducing, rather than a single coupling as in standard critical phenomena, a function, $\Delta(u)$, of the displacement field u, which flows to a fixed point (FP) $\Delta^*(u)$. This FP is non-analytic, as is the effective action of the theory. At a qualitative level, $\Delta(u)$ can be interpreted as the coarse-grained correlator of the random pinning force and its cusp singularity at the FP, $\Delta'(0^+) = -\Delta'(0^-)$, is related to shock singularities in the coarse grained force landscape, reponsible for pinning. Until now however, comparison between experiments, numerics and FRG was mostly about critical exponents.

The aim of this paper is to make precise statements concerning the physics of *dynamical* FRG and propose experimental and numerical tests. Recently a relation was found [17] between the FRG coupling functions $\Delta(u) = -R''(u)$ and *observables*, suggesting a method to measure these functions *in the statics*. The idea is to add to the disorder a parabolic potential (i.e., a mass m) with a variable minimum location w. The resulting sample-dependent free energy $\hat{V}(w)$ defines a renormalized random potential whose second cumulant is proved to be *the same* $R(w)$ function as defined in the replica field theory – deviations arising only in higher cumulants [17]. This holds for any internal dimension d of the elastic manifold, any number of components N of its displacement field $u(x)$, and any T. At $T = 0$, the (minimum energy) configuration $u(x; w)$ is unique and smoothly varying with w, except for a discrete set of shock positions where $u(x; w)$ jumps between degenerate minima. The limit of a single particle in a random potential ($d = 0$) maps to decaying Burgers turbulence, and the statistics of the shocks can in some cases be obtained, yielding exact result [17] for $\Delta(u)$.

This method was used recently [18] to compute numerically the zero-temperature FRG fixed-point function $\Delta(u)$ in the statics, for interfaces ($N = 1$), using powerful exact minimization algorithms. Random bond, random field and periodic disorder were studied in various dimensions $d = 0, 1, 2, 3$. The results were found close to 1-loop predictions and deviations consistent with 2-loop FRG. A linear cusp was found in any d and the functional shocks leading to this cusp were seen. The cross-correlation for two copies of disorder was also obtained and compared to a recent FRG study of chaos [19]. The main assumptions and central results of the FRG for the *statics* were thus confirmed. It is important to extend these methods to the dynamics of pinned objects and to the depinning transition.

In this Letter we extend the method of Ref. [17] to the dynamics. Using a slow, time-dependent, harmonic potential we show how the various terms in the effective dynamical action identify with the FRG functions. The $T > 0$ equilibrium dynamics reduces to the same definition as used for the statics. We describe the extension to depinning at $T = 0$. There the manifold is pulled by a quasi-static harmonic force (i.e. a spring of strength noted m^2), and we show how the statistics of the resulting jumps directly yields the critical force and the FRG functions, and how they converge to fixed forms as $m \to 0$. The model is similar to some stick-slip models used e.g. in dry friction [5, 20] and earthquake dynamics [7]. The present method provides a different way to look at these problems in numerics and experiments, in addition to giving a precise meaning to quantities computed in the field theory. In particular we discuss the identification of the critical force, the statistics of the jumps, using for illustration a graphical construction in $d = 0$. There we compute the FP functions for each universality class. These exhibit a cusp which we find is rounded by a finite velocity. These effects could be tested in experiments, as discussed at the end.

We consider the equation of motion for the overdamped dynamics of an elastic manifold parameterized by its time-dependent displacement field $u(x, t)$:

$$\begin{aligned} &\eta \partial_t u(x,t) = F_x[u(t); w(t)] \\ &F_x[u; w] = m^2(w - u(x)) + c\nabla_x^2 u(x) + F(x, u(x)) \end{aligned} \tag{1}$$

