Quantum field theory exam

M1 2010-2011

Friday January 28, 2011

The exam is made of an exercise and a problem. They are independent. The different parts of the problem are related. *However, all the results needed to tackle a part are given explicitly in the text.* So don't hesitate to start a new part in case you are stuck.

Questions marked with a $^\flat$ are not used anywhere in the sequel.

At the end you'll find a glossary of useful formulæ. Do not hesitate to read it once before you start the problem and refer to it thereafter.

Exercice : Projectors et normal ordered products.

Let $\mathcal{F}$ be the Fock space associated to a pair of operators $\hat{a}, \hat{a}^\dagger$ such that $[\hat{a}, \hat{a}^\dagger] = 1$. This space has an orthonormal basis $\{|\,n\,\rangle; n = 0, 1, \cdots\}$ such that $\hat{a}|\,0\,\rangle = 0$, $\hat{a}|\,n\,\rangle = \sqrt{n}|\,n-1\,\rangle$ for $n = 1, 2, \cdots$ and $\hat{a}^\dagger|\,n\,\rangle = \sqrt{n+1}|\,n+1\,\rangle$ for $a = 0, 1, \cdots$.

Les a^* and a be commuting variables. The normal ordering of a function f of the variables a^* and a with a series expansion

$$f = \sum_{m,n=0,1,\cdots} f_{m,n} (a^*)^m a^n$$

is defined to be

$$\boldsymbol{:} f \boldsymbol{:} \equiv \sum_{m,n=0,1,\cdots} f_{m,n} (\hat{a}^\dagger)^m \hat{a}^n.$$

I.1 : For $m = 0, 1, \cdots$, give the expansion of $\boldsymbol{:} e^{\lambda a^* a} \boldsymbol{:} |\,m\,\rangle$ on the states $|\,n\,\rangle$.

I.2 : Conclude that $\boldsymbol{:} e^{-a^* a} \boldsymbol{:} = |\,0\,\rangle\langle\,0\,|$, the orthogonal projector on the state $|\,0\,\rangle$.

We have the following generalization.

I.3 : Expanding the result of the first question around $\lambda = -1$, show that $\boldsymbol{:} \frac{(a^* a)^m}{m!} e^{-a^* a} \boldsymbol{:} = |\,m\,\rangle\langle\,m\,|$, the orthogonal projector on the state $|\,m\,\rangle$.

Problem :

A general result in quantum field theory is the so-called Kählen-Lehmann representation theorem, which reads as follows.

– If $\hat{\phi}$ is a real scalar field in a minkovskian Poincaré invariant quantum field theory with vacuum $|\,\Omega\,\rangle$ on space time [1] $\mathbb{R}^d$, its $2-$point function decomposes as a positive linear combination of the real scalar free field $2-$point function. More accurately, there exists a function $\rho \geqslant 0$ such that :

$$\langle\,\Omega\,|T\hat{\phi}(x)\hat{\phi}(y)|\,\Omega\,\rangle = \int_0^{+\infty} d\mu^2\, \rho(\mu^2)\Delta_F(x-y,\mu^2)$$

The function ρ, called the spectral function, depends on the quantum field theory and the field Φ at hand.

We recall that $\Delta_F(x,\mu^2)$ is the Feynman propagator in dimension d,

$$\Delta_F(x,\mu^2) \equiv \int \frac{d^dk}{(2\pi)^d}\frac{ie^{ikx}}{k^2-\mu^2+i0^+}.$$

In this formula, kx and k^2 stand for Minkovski space scalar products.

The Kählen-Lehmann representation theorem gives no explicit information on the spectral function. The goal in this problem is to compute it explicitly in a simple situation and use the result to tackle a renormalization problem.

Part A : Preliminaries.

