A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

August 2007

EPL, **79** (2007) 40004
doi: 10.1209/0295-5075/79/40004

www.epljournal.org

Escape from a potential well, stochastic resonance and zero frequency component of the noise

F. PÉTRÉLIS[1], S. AUMAÎTRE[1,3] and K. MALLICK[2]

[1] *Laboratoire de Physique Statistique, Ecole Normale Supérieure - 24 rue Lhomond, 75005 Paris, France*
[2] *SPhT, Orme des Merisiers, CEA-Saclay - 91191 Gif-sur-Yvette, France*
[3] *DSM/DRECAM/SPEC, Orme des Merisiers, CEA-Saclay - 91191 Gif-sur-Yvette, France*

received 9 April 2007; accepted in final form 4 July 2007
published online 27 July 2007

PACS 05.40.-a – Fluctuation phenomena, random processes, noise and Brownian motion
PACS 05.45.-a – Nonlinear dynamics and chaos

Abstract – For an overdamped particle evolving in a potential and submitted to colored noise, we study numerically the problems of first passage time at the top of the potential and of stochastic resonance. In the limit where the correlation time of the noise is small compared to the other characteristic times of the problem we show that properties of the solution related to long time behavior are controlled by the zero-frequency component of the noise.

Studies of the effect of noise on a system are generally motivated by physical situations in which one of the variables is submitted to the influence of some rapidly evolving others. As a first model of the rapidly evolving variables, it is natural to use random noises. These noises are usually assumed to be Gaussian random variables and, as a first step, one models their fast time evolutions by using noises that are delta-correlated, *i.e.* white noises. Of course, other noises than Gaussian white noises have been considered. A very large literature tackled the case of colored noises but, as advocated in [1], many researches have focused on Ornstein-Uhlenbeck (O-U) processes with only one characteristic time and a spectrum that tends to a constant for frequencies smaller than the inverse of this characteristic time. Experimentally, it is easy to generate a noise with a more complex spectrum and in many situations it is necessary to filter out the low-frequency part of the noise spectrum [2]. It is thus important to investigate the effect of colored noises having a spectrum richer than the one of the O-U process.

In [3], Doering and Hormsthemke studied simple dynamical systems driven either by time-periodic forcing or by a Markovian dichotomous process. The probability density functions (PDF) of the variable are in general different and the Markovian dichotomous process leads to more changes in the number of extrema of the PDF, *i.e.* more noise-induced transitions. They pointed out that these effects are related to the time-periodic forcing having no low-frequency components. Studying the escape of particle over a fluctuating barrier driven by a colored noise, Hänggi showed that the escape time can exhibit a minimum at the condition that the noise spectrum at zero frequency is an increasing function of the correlation time [4]. In [5], we identified that the spectrum of the noise at zero frequency controls the phenomena of on-off intermittency. In the limit of small noise amplitude multiplied with its correlation time, only the noise spectrum at zero frequency appears in the expression of the PDF of the on-off intermittent variable. In general, filtering out the low-frequency components of the noise destroys the on-off intermittent regime.

The purpose of this work is to investigate the influence of the spectrum of the noise on some classical phenomena such as the exit of an overdamped particle from a potential well and stochastic resonance. We do not restrict ourselves to O-U processes and will show that, under some assumptions, the zero frequency component of the spectrum is the relevant parameter.

The problem of the escape of a particle from a potential has motivated many researches due to its relation with a large variety of problems ranging from mechanics to chemistry. Effects of colored noise have been studied for a Brownian particle (the Kramers' problem) or for an overdamped particle. For the latter, most studies focused on an O-U noise and derived asymptotic expressions for the escape rate as a function of the noise correlation time, see [6,7] for reviews.

A related problem that has drawn lots of attention in the last twenty years is stochastic resonance: the ability of a system to use noise to synchronize its behavior with a weak periodic forcing. The phenomena is that, for fixed periodic forcing, at low noise amplitude, the particle oscillates in one of the well; at very large noise amplitude, the periodic forcing has no effect and the particle switches randomly in time between the wells. At intermediate noise, the particle synchronizes with the forcing and switches periodically between the two wells. Therefore it uses the noise to realize coherent motion. The mechanism was discovered in [8] and was first observed experimentally in an electronic bistable system [9]. These works were followed by a large number of theoretical, numerical or experimental studies, see [10] for a review.

