Laboratoire Kastler Brossel
Physique quantique et applications

LKB

Bose-Einstein condensates, phase-coherent dynamics & entangled states: to what extent and for what purpose?

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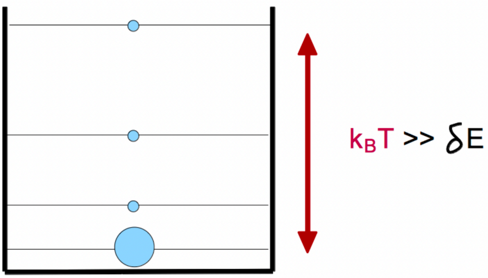
Max-Planck-Instutut für Quantenoptik, Garching, October 2024

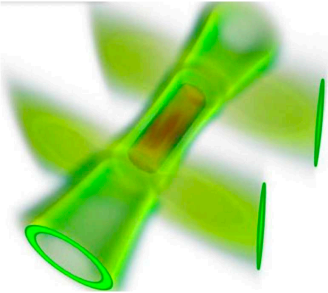
Plan

1. Introduction
   - BEC Phase coherence
   - Role of the interactions
   - Partition noise and Spin squeezing
2. Multiparameter quantum metrology
   - Imaging and compressed sensing
   - First experimental investigation
3. Theoretical questions
   - Quantum Josephson equation
   - Squeezing limit due to finite temperature
   - BEC intrinsic coherence time

Bose-Einstein condensate

Bosons $T < T_c$ macroscopic population of a single particle state





$\Rightarrow$ Macroscopic coherence properties : spatial and temporal

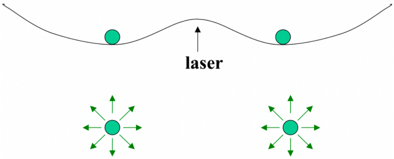
Typical number for a BEC : $\Delta x = 50\mu m,\ N = 10^6,\ T = 100nK$

$\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{mkT}} \sim 0.6\mu m,\ \rho = 10^{19}$ at/m^3, $(\rho\lambda_{th} > 1)$; Lifetime $\tau = 100s$

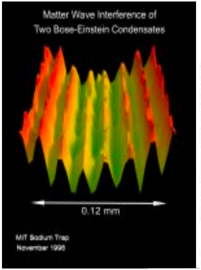
First evidence for phase coherence

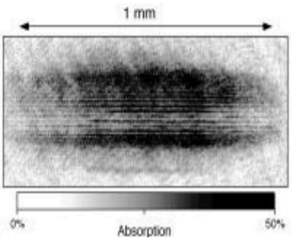
Interference between two BEC, MIT 1997

Set-up:



Result:

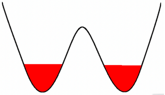




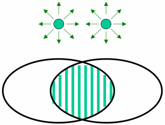
Phase dynamics

For how long do the condensats remember their (relative) phase ?

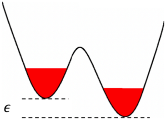
Two BEC with a well defined relative phase at time $t = 0$

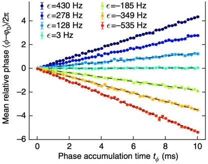


Interferometric measurement of the relative phase at time t



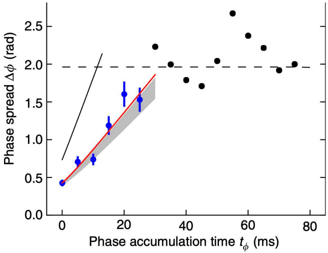
Mean relative phase dynamics





$$\dot{\theta}_a - \dot{\theta}_b = -(\mu_a - \mu_b)/\hbar$$

Relative phase spreading



Berrada, van Frank, Bücker, Schumm, Schaff, Schmiedmayer, Nat. Comm. 1013

The condensate phase operator $\hat{\theta}$

Modulus-phase representation (condensate mode)

$$\hat{a} = e^{i\hat{\theta}}\sqrt{\hat{n}} \quad \textbf{and} \quad \hat{a}^{\dagger} = \sqrt{\hat{n}}e^{-i\hat{\theta}} \quad \textbf{with} \quad [\hat{n}, \hat{\theta}] = i$$

