Quantum Physics Exam, M2 ICFP 2018-2019

Alice Sinatra and David Papoular

Duration 3 hours. The notes from lectures and tutorials as well as the printed materials distributed by the teachers are authorized.

Coherence time of a Bose-Einstein condensate

At the thermodynamic limit, the Bose-Einstein condensation confers infinite temporal coherence to the Bose gas at very low temperature. However, in a finite size system, even isolated from the environment during its evolution, the interacting gas acquires a finite coherence time that we will calculate.

We consider a diluted gas of non-relativistic bosons of mass m at thermal equilibrium, in the very low temperature regime, with a well-formed Bose-Einstein condensate. The gas, isolated and homogeneous in three dimensions, consists of N atoms in a V quantization volume with periodic boundary conditions. The cold atoms interact by a short-range potential characterized by the parameter a, the s-wave scattering length. We limit ourselves to the case $a > 0$ and the weakly interacting regime $(\rho a^3)^{1/2} \ll 1$ with $\rho = N/V$ the mean density. To describe the interactions, we use the lattice model, introduced in the course, with an interaction potential between two particles

$$V_{12} = g_0 \frac{\delta_{\mathbf{r}_1,\mathbf{r}_2}}{b^3} \tag{1}$$

where b is the lattice constant and g_0 is the "bare" coupling constant that is however close to the true coupling constant $g = \frac{4\pi\hbar^2 a}{m}$ in the regime considered here.

1 Phase operator and phase spreading in a one-mode model

1.1 Condensate phase operator

In the presence of a condensate, it is convenient to introduce a "module-phase" representation for the operator a_0 which annihilates an atom in the condensate mode :

$$a_0 = e^{i\hat{\theta}}\sqrt{\hat{n}_0} \tag{2}$$

where $\hat{\theta}$ is an hermitien operator, $\hat{n}_0 = a_0^\dagger a_0$ and

$$[\hat{n}_0, \hat{\theta}] = i \tag{3}$$

As we will show, this representation is justified if we neglect the possibility that the condensate mode is empty.

1. (a) Justify that, in the Fock basis $|n_0\rangle$ of the condensate mode, the operator $\hat{n}_0$ is written

$$\hat{n}_0 = \sum_{n_0=0}^{\infty} n_0 \, |n_0\rangle\langle n_0| \tag{4}$$

(b) We define the operator $\sqrt{\hat{n}_0}$ in the following way

$$\sqrt{\hat{n}_0} = \sum_{n_0=0}^{\infty} \sqrt{n_0}\, |n_0\rangle\langle n_0| \tag{5}$$

verify that $\sqrt{\hat{n}_0}$ is hermitian.

2. Starting from the definition (2), where however we still do not know if $\hat{\theta}$ is hermitian, calculate the action of the operators $e^{i\hat{\theta}}$ and $\left(e^{i\hat{\theta}}\right)^\dagger$ on the Fock states $|n_0\rangle$ with $n_0 \neq 0$.

3. Show that in the Fock space where the state $|n_0 = 0\rangle$ has been subtracted, one has $e^{i\hat{\theta}}\left(e^{i\hat{\theta}}\right)^\dagger = 1$ and $\left(e^{i\hat{\theta}}\right)^\dagger e^{i\hat{\theta}} = 1$, so that the operator $e^{i\hat{\theta}}$ is unitary.

4. Justify the fact that, under the conditions specified at the beginning of the text, provided that $N \gg 1$, one can neglect the possibility that the mode of the condensate is empty, and thus introduce an hermitian operator $\hat{\theta}$ for the condensate phase.

5. Starting from the commutation rules (3), with $\sqrt{\hat{n}_0}$ and $\hat{\theta}$ hermitian :

   - show that $[\hat{n}_0, \hat{\theta}^p] = i\, p\, \hat{\theta}^{p-1}$
   - show that $[\hat{n}_0, F(\hat{\theta})] = i\, F'(\hat{\theta})$
   - recover from (2) the expected commutation rules $[a_0, a_0^\dagger] = 1$.

6. Highlight the analogy between the operators $-\hat{\theta}$ and $\hat{n}_0$ and the operators $\hat{x}$ and $\hat{p}$, position and momentum of a one-dimensional particle.