In this letter, we first study the problem of escape from a potential. An analytical expansion is performed and its prediction is compared to numerical simulations that use harmonic noise [11,12]. Then we perform numerical simulations of the problem of stochastic resonance and show that a result similar to the former prediction applies. Finally we give our interpretation of the phenomenon.

We start with the problem of first passage time at the top of a potential of an overdamped particle submitted to a colored noise. Let x be the position of the particle, $V(x)$ the potential and ζ the colored noise with autocorrelation function $K(t)=\langle\zeta(t)\zeta(0)\rangle_s$. The particle position evolves as follows:

$$\dot{x}=-\partial_x V(x)+\zeta(t)\,. \tag{1}$$

We are interested in the statistics of the first passage time at the top of the potential for a particle initially located at the potential minimum. As an example we choose $V(x)=-\cos x$ and study the statistics of the first passage time T_e defined by the event

$$\{x(0)=0, |x(t)|\leqslant\pi \ \text{ for }\ 0\leqslant t\leqslant T_e, |x(T_e)|=\pi\}\,.$$

A first step in order to calculate the statistics of the first passage time is to derive an evolution equation for the PDF of the variable x. This equation takes the form of a Fokker-Planck equation if the noise is white. In other cases, assumptions on the noise have to be made and equations similar to the Fokker-Planck equation can only be derived through expansions. We perform the Van-Kampen expansion [13] from which we derive the evolution equation for the PDF of the variable $P(x,t)$,

$$\partial_t P=-\partial_x\left(f(x)P\right)+\partial_{x,x}\left(S(x)P\right), \tag{2}$$

where the force experienced by the particle is $f(x)=-\partial_x V=-\sin x$ and $S(x)=\int_0^\infty K(t)e^{-t}\,\mathrm{d}t+(1-\cos x)\int_0^\infty K(t)\sinh t\,\mathrm{d}t$. Now, having in mind that ζ is a noise, we assume that correlations are lost faster than any characteristic time of the deterministic system. This amounts to its autocorrelation function tending to zero on a time small compared to unity. In that limit, the expression of S simplifies to a constant

$$S=\int_0^\infty K(t)\,\mathrm{d}t\,. \tag{3}$$

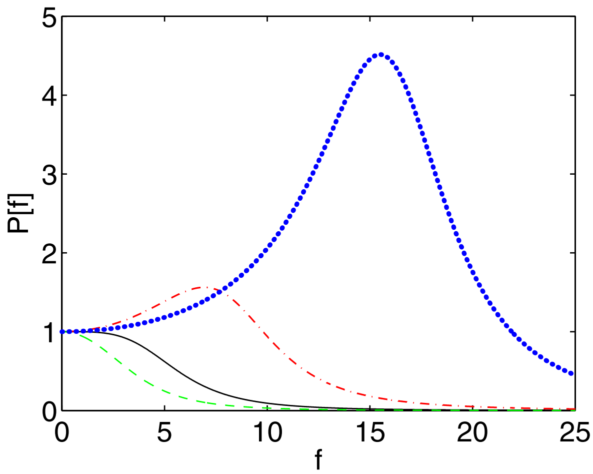

Fig. 1: Spectra of the noises given by eq. (5) for $\eta=4$, (dashed line): $\Omega=2$, (solid line): 4, (dot-dashed line): 8, (dotted line): 16 and the same value of the power spectra at zero frequency, *i.e.* $8\eta\alpha^2=\pi(\Omega^2+\eta^2)$.

Therefore, the evolution of the PDF of the variable is completely controlled by the integral of the autocorrelation function of the noise, equivalently by the noise spectrum at zero frequency as stated by the Wiener-Khintchin theorem. We recall that this expansion is valid when the product of the noise amplitude with its correlation time is small [13]. However, we will show that even out of this limit, S still controls the exit time even if it does not control the whole PDF of the variable.