Correct matrix elements on Fock states except on vacuum

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \Rightarrow e^{i\hat{\theta}}|n-1\rangle \quad \textbf{for all} \quad n > 0$$

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \Rightarrow e^{-i\hat{\theta}}|n+1\rangle \quad \textbf{for all} \quad n \in \mathbb{N}$$

Analogy with position and momentum operator of a particle

$$e^{-i\hat{p}}|x\rangle = |x+1\rangle \quad \textbf{translations generator} \quad [\hat{x}, \hat{p}] = i$$

One-mode model (mean field, zero temperature)

- **Hamiltonian with atomic interactions**

$$H = \frac{g}{2}\frac{N^2}{V} \quad \textbf{chemical potential} \quad \mu(N) = \frac{dE}{dN} = g\frac{N}{V}$$

- **Time derivative of the phase**

$$\frac{d\theta}{dt} = \frac{1}{i\hbar}[\theta, H] = -\frac{g}{\hbar}\frac{N}{V} = -\frac{1}{\hbar}\mu(N) \quad \textbf{and} \quad N = \text{constant}$$

- **For an initial state state with fluctuations of N**

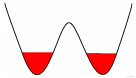
$$\mu(N) = \mu(\bar{N}) + \frac{d\mu}{dN}(N - \bar{N}) + \dots$$

$$\text{Var}(\theta)(t) = \text{Var}(\theta)(0) + t^2\left(\frac{1}{\hbar}\frac{d\mu}{dN}\right)^2 \text{Var}(N - \bar{N})$$

- **For a binomial distribution of $(N - \bar{N})$ (gaussian for large N), Phase spreads in time as the particle's gaussian wave packet**

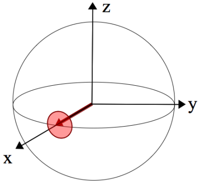
Relative phase dynamics in terms of a collective spin

Two bosonic modes (internal or spatial) with $\bar{N}_a = \bar{N}_b = N/2$



$$|\psi(0)\rangle = \frac{1}{\sqrt{N}} \left(\frac{a^\dagger + b^\dagger}{\sqrt{2}} \right)^N |0\rangle$$

Collective spin & Relative phase



$$S_z = \frac{\hat{N}_a - \hat{N}_b}{2}$$

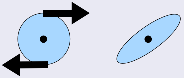
$$S_x = \frac{N}{2}$$

$$S_y = \frac{1}{2i}(a^\dagger b - b^\dagger a) \simeq \frac{N}{2}(\hat{\theta}_a - \hat{\theta}_b)$$

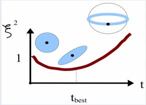
Relative phase evolution

$$(\hat{\theta}_a - \hat{\theta}_b)(t) = (\hat{\theta}_a - \hat{\theta}_b)(0) - \chi t(\hat{N}_a - \hat{N}_b) \quad \textbf{with} \quad \chi = \frac{1}{\hbar}\frac{d\mu_a}{dN_a}$$

$H_{OAT} = \hbar\chi S_z^2$ — Spin squeezing for $1/N \ll \chi t \ll 1/\sqrt{N}$

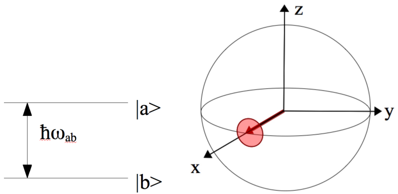


$$S_y(t) = S_y(0) - (N\chi t)S_z$$



Spin squeezing for phase estimation $\phi = \omega_{ab} t$

N two-level atoms (non correlated) $\Rightarrow$ collective spin $N/2$

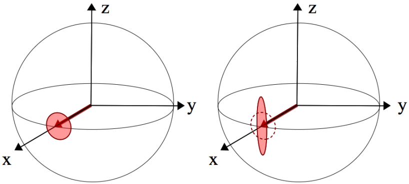


$$\Delta S_y \Delta S_z = \frac{|\langle S_x \rangle|}{2} = \frac{N}{4}$$

Angular position uncertainty

$$(\Delta\phi)_{NC} = \frac{\Delta S_y}{\langle S_x \rangle} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}.$$