1.2 Ballistic spread of the phase in a one-mode model

In this section, we will study the phase dynamics of the condensate in a simple model where all gas particles are in the condensate mode $\hat{n}_0 = \hat{N}$. The system Hamiltonian is

$$H^{1\text{ mode}} = \frac{g\hat{N}^2}{2V} \qquad \text{with} \qquad [\hat{N}, \hat{\theta}] = i \tag{6}$$

and we assume that the initial state of the system is a statistical mixture with small fluctuations in the number of atoms around $N = \bar{N}$:

$$\hat{\sigma} = \sum_N \Pi_N |N\rangle\langle N| \tag{7}$$

with $\hat{N}|N\rangle = N|N\rangle$ and Π_N a narrow enough function of $N - \bar{N}$ to have $\mathrm{Var}[\hat{N}] \ll \bar{N}$. We recall that for any operator $\hat{A}$, one defines the variance $\mathrm{Var}[\hat{A}]$ and the standard deviation ΔA:

$$\mathrm{Var}[\hat{A}] = \langle \hat{A}^2\rangle - \langle \hat{A}\rangle^2 \quad \text{et} \quad \Delta A = \sqrt{\mathrm{Var}[\hat{A}]} = \left(\langle \hat{A}^2\rangle - \langle \hat{A}\rangle^2\right)^{1/2}\ , \tag{8}$$

where the average $\langle\cdot\rangle$ is taken in the state represented by the density matrix $\hat{\sigma}$.

7. (a) What is the normalization condition on the density matrix $\hat{\sigma}$?
   Express it in terms of the populations Π_N.
   (b) Draw qualitatively the shape of the coefficients Π_N as a function of N.

8. We will use throughout the problem the Heisenberg picture, in which each operator $\hat{A}(t)$ evolves in time according to the equation

$$i\hbar \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] \ . \tag{9}$$

   (a) Show that for this simple model, the operator $\hat{N}$ is a constant of motion.
   Does this mean that there are no fluctuations in the number of particles?
   (b) Derive the equation of motion for the operator $\hat{\theta}(t)$, and write the operator $d\hat{\theta}/dt$ explicitly in terms of the $\hat{N}$ operator.
   (c) Show that the operator $d^2\hat{\theta}/dt^2 = 0$.

9. We will introduce

$$\mu_0(\hat{N}) \equiv -\hbar \frac{d\hat{\theta}(t)}{dt}(\hat{N}) \tag{10}$$

   Express $\hat{\theta}(t) - \hat{\theta}(0)$ as a function of $\mu_0(\hat{N})$ and of t. It should be noted in passing that in this simple one-mode model, $\mu_0(N)$ coincides with the chemical potential of a N particle gas.

10. Show that the standard deviation $\Delta\phi = \Delta[\hat{\theta}(t) - \hat{\theta}(0)]$ of the operator $\hat{\theta}(t) - \hat{\theta}(0)$ satisfies:

$$\Delta\phi = \Delta[\hat{\theta}(t) - \hat{\theta}(0)] = \frac{1}{\hbar}\frac{g}{V}\,\Delta N\, t \ , \tag{11}$$

   where the standard deviation ΔN reflects the uncertainty about the number of particles in the statistical mixture described by σ.

11. Give a simple physical interpretation of this phase spreading due to the fluctuations of $\hat{N}$. One can draw inspiration from the last question of section 1.1. Why do we talk about the "ballistic spreading" of the condensate phase?

12. In Schmiedmayer's group in Vienna they measured the variation of $\mu_0(N)$ with the number of atoms around $N = \bar{N}$ in a gas at very low temperature. The results are shown in figure 1, left panel, where

$$\Delta\mu = \mu_0(\bar{N}(1-z)) - \mu_0(\bar{N}(1+z))) \tag{12}$$

   Deduce the value of $\frac{1}{\hbar}\frac{d\mu_0}{dN}|_{(N=\bar{N})} \times \bar{N}$ for a condensate with $\bar{N}$ atomes.