Let $\hat{\varphi}(x)$, $x=(x^0,\cdots,x^D)\in\mathbb{R}^d$ be an operator acting on a Hilbert space $\mathcal{H}$ containing a vector $|\,\Omega\,\rangle$, such that the following properties hold : one can decompose $\hat{\varphi}(x)$ as $\hat{\varphi}(x)=\hat{\varphi}_+(x)+\hat{\varphi}_-(x)$ with

– $\hat{\varphi}_-(x)|\,\Omega\,\rangle=0$ and $\langle\,\Omega\,|\hat{\varphi}_+(x)=0$,

– $[\hat{\varphi}_-(x),\hat{\varphi}_+(y)]$ is proportional to the identity operator Id (and henceforth commutes with any operator acting on $\mathcal{H}$).

Set $\hat{\phi}(x)\equiv\hat{\varphi}_+(x)^2+2\hat{\varphi}_+(x)\hat{\varphi}_-(x)+\hat{\varphi}_-(x)^2$.

A.1 : $^\flat$ Recall briefly why the two properties above hold if $\hat{\varphi}(x)$ is a real scalar free field (in the Heisenberg picture) on $\mathbb{R}^d$ with mass m

1. We consider in the problem a generic dimension $d = D + 1$. Apart from this slight generalization, the definitions and notations that follows should be familiar. For the record, $D \simeq 3$ in the real world.

acting on the Fock space. What is the interpretation of $\hat{\phi}(x)$ in that case ? ✎

A.2 : Show that

$$[\hat{\varphi}_-(x), \hat{\varphi}_+(y)] = \langle\, \Omega\, |\hat{\varphi}_-(x)\hat{\varphi}_+(y)|\, \Omega\,\rangle\, \mathrm{Id}$$

and

$$\langle\, \Omega\, |\hat{\varphi}(x)\hat{\varphi}(y)|\, \Omega\,\rangle = \langle\, \Omega\, |\hat{\varphi}_-(x)\hat{\varphi}_+(y)|\, \Omega\,\rangle$$

✎

A.3 : Show that

$$\langle\, \Omega\, |\hat{\phi}(x)\hat{\phi}(y)|\, \Omega\,\rangle = \langle\, \Omega\, |\hat{\varphi}_-(x)^2\hat{\varphi}_+(y)^2|\, \Omega\,\rangle$$

✎

A.4 : Use the commutation relation to derive that

$$\langle\, \Omega\, |\hat{\phi}(x)\hat{\phi}(y)|\, \Omega\,\rangle = 2\langle\, \Omega\, |\hat{\varphi}_-(x)\hat{\varphi}_+(y)|\, \Omega\,\rangle^2.$$

✎

A.5 : Conclude from the previous questions that

$$\langle\, \Omega\, |T\hat{\phi}(x)\hat{\phi}(y)|\, \Omega\,\rangle = 2\langle\, \Omega\, |T\hat{\varphi}(x)\hat{\varphi}(y)|\, \Omega\,\rangle^2.$$

Distinguish the cases $x^0 > y^0$ and $x^0 < y^0$. ✎

A.6 : ♭ In case $\hat{\varphi}$ is a real free scalar field, compare the previous result to $\langle\, \Omega\, |T\hat{\varphi}(x)^2\hat{\varphi}(y)^2|\, \Omega\,\rangle$ given by application of Wick's theorem. Discuss the discrepancy. ✎

In the rest of the problem, we assume moreover that the 2−point function of $\hat{\varphi}$ is that of a real scalar field of mass m, i.e.

$$\langle\, \Omega\, |T\hat{\varphi}(x)\hat{\varphi}(y)|\, \Omega\,\rangle = \Delta_F(x-y, m^2).$$

So the Kählen-Lehmann representation theorem guaranties the existence of a spectral function ρ such that

$$2\Delta_F(x-y, m^2)^2 = \int_0^{+\infty} d\mu^2\, \rho(\mu^2)\Delta_F(x-y, \mu^2),$$

and the goal is to give an explicit formula for ρ.

