ICFP M2 – Selected Topics in Statistical Field Theory
TD n° 7 – Model A Dynamics

Kay Wiese & Camille Aron

March 9, 2018

Let us consider the non-conserved order-parameter field $\phi(\boldsymbol{x}, t)$. Its evolution is governed by the following time-dependent Landau-Ginzburg equation, the so-called model A dynamics,

$$\frac{\partial \phi(\boldsymbol{x}, t)}{\partial t} = -\frac{\delta \mathcal{H}[\phi]}{\phi(\boldsymbol{x}, t)} + \zeta(\boldsymbol{x}, t)\,. \tag{1}$$

$\mathcal{H}[\phi]$ is the Landau-Ginzburg Hamiltonian of the static theory, and $\zeta(\boldsymbol{x}, t)$ is a Gaussian white noise,

$$\langle \zeta(\boldsymbol{x}, t) \rangle = 0 \qquad \langle \zeta(\boldsymbol{x}, t) \zeta(\boldsymbol{x}', t') \rangle = 2D\, \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t')\,, \tag{2}$$

where the constant D will be fixed below.

A – Relaxation to equilibrium

(1) Fokker-Planck. Derive the evolution equation on $G_t[\phi]$, the probability of a configuration $[\phi]$ at time t.

(2) Express the condition on D such that the Gibbs-Boltzmann distribution $G_{\text{GB}}[\phi] = Z^{-1} \exp\left(-\mathcal{H}[\phi]/T\right)$ is a possible steady-state solution of the above Fokker-Planck equation.

(3) H-theorem. Consider the time-dependent quantity

$$H(t) = \int \mathcal{D}[\phi]\, G_t[\phi]\, (\ln G_t[\phi] - \ln G_{\text{GB}}[\phi])\,. \tag{3}$$

Using the above Fokker-Planck equation, show that $\partial_t H(t) \leq 0$ for all t. Conclude that $\lim_{t\to\infty} G_t[\phi] = G_{\text{GB}}[\phi]$.

B – ϕ^4-theory: dynamical scaling

Let us consider the following Ising-type Landau-Ginzburg Hamiltonian,

$$\mathcal{H}[\boldsymbol{\phi}] = \int \mathrm{d}^d \boldsymbol{x} \, \left[\frac{1}{2} (\boldsymbol{\nabla}\phi)^2 + \frac{r}{2}\phi^2 + \lambda\, \phi^4 - h\phi \right]\,,$$

where the "mass^2" r tunes the distance to criticality, and $\lambda \geq 0$ is the interaction parameter. Close to the critical point, the correlation length ξ diverges: $\xi \sim |r|^{-\nu}$. We also expect the dynamical timescales to diverge, the so-called critical slowing down. The dynamical scaling hypothesis assumes that the dynamics of the order parameter close to criticality are governed by a characteristic time $\tau \sim \xi^z$, where z is called the dynamical exponent.

(1) Give the dimensions of the field $\phi(\boldsymbol{x})$, r, and λ, in terms of the microscopic cutoff scale Λ^{-1}. In which dimensions is $\lambda\,\phi^4$ an irrelevant operator in the UV?

(2) By simple dimensional analysis, one expects the typical relaxation time of a Fourier mode k of the field to scale as

$$t_{\mathrm{R}}(k) = \tau\, f(k\xi)\,, \tag{4}$$

where $f(u)$ is dimensionless. How do you expect $t_{\mathrm{R}}(k \to \infty)$ to depend on the distance to criticality? Deduce that $f(u \to \infty) \sim u^{-z}$. Close to criticality, deduce that $t_{\mathrm{R}}(k) \propto k^{-z}$.

(3) We recall that in equilibrium, and close to criticality, the spontaneous ($h = 0$) order parameter $\langle\phi\rangle_{\mathrm{eq}} \sim |r|^\beta$. At $t = 0$, the field is set to a value $\phi_0 \neq \langle\phi\rangle_{\mathrm{eq}}$, and we follow the relaxation of $\phi(t)$. Propose a scaling law for $\phi(t)$ and show that close to criticality, $\phi(t) \sim t^{-\theta}$ with $\theta = \beta/z\nu$.

(4) The scaling law for the correlation function reads

$$C(x,t) = x^{-(d-2+\eta)} f_C(x/\xi, t/\tau)\,, \tag{5}$$

where $f_C(u,v)$ is dimensionless. Use the fluctuation-dissipation theorem to obtain the scaling law for the susceptibility v,*i.e.* the linear response function. Show that

$$\chi(k,\omega) = k^{-2+\eta} f_\chi(k\xi, \omega\tau)\,, \tag{6}$$

where $f_\chi(u,v)$ is dimensionless.

C – Gaussian approximation

Let us momentarily work with the Gaussian theory: $\lambda = 0$.

(1) Compute the dynamical susceptibility $\chi_0(\boldsymbol{k},\omega) \equiv \left.\frac{\partial\langle\phi(\boldsymbol{k},\omega)\rangle}{\partial h(\boldsymbol{k},\omega)}\right|_{h=0}$.

(2) By comparing with Eq. (6), deduce the exponents ν, η, and z. Extract the relaxation timescale of a mode k of the field, $t_{\mathrm{R}}(k)$.

(3) Compute the dynamical correlation $C_0(\boldsymbol{k},\omega)$ defined as

$$\langle\phi(\boldsymbol{k},\omega)\phi(\boldsymbol{k}',\omega')\rangle \equiv C_0(\boldsymbol{k},\omega)(2\pi)^{d+1}\delta^d(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega')\,. \tag{7}$$

(4) Check that the fluctuation-dissipation theorem holds between $\chi_0(\boldsymbol{k},t)$ and $C_0(\boldsymbol{k},t)$.

D – Perturbative approach

(1) Introducing the shorthand notation $Q \equiv (\boldsymbol{k},\omega)$, show that

$$\phi(Q) = \chi_0(Q)\,[h(Q)+\zeta(Q)] - 4\lambda\,\chi_0(Q)\int_{Q'}\int_{Q''}\phi(Q')\phi(Q'')\phi(Q-Q'-Q'')\,. \tag{8}$$

(2) Propose a corresponding diagrammatics and use it to compute $\chi(Q)$ to first order in λ.

E – Challenge

Repeat the analysis for the diffusive relaxation dynamics of a conserved order parameter, the so-called model B dynamics,

$$\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = \nabla^2_{\boldsymbol{x}} \frac{\delta\mathcal{H}[\phi]}{\phi(\boldsymbol{x},t)} + \zeta(\boldsymbol{x},t)\,. \tag{9}$$

$\mathcal{H}[\phi]$ is the Landau-Ginzburg Hamiltonian of the static theory, and $\zeta(\boldsymbol{x},t)$ is a Gaussian white noise correlated in space,

$$\langle\zeta(\boldsymbol{x},t)\rangle = 0 \qquad \langle\zeta(\boldsymbol{x},t)\zeta(\boldsymbol{x}',t')\rangle = -2D\,\nabla^2_{\boldsymbol{x}}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')\,. \tag{10}$$