13. In the figure 1, right panel, we show the spread of the condensate phase as a function of time due to the fluctuations of N in the initial mixture

$$\Delta\phi = \left\{\mathrm{Var}[\hat{\theta}(t) - \hat{\theta}(0)]\right\}^{1/2} \tag{13}$$

   The black curve corresponds to Poisson fluctuations with $\mathrm{Var}[\hat{N}] = \bar{N}$ and the red curve to sub-poissonian fluctuations with $\mathrm{Var}[\hat{N}] \simeq \bar{N}/\ 2$. What is your best estimation for the number of atoms in this experiment?

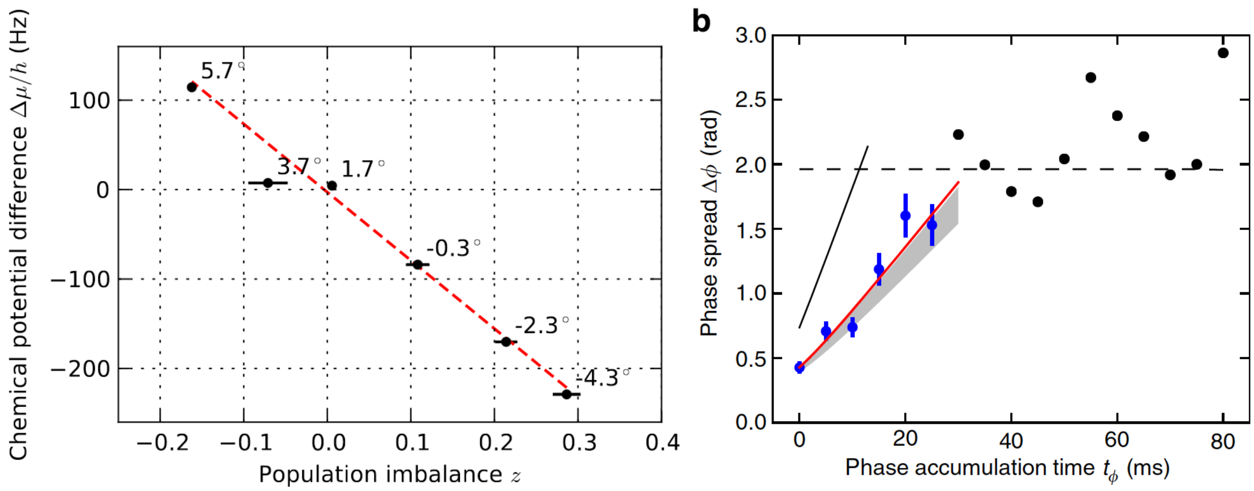


Figure 1: From Tarik Berrada et al. Nat. Comm. (2013).

2 Time correlation function and link with the PGP

In the previous section, with a one-mode model, we saw that in the presence of fluctuations in the number of particles (quantity preserved by the Hamiltonian evolution), the phase of the condensate spreads, with a variance $\mathrm{Var}[\hat{\theta}(t) - \hat{\theta}(0)]$ that grows in time.

In the following we will consider an initial state with fixed number of particles equal to N, and we will focus on non-zero temperature multimode effects. We will see that if the system is prepared in the *canonical ensemble*, the condensate phase spreading is still ballistic, due to the fluctuations of the total energy of the gas E (another conserved quantity). If, on the other hand, the system is prepared in the *micro canonical* ensemble with fixed N and E, the phase spreading becomes diffusive.

Instead of directly looking at $\mathrm{Var}[\hat{\theta}(t) - \hat{\theta}(0)]$ as in the previous section, we will study the time correlation function $g_1(t)$. The two expressions are related by the following relation, valid for a Gaussian distribution of $\hat{\theta}(t) - \hat{\theta}(0)$:

$$g_1(t) \simeq \langle \hat{n}_0 \rangle \langle e^{-i\hat{\theta}(t)} e^{i\hat{\theta}(0)} \rangle \simeq \langle \hat{n}_0 \rangle \langle e^{-i[\hat{\theta}(t) - \hat{\theta}(0)]} \rangle = \langle \hat{n}_0 \rangle e^{-\frac{1}{2}\mathrm{Var}[\hat{\theta}(t) - \hat{\theta}(0)]} \tag{14}$$

the demonstration of which we do not ask.

