ICFP M2 – Selected Topics in Statistical Field Theory
TD n° 8 – Scale Invariance in Driven Interfaces

Kay Wiese & Camille Aron

March 16 & 23, 2018

The Kardar-Parisi-Zhang (KPZ) equation is the field theory of many surface growth processes. For a surface parametrized by $h(\boldsymbol{x}, t)$, it reads

$$\partial_t h(\boldsymbol{x}, t) = \boldsymbol{\nabla}^2 h(\boldsymbol{x}, t) + \frac{\lambda}{2} \left[\boldsymbol{\nabla} h(\boldsymbol{x}, t)\right]^2 + \zeta(\boldsymbol{x}, t) \tag{1}$$

where $h(\boldsymbol{x}, t)$ is assumed to be single valued, $\lambda \geq 0$, and $\zeta(\boldsymbol{x}, t)$ is a Gaussian white noise,

$$\langle \zeta(\boldsymbol{x}, t) \rangle = 0 \,, \qquad \langle \zeta(\boldsymbol{x}, t) \zeta(\boldsymbol{x}', t') \rangle = 2D \, \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t') \,. \tag{2}$$

A – Scaling analysis

(1) Write down the Martin-Siggia-Rose (MSR) action functional, $S[h, \hat{h}]$, associated to the dynamics of the KPZ equation.

(2) Measuring lengths in units of $[x] = \Lambda^{-1}$, compute the scaling dimensions for the fields h and $\hat{h}$, and the coupling constant λ. In which dimensions is λ relevant? When is $\lambda = 0$ expected to be a stable fixed point?

(3) Edwards-Wilkinson model. At $\lambda = 0$, notice that the dynamics coincides with those of a Gaussian Model A at criticality. Compute the steady-state probability distribution $P_{t \to \infty}[h]$, and the correlation function $C_0(\boldsymbol{k}, \omega)$.

(4) The scaling law for a critical correlation function generically reads

$$C_*(x, t) = |x|^{2\chi} f_{C_*}(t/|x|^z) \,, \tag{3}$$

where $f_{C_*}(y)$ is a dimensionless function, z the dynamical exponent, and we introduced the static exponent $\chi = -(d - 2 + \eta)/2$. Express the behavior of $f_{C_*}(y)$ when $y \to 0$ and $y \to \infty$. Discuss why χ is called the roughness exponent. Compute χ and z in the Edwards-Wilkinson model. In which dimensions does one get a rough surface?

(5) For $\lambda > 0$, check that the KPZ equation is invariant under the infinitesimal tilting of the surface

$$h(\boldsymbol{x}, t) \mapsto h'(\boldsymbol{x}', t') = h(\boldsymbol{x} - \lambda \, \delta\boldsymbol{v} \, t, t) - \delta\boldsymbol{v} \cdot \boldsymbol{x} \,. \tag{4}$$

Applying the scale transformations $\boldsymbol{x} \to b\,\boldsymbol{x}$, $t \to b^z \, t$ to Eq. (4). Show that at any *finite* fixed point $(0 < \lambda^* < \infty)$, we have the relation

$$\chi + z = 2 \,, \tag{5}$$

which reduces the number of independent exponents to one.

B – Mapping to linear diffusion

For $\lambda > 0$, we can eliminate the non-linear term by the so-called Cole-Hopf transformation, *i.e.* working in terms of the field

$$Z(\boldsymbol{x}, t) = \exp\left[\frac{\lambda}{2} h(\boldsymbol{x}, t) - \frac{\lambda^2 D}{4} t\right] . \tag{6}$$

(1) Using stochastic calculus, show that the KPZ equation maps onto the following Itô stochastic differential equation,

$$\partial_t Z(\boldsymbol{x}, t) = \nabla^2 Z(\boldsymbol{x}, t) + \frac{1}{2} V(\boldsymbol{x}, t) Z(\boldsymbol{x}, t) , \tag{7}$$

where V is a Gaussian white noise with $\langle V(\boldsymbol{x}, t) V(\boldsymbol{x}', t')\rangle = 2\lambda^2 D \delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$. Equation (7) is a linear diffusion equation with multiplicative noise.

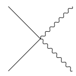
(2) Show that the MSR action functional associated to the dynamics of the equation (7) reads

$$S[Z, \hat{Z}] = \int \mathrm{d}t \int \mathrm{d}^d\boldsymbol{x}\; \mathrm{i}\hat{Z}(\boldsymbol{x}, t) \left[\partial_t Z(\boldsymbol{x}, t) - \boldsymbol{\nabla}^2 Z(\boldsymbol{x}, t)\right] - \frac{g}{2}\left[\mathrm{i}\hat{Z}(\boldsymbol{x}, t) Z(\boldsymbol{x}, t)\right]^2 , \tag{8}$$

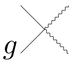
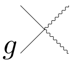
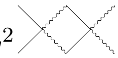
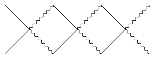
with the coupling $g \equiv \lambda^2 D/2$.

C – Roughening transition *via* RG

Let us now calculate the β-equation associated to the renormalization of the coupling g in $d > 2$ by concentrating on the vertex

$$\left[\mathrm{i}\hat{Z}(\boldsymbol{x}, t) Z(\boldsymbol{x}, t)\right]^2 =$$  . (9)

(1) Show that

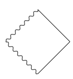
$\exp g$  $= 1 + g$  $+ g^2$  $+ g^3$  ... (10)

where the higher-order vertices may be discarded since the only divergencies they may contain are sub-chains as those depicted in Eq. (10).

(2) By resuming the series in Eq. (10), show that the effective 4-point function is

$\Gamma(\boldsymbol{k}, \omega) = g \dfrac{1}{1 - g\,\text{[diagram]}(\boldsymbol{k}, \omega)}$, with the diagram  (11)

(3) Show that

 $$(\boldsymbol{k}, \omega) = \frac{1}{(8\pi)^{d/2}} \Gamma(1 - d/2) \left(\frac{1}{2}\boldsymbol{k}^2 + \mathrm{i}\omega\right)^{d/2-1} . \tag{12}$$

(4) The theory is renormalizable if we can make the 4-point function finite as a function of g_{R} instead of g by setting

$$g = Z_g\, g_{\mathrm{R}}\, \Lambda^{-\epsilon} \text{ with } Z_g \equiv \frac{1}{1 + a g_{\mathrm{R}}} \text{ and } \epsilon \equiv d - 2 , \tag{13}$$

where Λ is an arbitrary scale. Fix a by demanding that $\Gamma(\boldsymbol{k},\omega)$ evaluated at $\Lambda^2 = \frac{1}{2}\boldsymbol{k}^2 + \mathrm{i}\omega$ be equal to g_R. Show that

$$g_\mathrm{R} = \frac{g}{\Lambda^{-\epsilon} - ag}\,. \tag{14}$$

(5) Derive the β-equation

$$\beta(g_\mathrm{R}) = (d-2)g_\mathrm{R} - \frac{2}{(8\pi)^{d/2}\Gamma(2-d/2)} g_\mathrm{R}^2\,, \tag{15}$$

and show that there is a non-trivial unstable perturbative fixed point in $d > 2$ which separates a smooth Edwards-Wilkinson phase from a rough phase at strong coupling.