Both sides depend only on $x-y$ (translation invariance), so there is no loss of generality in taking $y = 0$. After a Wick rotation one is led to write

$$2\Delta_E(x, m^2)^2 = \int_0^{+\infty} d\,\mu^2\rho(\mu^2)\Delta_E(x, \mu^2), \tag{1}$$

where $\Delta_E(x, \mu^2)$ is the euclidean propagator

$$\Delta_E(x, \mu^2) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{e^{ikx}}{k^2 + \mu^2}. \tag{2}$$

In this formula, kx and k^2 stand for euclidean space scalar products.

Part B : Identities for the euclidean propagator

B.1 : Starting from formula (2) for the euclidean propagator, show that $\Delta_E(x, m^2)^2 = \int \frac{d^d k}{(2\pi)^d} e^{ikx} I(k, m^2)$ where

$$I(k, m^2) \equiv \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} \frac{1}{(k-q)^2 + m^2}.$$

✎

B.2 : Using the Kählen-Lehmann representation (1) infer that

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} \frac{1}{(k-q)^2 + m^2} = \frac{1}{2} \int_0^{+\infty} d\,\mu^2 \rho(\mu^2) \frac{1}{k^2 + \mu^2}. \tag{3}$$

For which values of the space-time dimension d is the integral on the left-hand side convergent? ✎

B.3 : Starting from formula (2) for the euclidean propagator, show, via the representation (10) of $(k^2 + m^2)^{-1}$ and the computation of the resulting gaussian integral of dimension d, that

$$\Delta_E(x, m^2) = \int_0^{+\infty} dw \, \frac{1}{(4\pi w)^{d/2}} e^{-m^2 w - x^2/(4w)}.$$

Infer that

$$\Delta_E(x, m^2) = \int_0^{+\infty} dt \, \frac{t^{d/2-2}}{(4\pi)^{d/2}} e^{-m^2/t - tx^2/4}. \tag{4}$$

and that

$$\Delta_E(x, m^2)^2 = \frac{1}{(4\pi)^d} \int_0^{+\infty} du dv \, (uv)^{d/2-2} e^{-m^2(u+v)/(uv) - (u+v)x^2/4}.$$

✎

We insert in this formula the identity $1 = \int_0^{+\infty} dt \, \delta(t - u - v)$ and then, for fixed t, make the change of variables $u = t\alpha$, $v = t\beta$. Note that a priori $\alpha, \beta \in [0, +\infty[$ but, due to the δ function, their effective domain of variation is smaller.

B.4 : Check that this leads to

$$2\Delta_E(x, m^2)^2 = \int_0^{+\infty} dt\, e^{-tx^2/4}\, L(t)$$

with

$$L(t) \equiv \frac{2t^{d-3}}{(4\pi)^d} \int_0^1 d\alpha\, (\alpha(1-\alpha))^{d/2-2} e^{-m^2/(t\alpha(1-\alpha))}.$$

✎

B.5 : Using formula (4) of question B.3 (with μ^2 instead of m^2) check that

$$\int_0^{+\infty} d\mu^2 \rho(\mu^2) \Delta_E(x, \mu^2) = \int_0^{+\infty} dt\, e^{-tx^2/4}\, R(t)$$

with

$$R(t) \equiv \frac{t^{d/2-2}}{(4\pi)^{d/2}} \int_0^{+\infty} d\mu^2\, \rho(\mu^2) e^{-\mu^2/t}.$$

✎

So the Kählen-Lehmann representation reads :

$$\int_0^{+\infty} dt\, e^{-tx^2/4} L(t) = \int_0^{+\infty} dt\, e^{-tx^2/4}\, R(t).$$

B.6 : Interpreting this identity as an equality of Laplace-Fourier transforms, conclude that for all $t \geqslant 0$

$$\frac{2t^{d/2-1}}{(4\pi)^{d/2}} \int_0^1 d\alpha\, (\alpha(1-\alpha))^{d/2-2} e^{-m^2/(t\alpha(1-\alpha))} = \int_0^{+\infty} d\mu^2\, \rho(\mu^2) e^{-\mu^2/t}. \quad (5)$$

✎

Part C : Computation of the spectral density

This part exploits the relation (5) to compute ρ. *We assume that* $d > 2$.