2.1 Time correlation function

At long times, the temporal coherence of the Bose gas is dominated by the contribution of the condensate. It is then described by the condensate time correlation function

$$g_1(t) = \langle a_0^\dagger(t) a_0(0) \rangle \tag{15}$$

where all the operators are in the Heisenberg picture.

14. Let $\hat{U}$ be a unitary operator and F a function that can be expanded in series. Prove the identity :

$$\hat{U} F(\hat{B}) \hat{U}^\dagger = F(\hat{U} \hat{B} \hat{U}^\dagger) \tag{16}$$

15. In the following, we will neglect the contribution of fluctuations of n_0 which are low in relative value. Show that by replacing $\hat{n}_0(t)$ by $\langle\hat{n}_0\rangle$ in (15) one obtains :

$$g_1(t) \simeq \langle\hat{n}_0\rangle\langle e^{-i\hat{\theta}(t)}e^{i\hat{\theta}(0)}\rangle \tag{17}$$

with $\hat{\theta}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{\theta}e^{-\frac{i}{\hbar}\hat{H}t}$ and $\hat{\theta}(0) = \hat{\theta}$.

16. By using twice the relation (16), show that we can rewrite the correlation function (17) in the form

$$g_1(t) \simeq \langle\hat{n}_0\rangle\langle e^{\frac{i}{\hbar}\hat{H}t}e^{-\frac{i}{\hbar}\hat{H}_\theta t}\rangle \tag{18}$$

where $\hat{H}$ is the system hamiltonian and where we introduced

$$\hat{H}_\theta \equiv e^{-i\hat{\theta}}\hat{H}e^{i\hat{\theta}} \tag{19}$$

2.2 Link with the PGP

In the most general case, which includes both the canonical and the micro canonical ensemble, it is assumed that the system is prepared in a statistical mixture of eigenstates of the Hamiltonian $\hat{H}$ for N-particles, described by the N-body density operator :

$$\hat{\sigma} = \sum_\lambda \Pi_\lambda|\psi_\lambda\rangle\langle\psi_\lambda| \tag{20}$$

where $\hat{H}|\psi_\lambda\rangle = E_\lambda|\psi_\lambda\rangle$ and E_λ is the total energy of the gas in the state $|\psi_\lambda\rangle$. It is therefore necessary first to calculate the correlation function in a state $|\psi_\lambda\rangle$.

17. Show that according to equation (18), one has

$$g_1^\lambda(t) \equiv \langle\psi_\lambda|a_0^\dagger(t)a_0(0)|\psi_\lambda\rangle \simeq \langle\hat{n}_0\rangle e^{\frac{i}{\hbar}E_\lambda t}\langle\psi_\lambda|e^{-\frac{i}{\hbar}\hat{H}_\theta t}|\psi_\lambda\rangle \tag{21}$$

One then writes $\hat{H}_\theta$ as a sum of the gas hamiltonian $\hat{H}$ plus a difference $\hat{W}$ that we will see being N times smaller than $\hat{H}$ at the limit of a large system

$$\hat{H}_\theta = \hat{H} + (\hat{H}_\theta - \hat{H}) \equiv \hat{H} + \hat{W} \tag{22}$$

We then recognize in (21) the probability amplitude for the system to remain in the state $|\psi_\lambda\rangle$, eigenstate of $\hat{H}$, after evolution with the "perturbed Hamiltonian" $\hat{H}_\theta$.

18. We want to calculate $g_1^\lambda(t)$ in (21) using the resolvent formalism and the projectors method. Of course we are speaking here about $G_\theta(z)$: the resolvent of the Hamiltonian $\hat{H}_\theta$. What is the correct choice for the projector P ?

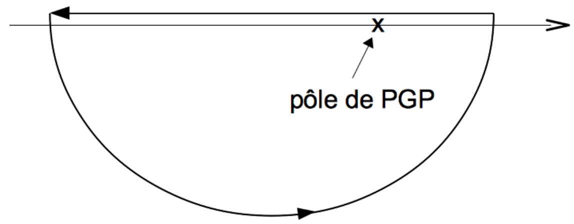
19. Recall the form of $PG_\theta(z)P$ in terms of the effective Hamiltonian $\hat{H}_{\text{eff}}(z)$, and give the form of $\hat{H}_{\text{eff}}(z)$ in terms of P, $Q = 1 - P$, $\hat{W}$ and $\hat{H}_\theta$.