where $F_x[u(t); w(t)]$ is the total force exerted on the manifold (we note $u(t) = \{u(x,t)\}_{x\in\mathbb{R}^d}$ the manifold configuration, x being its d-dimensional internal coordinate); η is the friction coefficient and c the elastic constant. Here at the bare level, the random pinning force is $F(x, u) = -\partial_u V(x, u)$ and the random potential V has correlations $\overline{V(0,x)V(u,x')} = R_0(u)\delta^{(d)}(x - x')$. We consider first bare random bond disorder with a short-ranged $R_0(u)$. At non-zero temperature one adds the thermal noise $\langle \xi(x,t)\xi(x',t')\rangle = 2\eta T\delta(t - t')\delta^d(x - x')$. We have added a harmonic coupling to an external variable $w(t)$, a given function of time (in most cases we choose it uniformly increasing in t). This is the simplest generalization of the statics, where $w(t) = w$ is time-independant. It is useful to define the fixed-w energy

$$\mathcal{H}_w[u] = \int \mathrm{d}^d x \, \frac{m^2}{2}(u(x) - w)^2 + V(x, u(x)) \tag{2}$$

associated to the force $F_x[u;w] = -\frac{\delta H_w[u]}{\delta u(x)}$. If $w(t)$ is an increasing function of t the model represents an elastic manifold "pulled" by a spring. Quasi-static depinning is studied for $dw/dt \to 0^+$.

We first describe qualitatively how to measure the FRG functions and later justify why the relation is expected to be exact. Consider the observable $w(t) - \langle \bar{u}(t) \rangle$, where $\bar{u}(t) = L^{-d} \int \mathrm{d}^d x\, u(x,t)$ is the center of mass position, and $\langle \ldots \rangle$ denotes thermal averages, i.e. the ground state at zero temperature. It is the shift between the translationally averaged displacement and the center of the well, i.e. the extension of the spring. It is proportional to the pulling force on the manifold, hence to the translationally averaged pinning force minus the friction force, i.e. $w(t) - \bar{u}(t) = m^{-2}(\eta v(t) - \int_x F(x, u(x,t)))$ (if we use periodic boundary conditions inside the manifold). Of particular interest are:

$$
\begin{aligned}
&\overline{w(t) - \langle \bar{u}(t) \rangle} = m^{-2} f_{av}(t) \\
&\overline{[w(t) - \langle \bar{u}(t) \rangle][w(t') - \langle \bar{u}(t') \rangle]}^{c} = m^{-4} L^{-d} D_w(t,t')\ ,
\end{aligned}
\tag{3}
$$

where connected means w.r.t. the double average $\overline{\langle ... \rangle}$. If we consider a function $w(t)$ such that $dw(t)/dt > 0$, one can also write: $D_w(t,t') = \Delta_w(w(t), w(t'))$. As written, the function Δ_w may in general depend on the history $w(t)$. However we expect that for fixed L, m and slow enough $w(t)$, e.g. $w(t) = vt$ with $v \to 0^+$, one has $\Delta_w(w(t), w(t')) \to \Delta(w(t) - w(t'))$. This function $\Delta(w - w')$, which is independent of the process $w(t)$, is the one defined in the F.T., as we will justify below.

Let us start with non-zero temperature, $T > 0$, and consider a process $w(t)$ so slow that the system (with a finite number of degrees of freedom $(L/a)^d$) remains in equilibrium. In practice it means that $\dot{w} t_L \ll u(L)$ where t_L is the largest relaxation time of the system, and $u(L)$ its width. The above definition is then consistent with the one from the statics, where it was shown that one can measure the equilibrium free energy in a harmonic well with fixed w (or its generalization to an arbitrary $w(x)$), defined through $e^{-\hat{V}(w)/T} = \int \mathcal{D}[u]\, \mathrm{e}^{-\mathcal{H}_w[u]/T}$, and extract from it the pinning energy correlator $R(w)$. This can be done by measuring the second cumulant [21] $\overline{\hat{V}(w) \hat{V}(w')}^{c} = \hat{R}[w - w']$, with $\hat{R}[w] = L^d \hat{R}(w)$ for a uniform parabola $w(x) = w$, and using that $\hat{R} = R$ [17]. One equivalently obtains the force correlator $\Delta(w)$ via the equilibrium fluctuations of the center of mass $\langle \bar{u} \rangle_w$ at fixed w, i.e. $\overline{(w - \langle \bar{u} \rangle_w)(w' - \langle \bar{u} \rangle_{w'})}^{c} = m^{-4} L^{-d} \Delta(w - w')$. In the statics it is easy to show that $\Delta(w) = -R''(w)$. The potentiality of this function breaks down in the driven dynamics, or at depinning, as discussed below.