Correlations reduce the collective spin fluctuations



Spin squeezed ensemble $\xi < 1$

$$(\Delta\phi)_{\mathrm{SQ}} = \xi\, (\Delta\phi)_{\mathrm{NC}} = \frac{\xi}{\sqrt{N}}$$

and reduce the statistical error in the estimation of ϕ

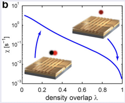
Exemples of experimental realizations

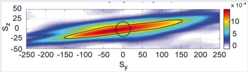
Non linear Hamiltonian $H_{NL} = \hbar\chi S_z^2$

- **Two components condensates :** $N \simeq 10^2 - 10^3$, $\xi^2 \simeq 0.15$

  Oberthaler, *Nonlinear atom interferometer surpasses classical precision limit* Nat. (2010).

  Treutlein, *Atom-chip-based generation of entanglement for quantum metrology* Nat. (2010).





- **Cavity backaction :** $N \simeq 10^2 - 10^4$, $\xi^2 \simeq 0.3$

  Vuletìc, *Implementation of cavity squeezing of a collective atomic spin* PRL (2010).

  Vuletìc, *Entanglement on an optical atomic-clock transition* Nat. (2020).

Spin squeezing by QND measurement

- **Large amount of squeezing :** $N \simeq 10^5$, $\xi^2 \simeq 10^{-2}$

  Kasevich, *Measurement noise 100 times lower than the quantum-projection limit using entangled atoms* Nat. (2016).

- **Observation of spin squeezed state for one second**

  Reichel, *Self-amplifying spin measurement in a long-lived spin-squeezed state* PRXQ (2023).

Quantum technologies : potential and limitations

Potentially useful

- Miniaturized sensors as $(\delta\omega)_{\mathrm{CSS}} = \frac{1}{T\sqrt{N}}$
- Applications in fundamental physics

From demonstations to applications if ...

- The quantum advantage becomes "easy" or necessary
- Example : squeezed states of light studied in the 1980 used today in interferometry for gravitational wave detection.

  M. Tse et al., *" Quantum-enhanced advanced LIGO detectors in the era of gravitational-wave astronomy"* Phys. Rev. Lett. 123, 231107 (2019)

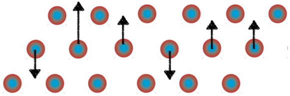
Limitations dues to decoherence

For a squeezed state to be useful, for a measurement time T :

- The uncertainty $\delta\omega$ should be dominated by quantum noise
- Quantum correlations should survive : $\gamma T \ll 1$

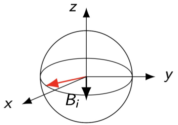
Single atoms for sensing an extended field

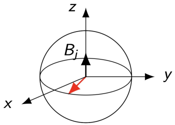
- **Estimation of N parameters : $\theta_i \propto \mathrm{B(i)}$**



Encoding the N parameters

$$\hat{U}(\vec{\theta}) = e^{-i\sum_i \theta_i \hat{s}_{i,z}}$$

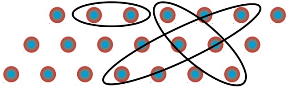




Initial state of the atoms (CSS)

$$|\psi_0\rangle = |Ox\rangle^{\otimes N}$$

- **With spin squeezing**

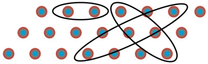


Initial state of the atoms (SSS)

$$|\psi_0\rangle = e^{-it\left(\sum \hat{s}_z^i\right)^2}\left[|Ox\rangle^{\otimes N}\right]$$

Quantum enhanced multiparameter estimation protocol

- **Use correlated spins and perform collective measurements**



- **Mesure N independent linear combinaisons of $\theta_1, \ldots \theta_N$**

For example, mesure of $\hat{S}_y = \sum \hat{s}_y^i \rightarrow \sum_k \theta_k$ with quantum gain

- **Mesure the other combination by spin flips**

$$e^{i\pi\hat{s}_x} e^{-i\theta\hat{s}_z} e^{-i\pi\hat{s}_x} = e^{i\theta\hat{s}_z}$$