20. Write $PG_\theta(z)P$ to the lowest order (non-zero) in $\hat{W}$

21. Recall the link between the resolvent and the evolution operator, and show that at this order

$$g_1^\lambda(t) \simeq \langle \hat{n}_0 \rangle e^{-\frac{i}{\hbar} W_\lambda t} \tag{23}$$

with $W_\lambda \equiv \langle \psi_\lambda | \hat{W} | \psi_\lambda \rangle$. One will use the Cauchy residue theorem, integrating in the complex plane along a closed contour as indicated below



2.3 Identification of $\hat{W}$

To obtain explicit results, we must give a physical meaning to the $\hat{W}$ operator. To this aim, we will identify $\hat{H}$ with the Bogoliubov Hamiltonian $H_{\rm Bog}(\hat{N}, \hat{\Lambda}, \hat{\Lambda}^\dagger)$, see the equation (27) below, which depends on the $\hat{\Lambda}$ and $\hat{\Lambda}^\dagger$ operators [1] and on the total number of atoms $\hat{N} = \hat{n}_0 + \hat{N}_\perp$ where $\hat{N}_\perp = \sum_{\mathbf{r}} b^3 \hat{\Lambda}^\dagger(\mathbf{r}) \hat{\Lambda}(\mathbf{r})$ is the number of particles in orthogonal to condensate modes.

22. We recall that $[\hat{\Lambda}(\mathbf{r}), \hat{\Lambda}^\dagger(\mathbf{r}')] = \frac{\delta_{\mathbf{r},\mathbf{r}'}}{b^3} - \frac{1}{V}$. Show that the operators $\hat{\Lambda}$ and $\hat{\Lambda}^\dagger$ conserve the total number of particles : $[\hat{N}, \hat{\Lambda}(\mathbf{r}')] = 0$. The two contributions $[\hat{N}_\perp, \hat{\Lambda}(\mathbf{r}')]$ and $[\hat{n}_0, \hat{\Lambda}(\mathbf{r}')]$, where we recall that $[\hat{n}_0, \hat{\theta}] = i$, can be calculated separately.

23. Justify the commutation relations $[\hat{\Lambda}^\dagger, \hat{\theta}] = 0$, $[\hat{\Lambda}, \hat{\theta}] = 0$, $[\hat{N}_\perp, \hat{\theta}] = 0$.

24. As we saw it in class, the number of particles in the condensate mode $\hat{n}_0$ is finally eliminated from the Bogoliubov theory, and $\hat{n}_0$ is replaced by $\hat{N} - \hat{N}_\perp$. Starting from $[\hat{n}_0, \hat{\theta}] = i$ deduce that $[\hat{N}, \hat{\theta}] = i$.

25. Show that

$$\hat{H}_\theta = e^{-i\hat{\theta}} \hat{H}(\hat{N}, \hat{\Lambda}, \hat{\Lambda}^\dagger) e^{i\hat{\theta}} = \hat{H}(\hat{N} - 1, \hat{\Lambda}, \hat{\Lambda}^\dagger) \tag{24}$$

26. Write the Heisenberg equation for $\hat{\theta}(t)$ and show that

$$i\hbar \frac{d\hat{\theta}(t)}{dt} = -i \frac{\partial \hat{H}(\hat{N}, \hat{\Lambda}, \hat{\Lambda}^\dagger)}{\partial N} |_{\Lambda, \Lambda^\dagger} \tag{25}$$

27. Show that to the leading order in $1/N$ one can identify :

$$\hat{W} = \hbar \frac{d\hat{\theta}(t)}{dt} \tag{26}$$

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[1] We recall that $\hat{\Lambda}(\mathbf{r}) = e^{-\hat{\theta}} \hat{\psi}_\perp(\mathbf{r})$, where $\hat{\psi}_\perp(\mathbf{r})$ is the projection of the field operator on the orthogonal modes to the condensate.