To test this prediction, we solve eq. (1) with the initial condition $x(0)=0$ and compute the first passage time T_e. We take for ζ the harmonic noise [11,12]. It is Gaussian with zero mean and its autocorrelation function is given by

$$K(t)=4\,\alpha^2\exp\left(-2\pi\eta t\right)\left(\cos\left(2\pi\Omega|t|\right)+\frac{\eta}{\Omega}\sin\left(2\pi\Omega|t|\right)\right), \tag{4}$$

where 2α is the noise standard deviation and η and Ω two parameters. Its spectrum is given by

$$P[f]=\frac{8\eta\alpha^2(\Omega^2+\eta^2)/\pi}{(\Omega^2+\eta^2-f^2)^2+4\eta^2 f^2}\,, \tag{5}$$

such that, depending on the values of η and Ω, the power spectra can be qualitatively different as displayed in fig. 1. Note that $S=P[0]/2=4\eta\alpha^2/(\pi(\Omega^2+\eta^2))$.

We first plot in fig. 2 the mean first passage time $\langle T_e\rangle$ as a function of α^{-2} for noises with different η and Ω. As expected, all these curves show an exponential dependence of $\langle T_e\rangle$ as a function of α^{-2}. Now we check our prediction that S controls the mean first passage time by plotting the results of fig. 2 as a function of S^{-1}. The data collapse in the lower plot of fig. 2 proves that this is indeed the case. The slope of these curve is close to 1/2, as in the case of a white noise of flat spectrum $2S$ for which $\langle T_e\rangle\propto e^{2/S}$ [6,7][1].

[1]We have checked that the prefactor tends to 2 with a standard deviation smaller than 10% for η larger than 8.

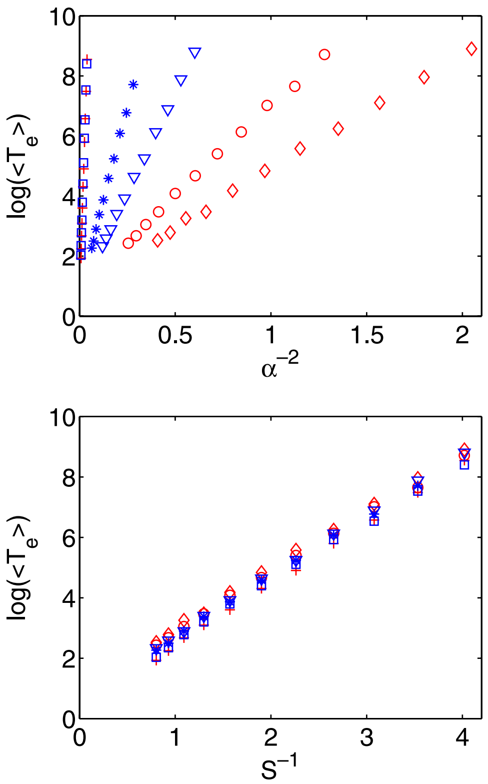

Fig. 2: Mean first passage time $\langle T_e \rangle$. In the top panel as a function of the inverse of the square of the noise standard deviation. In the bottom panel data collapse as a function of S^{-1}. The symbols are for $\eta = 2$, ($\circ$): $\Omega = 2$, ($+$): $\Omega = 16$, ($\Diamond$): $\Omega = 1$; for $\eta = 8$, ($*$): $\Omega = 8$, ($\Box$): $\Omega = 32$, (∇): $\Omega = 2$.

In our simulations, the correlation times $\tau_c = (2\pi\eta)^{-1}$ is smaller than one but α can be larger than unity. However, in the limit where the noise correlation time is smaller than the characteristic time of the deterministic evolution and when the noise intensity is not too large, we can state that the mean first passage time of an overdamped particle at the top of a potential is controlled by the noise spectrum at zero frequency. We have verified that a similar rescaling of the lowest order cumulants of T_e takes place when plotted as a function of S^{-1}. The fact that the PDF of the first passage time is fully controlled by S remains a matter of investigation.

It is tempting to inspect the importance of the noise at zero frequency in other dynamical systems submitted to noise. We now turn to the phenomena of stochastic resonance. A widely studied model is an overdamped particle evolving into a double-well potential submitted to a noise and a periodic forcing. The equation of motion is

$$\dot{x} = x - x^3 + \zeta(t) + A\cos\omega t\,. \qquad (6)$$

We study eq. (6) using the Gaussian colored noise defined by eq. (4). The stochastic resonance appears in the synchronization of the variable x with the