C.1 : Using the identity (9) for $a = 1/t$ and $s = d/2 - 1$, show that the left-hand side of relation (5) reads

$$\frac{2}{(4\pi)^{d/2}\Gamma(d/2-1)} \int_0^{+\infty} dw \int_0^1 d\alpha\, (w\alpha(1-\alpha))^{d/2-2} e^{-m^2/(t\alpha(1-\alpha))-w/t}.$$

✎

C.2 : Inserting in this expression the identity $1 = \int_0^{+\infty} d\mu^2\, \delta(\mu^2 - m^2/(\alpha(1-\alpha)) - w)$ and computing the w integral, check that the left-hand side of (5) can be rewritten as

$$\int_0^{+\infty} d\mu^2 \left(\frac{2}{(4\pi)^{d/2}\Gamma(d/2-1)} \int_0^1 d\alpha\, \frac{H(\mu^2\alpha(1-\alpha) - m^2)}{(\mu^2\alpha(1-\alpha) - m^2)^{2-d/2}} \right) e^{-\mu^2/t},$$

where H stands for the Heaviside step function, taking value 1 for positive values of the argument and 0 for negative values of the argument. ✎

C.3 : Interpreting now (5) with the help of the above formula as an equality of Laplace-Fourier transforms, conclude that

$$\rho(\mu^2) = \frac{2}{(4\pi)^{d/2}\Gamma(d/2-1)} \int_0^1 d\alpha\, \frac{H(\mu^2\alpha(1-\alpha) - m^2)}{(\mu^2\alpha(1-\alpha) - m^2)^{2-d/2}}.$$

✎

Setting $\alpha = (1+\gamma)/2$ leads to

$$\rho(\mu^2) = \frac{1}{(4\pi)^{d/2}\Gamma(d/2-1)} \int_{-1}^1 d\gamma\, \frac{H((\mu^2 - 4m^2) - \mu^2\gamma^2)4^{2-d/2}}{((\mu^2 - 4m^2) - \mu^2\gamma^2)^{2-d/2}}.$$

C.4 : By a rescaling of the integration variable, show that

$$\rho(\mu^2) \propto \frac{H(\mu^2 - 4m^2)}{\mu(\mu^2 - 4m^2)^{(3-d)/2}}. \tag{6}$$

✎

C.5 : The spectral function $\rho(\mu^2)$ vanishes below a certain threshold. Give an interpretation. ✎

C.6 : Compute the proportionality constant in (6) [2]. Check that for $d = 6$

$$\rho(\mu^2) = \frac{1}{192\pi^3} \frac{H(\mu^2 - 4m^2)}{\mu(\mu^2 - 4m^2)^{-3/2}}. \tag{7}$$

✎

Part D : "Graphology" for the ψ^3 theory in $d = 6$ dimensions.

The answers to questions in this part are mild adaptations of results established in the course. You are not expected to give detailed

2. Don't forget the glossary.

justifications. Use the analogy with the case of the ϕ^4 theory in dimension $d = 4$.

We consider a field theory with action

$$S(\psi) = \int d^6x \left(\frac{1}{2}(\partial_\mu \psi)(\partial^\mu \psi) - \frac{1}{2} m^2 \psi^2 - \frac{\lambda}{3!}\psi^3 \right).$$

for a real scalar field ψ.

D.1 : $^\flat$ What is the dimension of the coupling constant (the action is taken to be a dimensionless quantity)? ✎

Following the usual quantization procedure, one is led to a quantum theory for a field $\hat\psi$. Correlation functions of $\hat\psi$ are expressed as usual in terms of correlation functions of a free field $\hat\varphi$ in the Heisenberg representation. For instance, for the 2−point function, one has

$$\langle\, \Omega\, | T\hat\psi(x)\hat\psi(y) |\, \Omega\, \rangle = \frac{\langle\, 0\, | T\hat\varphi(x)\hat\varphi(y) \exp(-i\frac{\lambda}{3!}\int d^6z\, \hat\varphi(z)^3) |\, 0\, \rangle}{\langle\, 0\, | T \exp(-i\frac{\lambda}{3!}\int d^6z\, \hat\varphi(z)^3) |\, 0\, \rangle}$$

where $|\, 0\, \rangle$ is the Fock space vacuum for $\hat\varphi$. As usual, one can use Wick's theorem to get a series expansion in powers of the coupling constantλ, and reorganize the expansion as a sum of contributions associated to graphs.