3 Microscopic expression of the phase derivative

3.1 Derivative of the phase in the Bogoliubov theory

In this section we will calculate the time derivative of the condensate phase using the Bogolubov Hamiltonian

$$H_{\rm Bog}(\hat{N}) = \frac{g_0 \hat{N}^2}{2V} + \sum_{\mathbf{r}} b^3 \left[\hat{\Lambda}^\dagger \left(h_0 + \frac{g_0 \hat{N}}{V} \right) \hat{\Lambda} + \frac{g_0 \hat{N}}{2V} \left(\hat{\Lambda}^2 + \hat{\Lambda}^{\dagger\, 2} \right) \right] \tag{27}$$

where h_0 is the kinetic energy, with eigenvalues $\hbar^2 k^2/2m$ over plane waves. We recall the expansion of the fields $\hat{\Lambda}$ and $\hat{\Lambda}^\dagger$ over the eigenmodes of the linearized equations of motion :

$$\begin{pmatrix} \hat{\Lambda}(\mathbf{r}) \\ \hat{\Lambda}^\dagger(\mathbf{r}) \end{pmatrix} = \sum_{\mathbf{k}\neq\mathbf{0}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}} \left[\begin{pmatrix} U_k \\ V_k \end{pmatrix} \hat{b}_{\mathbf{k}} + \begin{pmatrix} V_k \\ U_k \end{pmatrix} \hat{b}^\dagger_{-\mathbf{k}} \right] \tag{28}$$

28. By using (25) and (27), explicitly write the derivative of the phase in terms of $\hat{N}$, $\hat{\Lambda}$ and $\hat{\Lambda}^\dagger$.

29. We remind the orthogonality relation for plane waves on the lattice :

$$b^3 \sum_{\mathbf{r}} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = V\, \delta_{\mathbf{k},\mathbf{k}'} \tag{29}$$

Calculate $\sum_{\mathbf{r}} b^3 \hat{\Lambda}^\dagger(\mathbf{r})\hat{\Lambda}(\mathbf{r})$ and $\sum_{\mathbf{r}} b^3 \hat{\Lambda}^2(\mathbf{r})$ and express the result a simple sum over $\mathbf{k}$.

30. Show that by neglecting the terms $b_{\mathbf{k}} b_{-\mathbf{k}}$ and $b^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}}$ (which are rapidly oscillating in the Heisenberg picture), we obtain the simple result:

$$\frac{d\hat{\theta}}{dt} = -\frac{\mu_0(\hat{N})}{\hbar} - \frac{g_0}{\hbar V} \sum_{\mathbf{k}\neq\mathbf{0}} (U_k + V_k)^2 \hat{n}_{\mathbf{k}} \tag{30}$$

with $\hat{n}_{\mathbf{k}} = \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}$ and where we introduced the zero temperature chemical potential

$$\mu_0(\hat{N}) = \frac{g_0}{V} \left[\hat{N} + \sum_{\mathbf{k}\neq\mathbf{0}} V_k (U_k + V_k) \right] \tag{31}$$

31. From the expressions of the Bogoliubov modes and the eigenenergy ϵ_k [2]

$$U_k \pm V_k = \left(\frac{E_k}{\epsilon_k} \right)^{\pm 1/2} \; ; \quad \epsilon_k = \sqrt{E_k(E_k + 2\rho g_0)} \; ; \quad E_k = \frac{\hbar^2 k^2}{2m} \; ; \quad \rho = \frac{N}{V} \tag{32}$$

show that one indeed has $\mu_0(N) = dE_0(N)/dN$ where $E_0(N)$ is the energy of the ground state in the Bogoliubov approximation

$$E_0(N) = \frac{g_0 N^2}{2V} - \sum_{\mathbf{k}\neq\mathbf{0}} \epsilon_k V_k^2 \tag{33}$$

[2] Careful: the notations used in the tutorials and in the lectures for ϵ_k and E_k are not the same. Here we use the lectures notations.

One can first show that $\epsilon_k V_k^2 = \frac{1}{2}(E_k + \rho g_0 - \epsilon_k)$ and that $V_k(U_k + V_k) = \frac{1}{2}\left(\frac{E_k}{\epsilon_k} - 1\right)$.