Let us note at this stage that a second definition can be given using two "copies". Consider two evolutions $u(x; w_1)$ and $u(x; w_2)$ driven by two (slow) processes $w_1(t) = w_2(t) + w$ of fixed separation, in the same disorder sample. Then define

$$
\overline{(w_1(t) - \langle \bar{u}_1(t) \rangle)(w_2(t) - \langle \bar{u}_2(t) \rangle)}^{c} = m^{-4} L^{-d} \Delta_t(w)
\tag{4}
$$

which is now an equal-time correlation. For a slow equilibrated motion at $T > 0$, it identifies with the static definition. The general case is discussed below and in [24].

Let us now describe $T = 0$ depinning, and restrict to $N = 1$. Quasi-static depinning is studied as the limiting case where $dw/dt \to 0^+$. One starts in a metastable state $u_0(x)$ for a given $w = w_0$, i.e. a zero-force state $F_x(u_0(x); w) = 0$ which is a local minimum of $H_{w_0}[u]$ with a positive barrier. One then increases w. For smooth short-scale disorder, the resulting deformation of $u(x)$ is smooth. At $w = w_1$, the barrier vanishes. For $w = w_1^+$ the manifold moves downward in energy until it is blocked again in a metastable state $u_1(x)$ which again is a local minimum of $H_{w_1}[u]$. We are interested in the center of mass (i.e. translationnally averaged) displacement $\bar{u} = L^{-d} \int \mathrm{d}^d x\, u(x)$. The above process defines a function $\bar{u}(w)$ which exhibits jumps at the set w_i. Note that time has disappeared: evolution is only used to find the next location. The first two cumulants

$$
\overline{w - u(w)} = m^{-2} f_c \quad , \quad \overline{(w - u(w))(w' - u(w')}^{c} = m^{-4} L^{-d} \Delta(w - w')
\tag{5}
$$

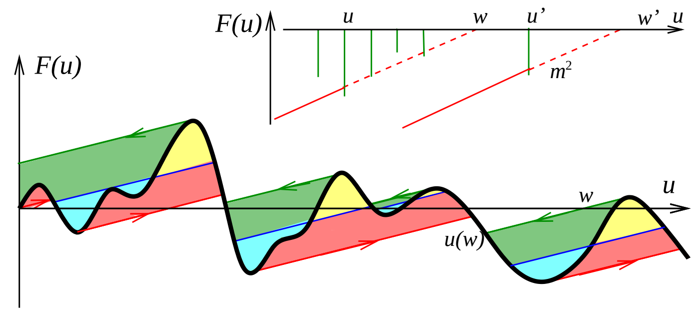

Fig. 1 – Main plot: Construction of $u(w)$ in $d = 0$. The pinning force $F(u)$ (bold black line). The two quasi-static motions driven to the right and to the left are indicated by red and green arrows, and exhibit jumps ("dynamical shocks"). The position of the shocks in the statics is shown, for comparison, from the Maxwell construction (equivalence of light blue and yellow areas, both bright in black and white). The critical force is $2/L$ times the area bounded by the hull of the construction. Inset: The same construction for the forward motion of the discretized model.

allow a direct determination (and definition) of the averaged (m-dependent) critical force f_c and of $\Delta(w)$, in analogy to the statics. Note that $u(w)$ depends a priori on the initial condition and on its orbit but at fixed m one expects an averaging effect when w is moved over a large region. This is further discussed below. Note that the definition of the (finite-size) critical force is very delicate in the thermodynamic limit [22]. Here the quadratic well provides a clear way to obtain a stationary state.