D.2 : Which Feynman graphs contribute to

$$\langle\, 0\, | T\hat\varphi(x)\hat\varphi(y) \exp(-i\frac{\lambda}{3!}\int d^6z\, \hat\varphi(z)^3) |\, 0\, \rangle$$

at order n in λ? ✎

D.3 : Which Feynman graphs contribute to

$$\langle\, \Omega\, | T\hat\psi(x)\hat\psi(y) |\, \Omega\, \rangle$$

at order n in λ? ✎

D.4 : Which Feynman graphs contribute to

$$\langle\, \Omega\, | T\hat\psi(x)\hat\psi(y) |\, \Omega\, \rangle_c \equiv \langle\, \Omega\, | T\hat\psi(x)\hat\psi(y) |\, \Omega\, \rangle - \langle\, \Omega\, | T\hat\psi(x) |\, \Omega\, \rangle \langle\, \Omega\, | T\hat\psi(y) |\, \Omega\, \rangle$$

at order n in λ? ✎

Using translation invariance, the 2−point function depends only on $x - y$ and we define

$$G_F(k) \equiv \int d^6x \, e^{-ikx} \langle \Omega | T\hat{\psi}(x)\hat{\psi}(0) | \Omega \rangle_c .$$

D.5 : Check that to second order in λ,

$$\begin{aligned} G_F(k) &= \frac{i}{k^2 - m^2 + i0^+} \\ &- \frac{\lambda^2}{2} \frac{i}{k^2 - m^2 + i0^+} I_F(k, m^2) \frac{i}{k^2 - m^2 + i0^+} \\ &+ O(\lambda^4). \end{aligned}$$

where

$$I_F(k, m^2) \equiv \int \frac{d^dq}{(2\pi)^d} \frac{i}{q^2 - m^2 + i0^+} \frac{i}{(k-q)^2 - m^2 + i0^+}$$

Interpret the λ^2 terms as the weight of a graph via Feynman rules. ✎

It is convenient to switch to the euclidean version. Remember that in the Wick rotation the Feynman propagator is replaced by the euclidean propagator . To take into account the Wick rotation $z^0 \to -iz^6$ also in the coupling term $\exp(-i\frac{\lambda}{3!}\int d^6z \, \hat{\varphi}(z)^3)$, the coupling constant $i\lambda$ is replaced by λ.

The euclidean version G of G_F is henceforth given, up to second order in λ, by

$$G(k) = \frac{1}{k^2 + m^2} + \frac{\lambda^2}{2} \frac{1}{k^2 + m^2} I(k, m^2) \frac{1}{k^2 + m^2} + O(\lambda^4).$$

where $I(k, m^2)$ is the function introduced in question B.1 (specialized to $d = 6$ of course).

The intuitive interpretation of the Feynman graph giving the λ^2 contribution is that a particle of momentum k disintegrates in 2 particles which collide again to reform one particle.

D.6 : Show that the Kählen-Lehmann representation (3) gives another intuitive interpretation of this process. ✎

D.7 : In general, which Feynman graphs contribute to $1/G(k)$? Give the expansion of $1/G(k)$ up to order 2 in λ. ✎

Part E : One loop renormalization.