- **Which combinaisons ? Those of the Hadamard transf.**

$$\tilde{\theta}_j = \sum_k [\mathcal{H}_m^{-1}]_{jk}\theta_k \quad \textbf{et} \quad \theta_k = \sum_j [\mathcal{H}_m]_{kj}\tilde{\theta}_j$$

size $M = N = 2^m, \qquad H_0 = 1, \qquad H_m = \begin{pmatrix} \mathcal{H}_{m-1} & \mathcal{H}_{m-1} \\ \mathcal{H}_{m-1} & -\mathcal{H}_{m-1} \end{pmatrix}$

Quantum gain in the measurement of a field

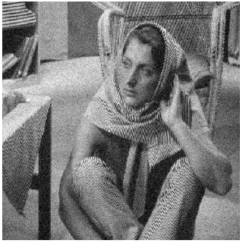
Original signal $\vec{\theta}$ $\Rightarrow$ $\hat{U}(\vec{\theta}) = e^{-\mathrm{i}\sum_i \theta_i \hat{s}_{i,z}}$ $N = 512 \times 512$

Reconstruction: measure N Hadamard coefficients ($\mu_c = 10$ repetitions)

- **Coherent spin state CSS**: $\Delta\theta_i^{\mathrm{CSS}} = \frac{1}{\sqrt{\mu}}$
- **Spin squeezed state SSS:** $H_{NL} = \chi S_z^2$

$$\Delta\theta_i^{\mathrm{SSS}} = \frac{\xi}{\sqrt{\mu}} \quad ; \quad t_{\mathrm{opt}} \sim \frac{1}{N^{2/3}} \quad ; \quad \xi(t_{\mathrm{opt}}) \sim \frac{1}{N^{1/3}} = \frac{1}{64}$$

Quantum gain : imperfect squeezing



Original signal $\Rightarrow$ $\hat{U}(\vec{\theta}) = e^{-\mathrm{i}\sum_i \theta_i \hat{s}_{i,z}}$ $N = 512 \times 512$

Imperfect spin squeezed SSSD, with collective dephasing $\gamma/\chi = 5$

$$\frac{\partial \rho}{\partial t} = -\mathrm{i}[\chi S_z^2, \rho] + \gamma \left[S_z \rho S_z - \frac{1}{2}\{S_z^2, \rho\} \right]$$

$$\Delta\theta_i^{\mathrm{SSSD}} = \frac{\xi}{\sqrt{\mu_c}} \quad ; \quad t_{\mathrm{opt}} \sim \frac{1}{N^{3/5}} \quad ; \quad \xi(t_{\mathrm{opt}}) \sim \frac{1}{N^{1/5}} \simeq \frac{1}{12}$$

Compressed sensing : ons mesures only $\mathcal{L}_{\mathcal{H}} < N$ Hadamard coefficients

$N = 512 \times 512$

CSS, $\mathcal{L}_{\mathcal{H}} = 64^2$

SSS, $\mathcal{L}_{\mathcal{H}} = 64^2$

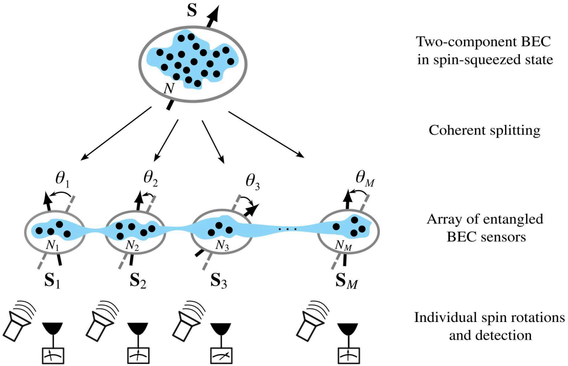
SSSD

Y. Baamara, M. Gessner, A. Sinatra
"Quantum-enhanced multiparameter estimation and compressed sensing of a field", SciPost Physics (2023)