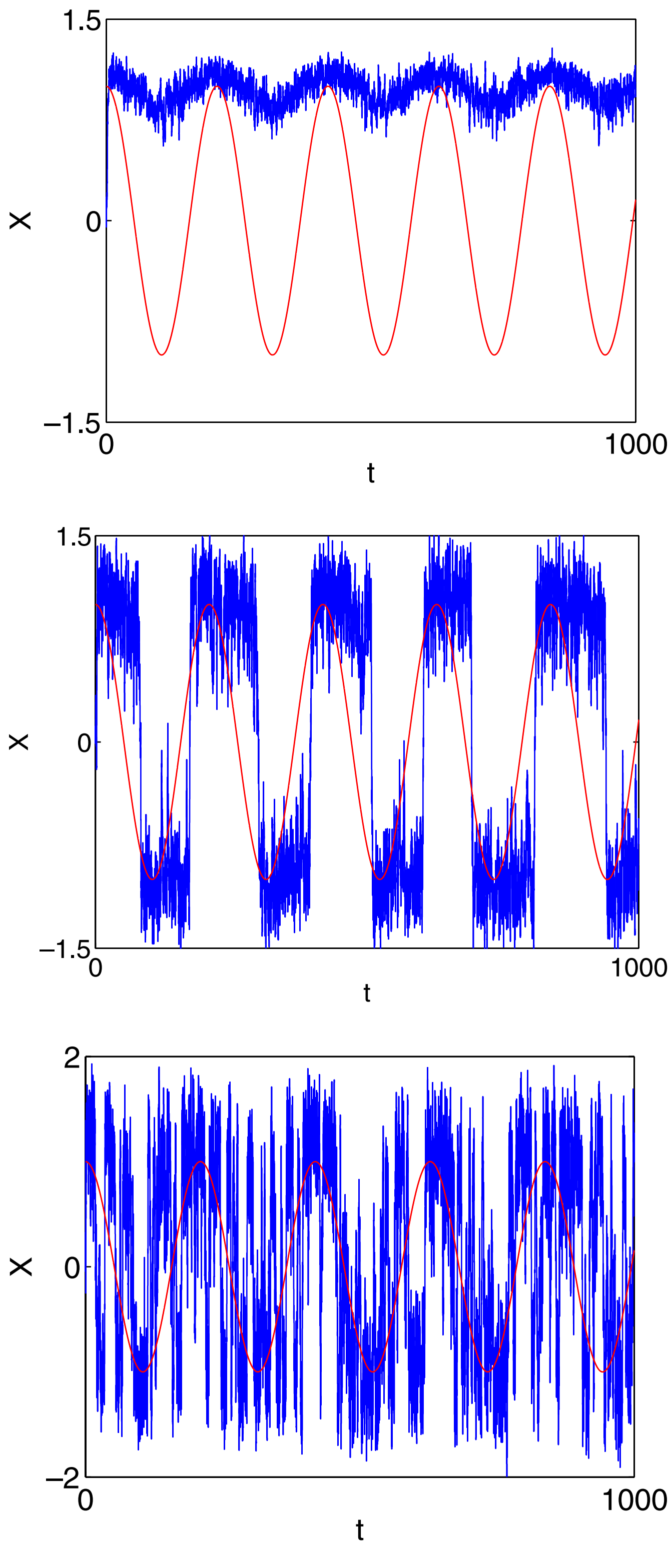

Fig. 3: Time series of x solution of eq. (6) with $A = 0.2$, $\omega = 0.03$, $\eta = 4$, $\Omega = 4$; from the top to the bottom $\alpha = 0.25, 0.75, 1.5$. In the middle figure, for $\alpha = 0.75$, stochastic resonance takes place.

forcing for intermediate values of the noise amplitude, see fig. 3. We characterize this coherent motion by computing the discrete Fourier transform of x at the