Elastic systems driven by a spring and stick-slip type motion were studied before, e.g. in the context of dry friction. The force fluctuations, and jump distribution were studied numerically for a string driven in a random potential [20]. However, the precise connection to quantities defined and computed in the field theory has to our knowledge not been made. The dependence on m for small m predicted by FRG, $\Delta(w) = m^{\epsilon-2\zeta}\tilde{\Delta}(wm^{-\zeta})$ is consistent with observations of [20] but the resulting $\tilde{\Delta}(w)$ has not been measured. Fully connected mean-field models of depinning also reduce to a particle pulled by a spring, together with a self-consistency condition, around which one can expand [10]. As discussed below, our main remarks here are much more general, independent of any approximation scheme, and provide a rather simple and transparent way to attack the problem.

For the qualitative discussion it is useful to study the model in $d = 0$, i.e. a particle with equation of motion

$$\eta\partial_t u = m^2(w-u) + F(u)\ . \tag{6}$$

In the quasi-static limit where w is increased slower than any other time-scale in the problem, the zero force condition $F(u) = m^2(u-w)$ determines $u(w)$ for each w. The graphical construction of $u(w)$ is well known from studies of dry friction [5]. When there are several roots one must follow the root as indicated in Fig. 1, where $F(u)$ is plotted versus $m^2(u-w)$. This results in jumps and a different path for motion to the right and to the left. Let us call A the area of this hysteresis loop (the area of all colored/shaded regions in Fig.1). It is the total work of the friction force when moving the center of the harmonic well quasi-statically once forth and back, i.e. the total dissipated energy. The above definition of the averaged critical force (5), assuming the landscape statistics to be translationally invariant (hence replacing disorder averages by translational ones over a large width M) gives

$$f_c = m^2\,\overline{(w-u_w)}^{\rm tr} = \frac{m^2}{M}\int_0^M \mathrm{d}w\ (w-u_w) = \frac{A}{2M}\ , \tag{7}$$

where we have used $\int u\,\mathrm{d}w = \int w\,\mathrm{d}u$. One can check that for $m \to 0$ this definition of f_c becomes identical to the one on a cylinder, f_d, which for a particle ($d = 0$) is $2f_d = f_d^+ - f_d^- = \max_u F(u) -$

$\min_u F(u)$ with $2f_d M = \lim_{m\to 0} A(m)$. (Since A depends on the starting point, this definition holds after a second tour, where the maximum (minimal) pinning force was selected). Finally, one can compare with the definition of shocks in the statics. There, the effective potential is a continuous function of w. Therefore, when making a jump, the integral over the force must be zero, which amounts to the Maxwell-construction of figure 1.