Joint parameter estimation experiment: P. Treutlein

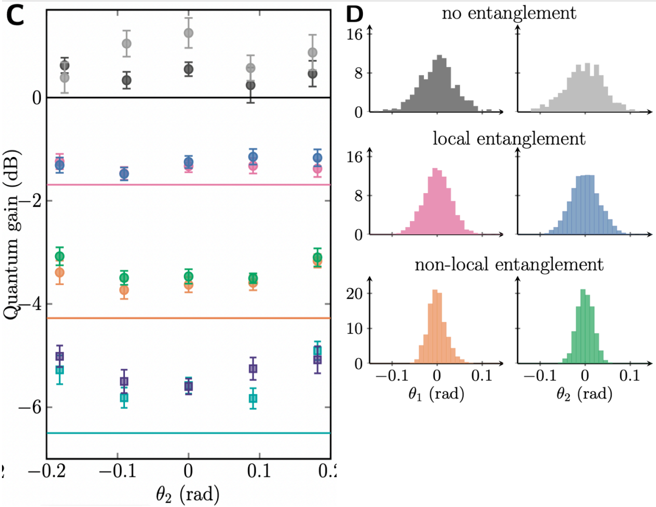
Many atoms per site and local measurements

- Spin-squeezed ensemble with N atoms distributed in $M = 2$ or $M = 3$ modes
- μ repetitions where the θ_k are locally imprinted and the collective spins S_k are locally manipulated and measured



Experimental results for $M = 2$ entangled sensors

Joint measurement of two parameters with quantum gain



Y. Li, L. Joosten, Y. Baamara, P. Colciaghi, A. Sinatra, P. Treutlein, T. Zibold, *und. ev.* (2024)

Quantum version of the 2^{nd} Josephson equation

Bogoliubov theory for weakly interacting, low T spinless bosons

A. Sinatra, Y. Castin, E. Witkowska PRA (2009), PRA (2010).

"Condensate phase derivative" = "chemical potential"

$$-\hbar\frac{d\hat{\theta}_0}{dt} = \mu_0(\hat{N}) + \sum_{\mathrm{k}\neq 0}(\partial_N \epsilon_k)\,\hat{n}_{\mathrm{k}} \equiv \hat{\mu}$$

$\hat{\mu}$ is the "adiabatic derivative" of $\hat{H}_{\mathrm{Bog}}$ with respect to N

$$\hat{H}_{\mathrm{Bog}} = E_0(N) + \sum_{\mathrm{k}\neq 0}\epsilon_k \hat{n}_{\mathrm{k}} \quad \text{and} \quad \hat{\mu} = \left.\frac{\partial \hat{H}_{\mathrm{Bog}}}{\partial N}\right|_{V,S}$$

Other frames in which we could derive the QJ equation

H. Kurkjian, Y. Castin, A. Sinatra PRA (2013), C. R. Physique (2016).

1. Pairs of fermions (time-dependent BCS theory including moving pairs Blaizot-Ripka 1985). Bosonic and fermionic excitations in $d\hat{\theta}/dt$.
2. Quantum hydrodymamics Landau and Khalatnikov 1949: low T, linear dispersion relation, exact equation of state (non mean field).

Quantum version of the 2^{nd} Josephson equation

"Condensate phase derivative" = "chemical potential"

$$-\hbar\frac{d\hat{\theta}_0}{dt} = \mu_0(\hat{N}) + \sum_{k\neq 0}(\partial_N\epsilon_k)\,\hat{n}_k \equiv \hat{\mu}$$

1. **Limit of spin squeezing due to finite temperature**

   Interactions via $\mu(N)$ generate squeezing. Fluctuations of $\hat{n}_k$ perturb the phase evolution and limit the squeezing.

2. **Time coherence of a condensate at finite temperature**

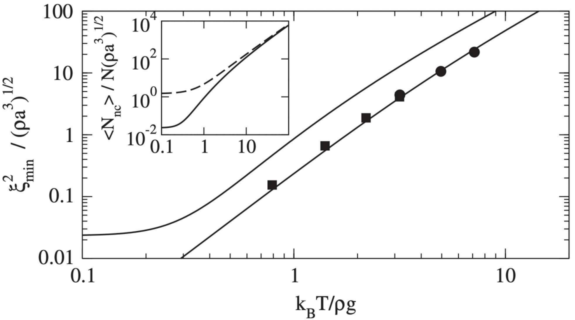
   Even in a single realization of the experiment, with N and E fixed, BEC coherence time is determined by the dynamics of the n_k.