32. Again starting from the expressions (32), show that one can rewrite the equation (30) in the form

$$-\hbar\frac{d\hat{\theta}}{dt} = \mu_0(\hat{N}) + \sum_{\mathbf{k}\neq\mathbf{0}} \frac{\partial\epsilon_k}{\partial N}\hat{n}_{\mathbf{k}} \tag{34}$$

3.2 Physical interpretation

We want to show that the second term on the right-hand side of (34) represents the contribution of thermally excited modes to the chemical potential. Let's place ourselves in the canonical ensemble with N atoms :

$$\hat{\sigma}_{\text{can}} = \frac{e^{-\beta\hat{H}_{\text{Bog}}}}{Z} \qquad \text{with} \qquad \hat{H}_{\text{Bog}} = E_0(N) + \sum_{\mathbf{k}\neq\mathbf{0}} \epsilon_k \hat{n}_{\mathbf{k}} \tag{35}$$

The free energy of the system is given by the ground state energy of the gas E_0 plus the free energy of Bogoliubov quasi-particles :

$$F = E_0(N) + k_B T \sum_{\mathbf{k}} \ln(1 - e^{-\beta\epsilon_k}) \tag{36}$$

33. At the order of Bogoliubov, the quasi-particles form an ideal gas of bosons. Recall the expression of the average occupation numbers $\bar{n}_{\mathbf{k}} = \langle\hat{n}_{\mathbf{k}}\rangle$ of Bogoliubov quasi-particles as a function of ϵ_k and of $\beta = 1/(k_B T)$.

34. Express the gas chemical potential $\mu_{\text{can}} = \left(\frac{dF}{dN}\right)_{V,T}$ and show that one finds the average value in the canonical ensemble of the operator $-\hbar\frac{d\hat{\theta}}{dt}$ given by the equation (34).

35. We will from now identify the right-hand side of the equation (34) with a "chemical potential operator":

$$\frac{d\hat{\theta}}{dt} = -\frac{\hat{\mu}}{\hbar} \tag{37}$$

Make the connection (similarities and differences) between the equation (37) and the second Josephson's equation for a superconductor.

4 Ballistic phase spreading for a system prepared in the canonical ensemble

A first mechanism that reduces the condensate coherence time, that is that causes a decay of the correlation function $g_1(t)$, comes in when taking the average of the dominant term of $g_1^\lambda(t)$, given by the equation (23), over the distribution (20).

4.1 Fluctuations of energy in the canonical ensemble

We will now consider a system initially prepared in the canonical ensemble, with N atoms and $\mathrm{Var}[\hat{N}] = 0$. We will calculate the coefficient $A = A_{\mathrm{can}}$ giving the ballistic spreading of the condensate phase

$$\mathrm{Var}[\hat{\theta}(t) - \hat{\theta}(0)] = A_{\mathrm{can}} t^2 \tag{38}$$

in this case. To start, we use the "eigenstate thermalization hypothesis'", which you will not try to prove, which says that in a N-body system that is "quantum ergodic", the expectation value of an operator in an eigenstate ψ_λ of the Hamiltonian, is well approximated by the micro canonical average at the energy E_λ of this operator. Applying this result to the operator $\hat{W}$, we have

$$W_\lambda \equiv \langle \psi_\lambda | \hat{W} | \psi_\lambda \rangle \simeq W_{\mathrm{mc}}(E_\lambda, N_\lambda) = -\mu_{\mathrm{mc}}(E_\lambda, N_\lambda) \tag{39}$$

and hence, for $N_\lambda = N$,

$$g_1^\lambda(t) \simeq \langle \hat{n}_0 \rangle e^{\frac{i}{\hbar}\mu_{\mathrm{mc}}(E_\lambda, N) t} \tag{40}$$

Since E_λ fluctuates over the distribution (20), a decay of $g_1(t)$ occurs. For Gaussian fluctuations of energy, which is the case in the canonical ensemble for a large system, we have

$$|g_1^{\mathrm{can}}(t)| = \langle \hat{n}_0 \rangle e^{-\frac{1}{2} A_{\mathrm{can}} t^2} \tag{41}$$

with $A_{\mathrm{can}} = \left(\frac{1}{\hbar}\, \partial_E \mu_{\mathrm{mc}}(E, N)|_{E=\bar{E}}\right)^2 \mathrm{Var} E$.