One can compute f_c and $\Delta(w)$ in $d = 0$ for a discrete force landscape, F_i, independently distributed with $P(F)$, and i integer. $u(w)$ is then integer and defined in the inset of figure 1. The process admits a continuum limit for small m, which depends on the behaviour of $P(F)$ in its tails (negative tail for forward motion). One obtains [24] the distribution of $u(w)$, $P_w(u)\mathrm{d}u = \mathrm{e}^{-a_w(u)}\mathrm{d}a_w(u)$ where $a'_w(u) = \int_{-\infty}^{m^2(u-w)} P(f)\mathrm{d}f$ and $a_w(-\infty) = 0$. One also obtains the joint distribution of $(u(w), u(w'))$, $P_{w;w'}(u, u') = (a'_w(u) - a'_{w'}(u))a'_{w'}(u')\mathrm{e}^{-a_w(u)-a_{w'}(u')+a_{w'}(u)}\theta(u' - u) + \delta(u' - u)a'_{w'}(u)\mathrm{e}^{-a_w(u)}$ for $w > w'$. Define $\Delta(w) =: m^4\rho_m^2\tilde{\Delta}(w/\rho_m)$ and $f_c =: f_c^0 + cm^2\rho_m$. This yields two main classes of universal behaviour at small m. The first contains (i) exponential-like distributions with unbounded support i.e. $\ln P(f) \approx_{f\to-\infty} -A(-f)^\gamma$ (for which $f_c^0 = ((\ln m^{-2})/A)^{\frac{1}{\gamma}}$) and (ii) distributions with exponential behaviour near an edge $P(f) \sim e^{-A(f+f_0)^\gamma}\theta(f + f_0)$ (with $\gamma < 0$ and $f_c^0 = f_0 - ((\ln m^{-2})/A)^{\frac{1}{\gamma}}$). For both (i) and (ii) the FP function is $\tilde{\Delta}(x) = \frac{x^2}{2} + \mathrm{Li}_2(1 - e^x) + \frac{\pi^2}{6}$ and $\rho_m = \rho_m^\gamma := 1/(|\gamma|A^{1/\gamma}m^2(\ln m^{-2})^{1-\frac{1}{\gamma}})$, $c = \gamma_E$ the Euler constant. The first class has $\zeta = 2$ up to log-corrections. The second class contains power-law distributions near an edge $P(f) = A\alpha(\alpha-1)(f+f_0)^{\alpha-2}\theta(f+f_0)$, $\alpha > 1$, for which $c = -\Gamma(1+\frac{1}{\alpha})$, $m^2\rho_m = (m^2/A)^{\frac{1}{\alpha}}$ and $f_c^0 = f_0$. The FP depends continuously on α with $\tilde{\Delta}(w) = -\Gamma(1 + \frac{1}{\gamma})\Gamma(1 + \frac{1}{\gamma}, w^\gamma) + w\Gamma(1 + \frac{1}{\gamma})e^{-w^\gamma} + \int_0^\infty \mathrm{d}y e^{-(y+w)^\gamma+y^\gamma}\Gamma(1 + \frac{1}{\gamma}, y^\gamma)$ where $\Gamma(a, x) = \int_x^\infty \mathrm{d}z z^{a-1}e^{-z}$; it has $\zeta = 2 - 2/\alpha$ [23]. Hence, despite the fact that $d = 0$ is dominated by extreme statistics (e.g. the distribution of $\rho_m^{-1}(w - u(w))$ converges to the Gumbel and Weibul distributions for class I and II respectively) it still exhibits some universality in cumulants and in all classes $\Delta(u)$ has a cusp non-analyticity at $u = 0$. We have checked the above scaling functions *and* amplitudes numerically, with excellent agreement.

We now come back to the interface, $d > 0$, and note that the manifold in the harmonic well can be approximated by $(L/L_m)^d$ roughly independent pieces with $L_m \sim 1/m$. The motion of each piece resembles the one of a particle, i.e. a $d = 0$ model, but with a rescaled unit of distance in the u direction, $u_m \sim L_m^\zeta \sim m^{-\zeta}$. The "effective-force" landscape seen by each piece becomes uncorrelated on such distances, and its amplitude scales as $F_m \sim m^2 u_m$. Hence one is in a bulk regime not dominated by extremes, i.e. $\Delta(w)$ probes only motion over about one unit. It is easy to check on Fig. 1 that an arbitrary initial condition joins the common unique orbit after about one correlation length. Hence the $d = 0$ model suggests that starting the quasi-static motion in u_0 and driving the manifold over $w \sim L_m^\zeta$ should then result in all orbits converging. Hence the definitions (4) and (5) are equivalent for $N = 1$. An interesting crossover to $d = 0$ behaviour and extremal statistics occurs if $L < L_m$.

Note that the averaged critical force, defined in (5), should, for $d > 0$, go to a finite limit, with $f_c(m) = f_c^\infty + Bm^{2-\zeta}$ from finite size scaling. Although f_c is not universal and depends on short-scale details, one easily sees that $-m\partial_m f_c(m)$ depends only on one unknown scale. We note that the definition (5) coincides with the one proposed recently as the maximum depinning force for all configurations having the same center of mass u_0 [22]. Since $\bar{u} - w$ is a fluctuating variable of order $(L/L_m)^{-d/2}$, the definition is the same as the above in the limit where $L \to \infty$, before $m \to 0$. The single w distribution is obtained from the distribution of $w - u(w)$ if all modes have a mass.