Spin squeezing limit at finite temperature

Condensate relative phase (mutually non interacting BEC, $\chi = \frac{g}{\hbar V}$)

$$\theta_a(t) - \theta_b(t) \underset{\text{large } t}{\simeq} -\overbrace{\frac{\partial_{\langle N_a \rangle} \mu_\Phi}{\hbar} t}^{\chi t} \left[N_a - N_b + \overbrace{\sum_k \partial_{\mu_\Phi} \epsilon_k (n_{ka} - n_{kb})}^{D} \right]$$

Squeezing limit (thermo. limit) $\xi^2_{\text{min}} = \frac{\langle D^2 \rangle}{N} = \sqrt{\rho a^3}\, F(k_B T/\rho g)$



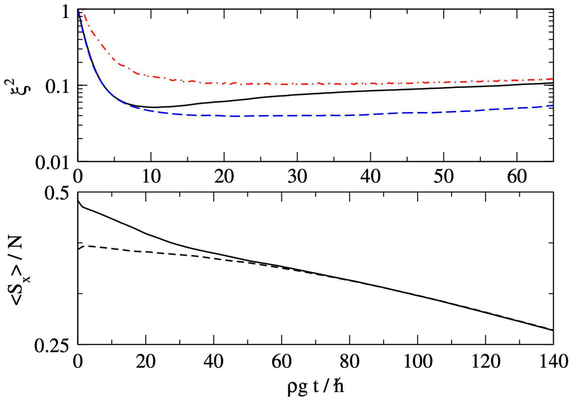
Lower line: classical field simulation (symbols) and analytics

Upper line : quantum result for ξ^2_{min}

Inset : ξ^2_{min} and the non condensed fraction

Evolution of spin squeezing and contrast

Thermalization due to interactions among Bogoliubov quasi particles



Top : Black Solid line = classical field simulation ; Blue dashed = Bogoliubov ; Red dotted = ergodic.

Parameters : $N = 3 \times 10^4$, $k_B T/\rho g = 3.16$, $\sqrt{\rho a^3} = 1.32 \times 10^{-2}$

Bottom : Black Solid line = Total contrast $\langle S_x \rangle$;

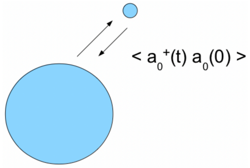
Dashed line = condensate contrast $\mathrm{Re}\langle b_0^* a_0 \rangle$.

A. Sinatra, E. Witkowska, J.-C. Dornstetter, Yun Li, Y. Castin, PRL (2011)

BEC coherence time

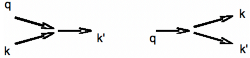
Fonction de correlation $g_1(t) = \langle \hat{a}_0^\dagger(t)\hat{a}(0)\rangle$

$$g_1(t) \simeq e^{-i\langle \hat{\theta}_0^\dagger(t) - \hat{\theta}(0)\rangle} e^{-\frac{1}{2}\mathrm{Var}[\hat{\theta}_0^\dagger(t) - \hat{\theta}(0)]}$$



System state $\hat{\rho} = \sum_\lambda \Pi_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|$ **with** $\hat{H}|\psi_\lambda\rangle = E_\lambda|\psi_\lambda\rangle$

In a single manybody eigenstate ψ_λ, at long times $\hat{n}_k$ decorrelate



Quantum ergodicity holds (ETH) $\langle\psi_\lambda|\hat{\mu}|\psi_\lambda\rangle = \mu_{\mathrm{mc}}(E_\lambda, N_\lambda)$