36. The first step in calculating A_{can} is to reduce it to *canonical* quantities, instead of micro canonical ones. Starting from the fact that, at the dominant order in $1/N$, one can identify the canonical chemical potential at the temperature T with the micro canonical chemical potential at an energy $E_{\mathrm{can}} \equiv \langle \hat{H} \rangle_{\mathrm{can}}$, corresponding to the mean energy in the canonical ensemble :

$$\mu_{\mathrm{can}}(T, N) \simeq \mu_{\mathrm{mc}}(E_{\mathrm{can}}(T, N), N) \tag{42}$$

express $\partial_E \mu_{\mathrm{mc}}$ as a function of $\partial_T \mu_{\mathrm{can}}$ and of $\partial_T E_{\mathrm{can}}$

37. By using the Bogoliubov Hamiltonian

$$H_{\mathrm{Bog}} = E_0 + \sum_{\mathbf{k} \neq \mathbf{0}} \epsilon_k \hat{n}_{\mathbf{k}} \tag{43}$$

show that one obtains

$$A_{\mathrm{can}} = \left(\frac{g_0}{\hbar V}\right)^2 \frac{\left(\sum_{\mathbf{k}} (U_k + V_k)^2 \epsilon_k \bar{n}_k (\bar{n}_k + 1)\right)^2}{\sum_{\mathbf{k}} \epsilon_k^2 \bar{n}_k (\bar{n}_k + 1)} \tag{44}$$

where $\bar{n}_k$ are the average occupation numbers of the Bogoliubov modes in the canonical ensemble.

5 Phase diffusion

If the system is prepared in the micro canonical ensemble, with $\mathrm{Var}[\hat{N}] = 0$ and $\mathrm{Var}[\hat{E}] = 0$, the coefficient A of ballistic spreading of the phase is zero. To find the coherence time of the condensate in this case, it is necessary to resume the calculation of the section 2.2, and include the next order in $\hat{W}$.

38. We will limit to second order in $\hat{W}$. Show that in the limit of a continuum spectrum and within the pole approximation, the effective hamiltonian $\hat{H}_{\mathrm{eff}}(E + i\eta)$ takes the form
$$\hat{H}_{\mathrm{eff}}^{\mathrm{pole}} = P\hat{H}_{\theta}P + \hbar\Delta_{\lambda} - \frac{i\hbar}{2}\Gamma_{\lambda} \tag{45}$$

39. In this case show that
$$g_1^{\lambda}(t) \simeq \langle \hat{n}_0 \rangle e^{-\frac{i}{\hbar}W_{\lambda}t} e^{-i\Delta_{\lambda}t} e^{-\frac{\Gamma_{\lambda}t}{2}} \tag{46}$$
so that
$$|g_1^{\lambda}(t)| \simeq \langle \hat{n}_0 \rangle e^{-\frac{\Gamma_{\lambda}t}{2}} \tag{47}$$

40. Give a physical interpretation to Δ_{λ} and Γ_{λ}.

41. For a quasi continuum, give the expression Γ_{λ} in terms of a sum over the eigenstates ψ_{μ} of $\hat{H}$, with $\mu \neq \lambda$, and recognize a Fermi golden rule.

42. Using the relation (14), deduce that the exponential damping of the correlation function described by the equation (47) corresponds to a diffusive spreading of the phase of the condensate, reminding of one-dimensional Brownian motion, with a diffusion coefficient D_{λ} that is related to Γ_{λ}.

43. Show that the diffusion coefficient of the phase D_{λ} is simply linked to the correlation function of $d\hat{\theta}/dt$:
$$D_{\lambda} = \mathrm{Re}\left\{ \int_0^{+\infty} d\tau \left[\left\langle \frac{d\hat{\theta}(\tau)}{dt}\frac{d\hat{\theta}(0)}{dt} \right\rangle_{\lambda} - \left\langle \frac{d\hat{\theta}}{dt} \right\rangle_{\lambda}^2 \right] \right\} \tag{48}$$
To this aim, one will use the equation (26) and one will insert in the expression (48) a closure relation over the eigenstates of the Hamiltonian $\hat{H}$.