Measurements of $\Delta(w)$ reveal interesting features in any d. At the bare level, the disorder of the system is of random-bond type (i.e. potential). As the mass is decreased, one should observe a crossover from random-bond to random-field disorder. Also a finite velocity should round the cusp singularity. These features are well visible in the $d = 0$ toy model as illustrated in Fig. 2a (quasi-static evolution in a model with $F_i = V_{i+1} - V_i$ and V_i uncorrelated) and Fig. 2b (Langevin dynamics at

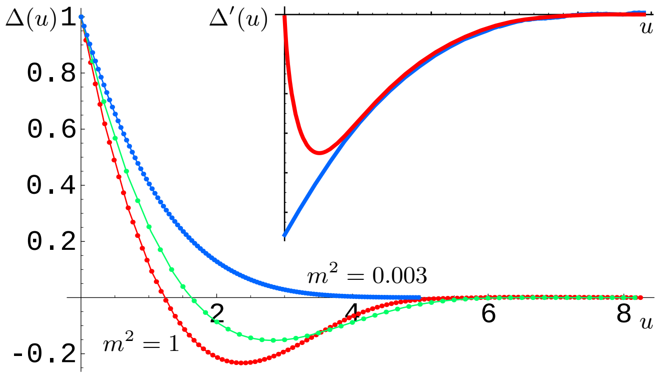

Fig. 2 – (a): Main plot. The measured $\Delta(u)$ for RB disorder distributed uniformly in $[0,1]$, and rescaled such that $\Delta(0)=1$ and $\int_0^\infty \mathrm{d}u|\Delta(u)|=1$. From bottom (which has $\int_0^\infty \mathrm{d}u\Delta(u)\approx 0$) to top the mass decreases from $m^2=1$ to 0.5 to 0.003. One observes a clear crossover from RB to RF.
(b): Inset. $\Delta'(u)$ for a driven interface at vanishing (blue, $\Delta'(0)<0$) and finite velocity (red, $\Delta'(0)=0$).

finite v) and can be obtained analytically in that case [24].