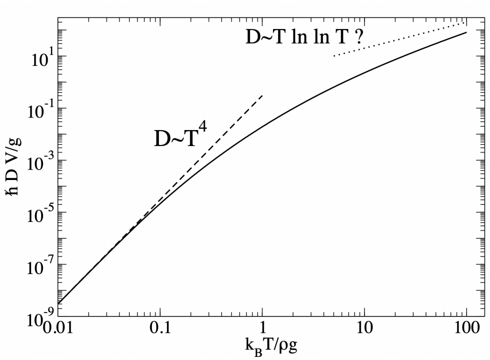
- **Phase factor:** $\langle\hat{\theta}_0^\dagger(t) - \hat{\theta}(0)\rangle = \mu_{\mathrm{mc}}(E_\lambda, N_\lambda)$
  (source of ballistic spreading if N or E fluctuate).
- **Phase diffusion in the microcanonical ensemble**

$$\mathrm{Var}[\hat{\theta}_0^\dagger(t) - \hat{\theta}(0)] \sim 2Dt \quad \textbf{with} \quad D = \int_0^t d\tau \left[\langle\frac{d\hat{\theta}}{dt}(t)\frac{d\hat{\theta}}{dt}(0)\rangle - \langle\frac{d\hat{\theta}}{dt}\rangle^2\right]$$

Phase Diffusion : D universal function of $k_B T/(\rho g)$

From kinetic equations for the $\hat{n}_k$, for $T \ll T_c$, $\sqrt{\rho a^3} \ll 1$

$$\frac{\hbar D V}{g} = G(k_B T/\rho g) \sim 0.36 \left(\frac{k_B T}{\rho g}\right)^4$$

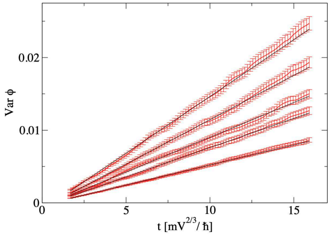


A. Sinatra, Y. Castin, E. Witkowska

PRA (2009)

Test against Classical field simulations

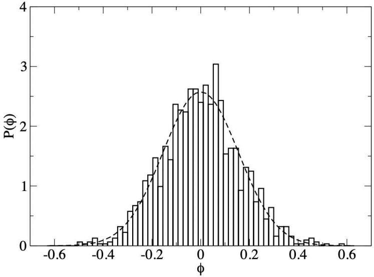
Classical field simulations in the microcanonical ensemble

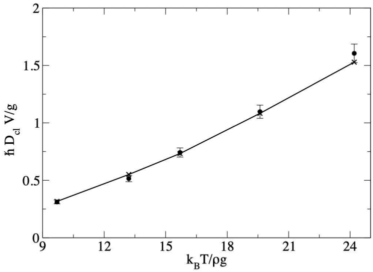


Diffusion coefficient

Squares with error bars : from numerics

Crosses linked by segments : from the classical field version of kinetic equations

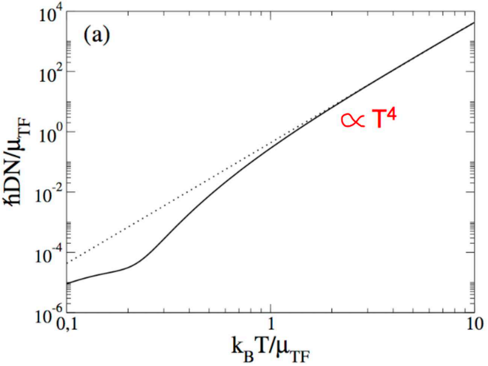




Phase Diffusion coefficient in a trap

Thermodynamic limit in the (unisotropic) harmonic trap

$$N \to \infty \quad \textbf{with} \quad \mu_{\rm GP} \quad \textbf{and} \quad T \quad \textbf{fixed} \to \omega_\alpha \propto 1/N^{1/3}$$

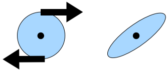


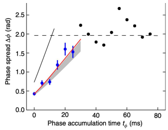
Y. Castin, A. Sinatra, C. R. Physique (2018)

Different scaling with T from the homogeneous case

Conclusions

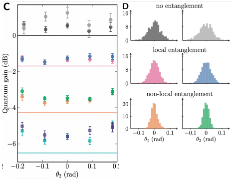
- Partition noise + interactions
  → squeezing at short times
  → phase spreading at long times





- Distributed squeezing can be used for imaging and compressed sensing

- First joint multi-parameter estimation with quantum gain



- Equation for the phase operator derivative (multimode description)

- Limit to spin squeezing and to the BEC coherence time for a large system at $T \neq 0$