The starting point is :

$$\begin{aligned} 1/G(k) &= k^2+m^2-\frac{\lambda^2}{2}I(k,m^2)+O(\lambda^4) \\ &= k^2+m^2-\frac{\lambda^2}{4}\int_0^{+\infty} d\,\mu^2\rho(\mu^2)\frac{1}{k^2+\mu^2}+O(\lambda^4). \end{aligned}$$

The function

$$I(k,m^2)=\int\frac{d^6q}{(2\pi)^6}\frac{1}{q^2+m^2}\frac{1}{(k-q)^2+m^2}$$

is in fact ill-defined.

E.1 : What type of divergence plagues the above expression? Show that the Kählen-Lehmann representation (3) of $I(k,m^2)$, where the spectral density ρ is given explicitly in (7), exhibits the same type of divergences, coming from large values of μ^2. ✎

Because of these divergences, the above computations have to be reinterpreted. We assume that in the "real" microscopic theory the exchange of very heavy particles is suppressed so that at low energies, the "true" microscopic spectral density r is close to ρ, but at high energies , r is small enough for

$$\int_0^{+\infty} d\,\mu^2 r(\mu^2)\frac{1}{k^2+\mu^2}$$

to be convergent. We want to infer the low energy physics of the theory, where only the properties of the 2−point function at "small" k are accessible.

In the free theory, i.e. at order 0 in λ, $1/G(k)=m^2+k^2$. In particular, m^2 is simply the value of $1/G(k)$ at $k^2=0$ and the derivative of $1/G(k)$ with respect to k^2 is simply 1.

In the interacting theory, nothing of this remains. However, changing the normalization of the field does not really change the physics. One can replace ψ with $\psi_r=Z^{-1/2}\psi$. This amounts to consider $G_r(k)\equiv G(k)/Z$ to which we could impose to resemble the free propagator, for instance by imposing that, for k^2 close to a certain fixed reference energy scale, denoted by K^2, the relation $1/G_r(k)-1/G_r(K)\sim k^2-K^2$ holds. One can also define a mass m_r by the condition $m_r^2 = 1/G_r(0)$. Note that the conditions are imposed directly on the correlation function, which is in principle measurable in experiments.

E.2 : Starting from the formula

$$1/G(k) = k^2 + m^2 - \frac{\lambda^2}{4}\int_0^{+\infty} d\,\mu^2 r(\mu^2)\frac{1}{k^2+\mu^2} + O(\lambda^4),$$

show that the normalization conditions are :

$$Z = 1 - \frac{\lambda^2}{4}\int_0^{+\infty} d\mu^2\, r(\mu^2)\frac{1}{(K^2+\mu^2)^2} + O(\lambda^4),$$

and

$$m_r^2 = m^2 - \frac{\lambda^2}{4}\int_0^{+\infty} d\mu^2\, r(\mu^2)\left(\frac{1}{\mu^2} + \frac{m^2}{(K^2+\mu^2)^2}\right) + O(\lambda^4).$$

✎

This last relation can be inverted (in perturbation theory) to get m^2 as a function of m_r^2.

In a more complete treatment, one would also have to define a coupling constant λ_r adapted to the low energy physics by some physical condition. The output would be that $\lambda_r = \lambda + O(\lambda^3)$.

E.3 : Show that, as a function of m_r and λ_r,

$$1/G_r(k) = k^2 + m_r^2 - \frac{\lambda_r^2}{4}\int_0^{+\infty} d\mu^2\, r(\mu^2)\left(\frac{1}{k^2+\mu^2} - \frac{1}{\mu^2} + \frac{k^2}{(K^2+\mu^2)^2}\right) + O(\lambda_r^4).$$

✎

E.4 : How does

$$\frac{1}{k^2+\mu^2} - \frac{1}{\mu^2} + \frac{k^2}{(K^2+\mu^2)^2}$$

behave for large μ^2 ? ✎

E.5 : Conclude that, if ρ is the spectral function given explicitly in (7) (after substitution of m_r for m), the integral

$$\int_0^{+\infty} d\mu^2\, \rho(\mu^2)\left(\frac{1}{k^2+\mu^2} - \frac{1}{\mu^2} + \frac{k^2}{(K^2+\mu^2)^2}\right)$$

is convergent. ✎

Assume that $\rho(\mu^2)$ (after substitution of m_r for m) and $r(\mu^2)$ are appreciably different only at energies larger than some energy scale of order Λ^2.