We now sketch the exact relation to the FT definitions. From now on we use condensed notations $u(x,t)=u_{xt}$ and so on. The bare MSR action functional can be parameterized as $\mathcal{S}[u,\hat u]=\hat u\cdot g^{-1}\cdot u+\hat u\cdot A^{(0)}[u]-\frac{1}{2}\hat u\cdot B^{(0)}[u]\cdot\hat u+O(\hat u^3)$ with g_{xy} an arbitrary (time independent) symmetric matrix (standard choice being $g_q^{-1}=q^2+m^2$), $A^{(0)}[u]_{xt}=\eta\partial_t u_{xt}$ and $B^{(0)}[u]_{xt,x't'}=2\eta T\delta_{xx'}\delta_{tt'}+\Delta_0(u_{xt}-u_{x't'})\delta_{xx'}$. We denote $u\cdot v:=\int_{xt}u_{xt}v_{xt}$ (and additional index contraction for $N>1$), $A^{(0)}$ and $B^{(0)}$ are respectively vector and matrix functionals. The effective action $\Gamma[u]$ can be parameterized identically with $A^{(0)}[u]\to A[u]$ and $B^{(0)}[u]\to B[u]$. It is obtained from the generating function: $W[w,\hat w]=\ln\int\mathcal{D}[u]\,\mathcal{D}[\hat u]\,\exp(-\mathcal{S}[u,\hat u]+\hat u\cdot g^{-1}\cdot w+\hat w\cdot g^{-1}\cdot u)$ through a Legendre transform: $W[w,\hat w]+\Gamma[u,\hat u]=\hat u\cdot g^{-1}\cdot w+\hat w\cdot g^{-1}\cdot u$. It can be expanded [21] as: $W[w,\hat w]=\hat w\cdot g^{-1}\cdot w-\hat w\cdot\hat A[w]+\frac{1}{2}\hat w\cdot\hat B[w]\cdot\hat w+O(\hat w^3)$. This functional directly generates correlations (5) and (5), in a more general form: $\overline{w_{xt}-\langle u_{xt}\rangle_w}=g_{xy}\hat A_{yt}[w]$ and $\overline{(w_{xt}-\langle u_{xt}\rangle_w)(w_{x't'}-\langle u_{x't'}\rangle_w)}^c=g_{xy}g_{x'y'}\hat B_{yt,y't'}[w]$. Hence $\hat A$ and $\hat B$ are observables which can be measured, i.e. for a uniform $w_{xt}=w_t$, $f_{av}(t)=\frac{1}{L^d}\int_y\hat A_{yt}[w]$ and $D_w(t,t')=\frac{1}{L^d}\int_{yy'}\hat B_{yt,y't'}[w]$, which for slow w_t should go to f_c and $\Delta(w_t-w_{t'})$ respectively. The question is how to relate them to the effective action, i.e. generalize the relation $\hat R=R$ from the statics. To this aim we perform a Legendre transform. Details are given in [24]. The result is $A[u[w]]=\hat A[w]$, (where we have defined $u[w]:=w-g\cdot\hat A[w]$, i.e. $u_{xt}[w]:=\langle u_{xt}\rangle_w$) and $\hat B[w]=(\mathrm{d}\hat u/\mathrm{d}\hat w)^t\cdot B[u[w]]\cdot\mathrm{d}\hat u/\mathrm{d}\hat w$ with $\mathrm{d}\hat u/\mathrm{d}\hat w=1-(\nabla_w\hat A[w])^t\cdot g$. Now consider w uniform in space $w_{xt}=w_t$. Then $\hat A_{yt}[w]$ is y-independent. In the limit of infinitely slow monotonous w_t one expects $A_{yt}[w]\to f_c\,\mathrm{sgn}(\dot w)$ hence $u_{xt}[w]\to w_t+\mathrm{sgn}(\dot w)f_c/m^2$ (with usually $f_c=0$ at $T>0$). From STS it implies $A[w]=\hat A[w]$ in that limit for uniform w. Similarly one finds $D_w(t,t')=\sum_{xy}B_{xt,yt'}[w]$ for infinitely separated times (with fixed $w_t-w'_t$) since $\mathrm{d}\hat u/\mathrm{d}\hat w=g\frac{\delta u[w]}{\delta w}\to 1$, i.e. the disorder-averaged response function becomes trivial at zero frequency (due again to STS). Hence the measured $\Delta(w)$ in (5) is – to all orders – the one defined in the FT.

We have generalized [24] the above method to a manifold driven in N dimensions (e.g. a flux line in a 3D superconductor). For a particle, and fixed m, we have seen numerically that different initial conditions converge. The Middleton theorem no longer holds, and particles can pass by each other. To probe transverse motion and correlations $\Delta(w)$ in the transverse direction, we use the two-copy definition (4). Finally thermal rounding of depinning, creep and crossover from statics to depinning can be studied more precisely by this method.

To conclude let us propose that $\Delta(w)$ be measured directly in experiments, which would represent an important test of the theory and the underlying assumptions. Creep and depinning of magnetic domains in thin films with surface step disorder have been investigated using imaging [1]: adding a magnetic field gradient should allow to confine the interface in an effective quadratic well, whose strength and position can be varied (hence probing both statics and dynamics). In contact lines of fluids it is capillarity and gravity which provide the quadratic well, and provided large scale inhomogeneities can be controlled, $\Delta(u)$ could be measured from statistics on lengths larger than the capillary length (i.e. L_m here). We hope this will stimulate further numerical [25] and experimental [26] studies.

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