E.6 : Infer, using naïve dimensional analysis, that for $k^2, K^2, m_r^2 << \Lambda^2$ (the low energy window) one has

$$1/G_r(k) = k^2 + m_r^2 - \frac{\lambda_r^2}{4}\int_0^{+\infty} d\mu^2\, \rho(\mu^2)\left(\frac{1}{k^2+\mu^2} - \frac{1}{\mu^2} + \frac{k^2}{(K^2+\mu^2)^2}\right) + O(\lambda_r^4),$$

up to terms of order[3] Λ^{-2}. ✎

In the low energy window one can henceforth take the limit $\Lambda^2 \to \infty$ and get a renormalized low energy theory which is finite and independent of the details of the "real" microscopic theory.

We apply this result to the case were the low energy theory is massless, i.e. $m_r = 0$.

E.7 : $^\flat$ Compute explicitly $1/G_r(k)$ (in the limit $\Lambda^2 \to \infty$) for $m_r = 0$ t order 2 in λ_r. What happens for $K^2 \to 0$? ✎

3. A more careful computation would lead to the appearance of $\Lambda^{-2}\log\Lambda$ corrections which do not change at all the discussion.

A glossary of concepts and formulæ

– The one dimensional gaussian integral :

$$\int_{-\infty}^{+\infty} \frac{du}{\sqrt{\pi}} e^{-u^2+iuv} = e^{-v^2/4}.$$

– the Dirac δ function :

If f is a differentiable function,

$$\delta(f(w)) = \sum_{a, f(a)=0} \frac{1}{|f'(a)|}\delta(w-a).$$

In the sum over a, a runs over the zeroes of f.

– Laplace-Fourier transform :

If f is a continuous function defined for $w > 0$, locally integrable near the origin and such that that for a certain real number A, $f(w) = 0(e^{Aw})$ when $w \to +\infty$, one defines its Laplace transform F by

$$F(a) = \int_0^{+\infty} dw\, e^{-aw} f(w), \;\; a > A.$$

The Fourier inversion formula leads to the following important result : if f and g have Laplace transforms F and G that coïncide for large a, then $f = g$.

– Some integral representations :

The Euler Γ function is defined for $s > 0$ by the formula

$$\Gamma(s) = \int_0^{+\infty} dw\, e^{-w} w^{s-1}. \tag{8}$$

Some special values of the Γ function are simple. For instance, one shows that $\Gamma(1) = 1$ (obvious) and $\Gamma(1/2) = \sqrt{\pi}$ (a bit less obvious).

One shows, using integration by parts for example, that $\Gamma(s+1) = s\Gamma(s)$.

Hence, one gets by recursion that

$$\Gamma(n+1) = n! \text{ pour } n = 0, 1, \cdots .$$

and that

$$\Gamma(n+1/2)=\sqrt{\pi}\frac{(2n-1)!!}{2^n}=\sqrt{\pi}\frac{(2n)!}{4^n n!} \text{ pour } n=0,1,\cdots.$$

By a change of variable one checks that, for $a, s > 0$,

$$\frac{1}{a^s}=\frac{1}{\Gamma(s)}\int_0^{+\infty} dw\, e^{-aw} w^{s-1}. \tag{9}$$

To be slightly pedantic, the Laplace-Fourier transform of $\frac{w^{s-1}}{\Gamma(s)}$ is $1/a^s$...

The special case $s=1$ is elementary :

$$\frac{1}{a}=\int_0^{+\infty} dw\, e^{-aw} \qquad a>0. \tag{10}$$

The Euler Γ function allows to express the value of many common integrals. For instance :

$$\int_{-1}^{1} dv \left(\frac{1}{1-v^2}\right)^{2-d/2}=\frac{\Gamma(1/2)\Gamma(d/2-1)}{\Gamma(d/2-1/2)}.$$