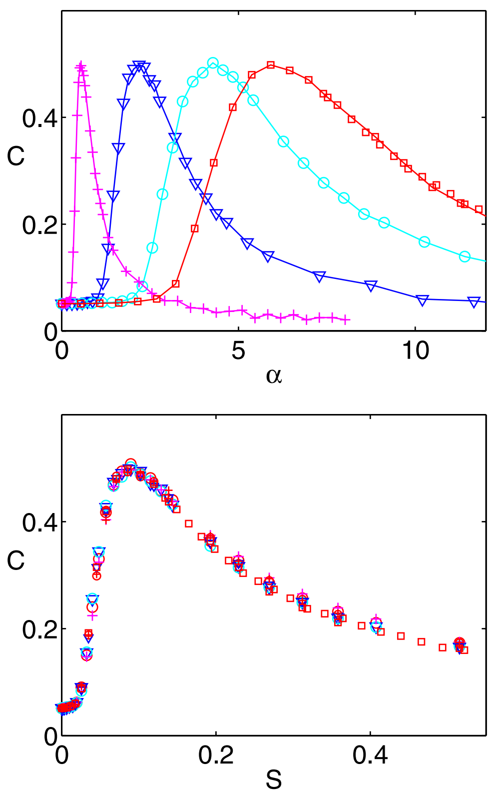

Fig. 4: Top: C as a function of the noise standard deviation α for (+): $\eta=4$, $\Omega=1$; ($\triangledown$): $\eta=4$, $\Omega=16$; ($\circ$): $\eta=4$, $\Omega=32$; ($\square$): $\eta=8$, $\Omega=64$. Bottom: data collapse of C as a function of S, same results as on top and new results for $\eta=\Omega=4$, $\eta=\Omega=8$, $\eta=\Omega/4=8$, $\eta=4\Omega=8$.

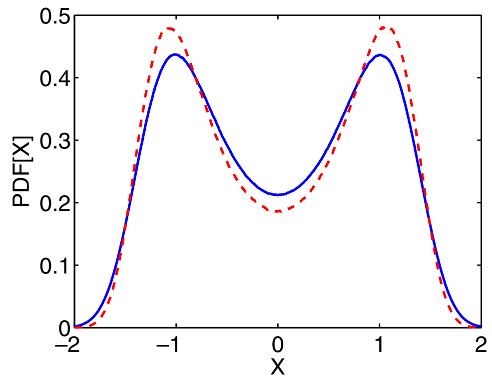

Fig. 5: PDF of the solution of eq. (6) for $A=0.2$, $\omega=0.03$, $\eta=4$ and (solid line): $\alpha=4.37$, $\Omega=16$; (dashed line): $\alpha=1.09$, $\Omega=1$. In both cases $S=0.36$.

forcing frequency and define

$$C=\lim_{T\to\infty}|\int_0^T x(t)\exp(i\,\omega\,t)\,\mathrm{d}t|/T\,. \tag{7}$$

Stochastic resonance amounts to C having a maximum as a function of the noise amplitude [14]. Therefore, in fig. 4, we plot C as a function of the noise standard deviation α for different values of η and Ω. It appears clearly that the noise standard deviation does not alone control C.

Inspired by our former result on the role of the zero frequency component of the noise for the problem of the exit time from a potential, we now plot C as a function of the noise spectrum at zero frequency S. These simulations are performed with the parameters such that $\tau_c\ll 1\ll\omega^{-1}$. In this limit, the curves of stochastic resonance are completely controlled by the noise spectrum at zero frequency (see the lower plot of fig. 4).

Due to the strong link between the problem of escape from a potential and the stochastic resonance, it is not completely surprising that in regimes where the escape process is controlled by the noise spectrum at zero frequency, so is the stochastic resonance. Indeed an approximate criteria for the stochastic resonance is given by the period of the forcing being twice the mean exit time from one of the wells of the potential. One can expect that if the escape time is controlled by the zero-frequency component of the noise, the same is true for the maximum of C. Our numerical simulations show that this is indeed the case and not only for the location of the maximum but *for all the values of* C in the range our simulations were performed.

One may argue that our results are a straightforward consequence of the correlation time of the noise being very small and the spectrum being almost white. In the case of O-U noise with correlation time τ_c, in the limit of small τ_c, the noise tends to be white and the mean first passage time at the top of a potential tends to the value given by a white noise [15]. Therefore, this time is controlled by the noise spectrum at zero frequency since this spectrum tends to a constant. However, for such noises, it is not possible to identify which part of the spectrum plays a role for different aspects of the dynamics. For the harmonic noise, we point out that when the noise amplitude is not too small, it is only the long time evolution of the process that is controlled by the noise spectrum at zero frequency. For instance, we plot in fig. 5, the PDF of x for same value of S, consequently of C but different noise spectra. The short time evolution of the variable and its PDF are different even if the long time evolution are the same in the sense that they lead to the same value of C. This is because C is mainly a function of S thus of the zero-frequency component of the noise, whereas the PDF depends on all the components of the noise.

Our interpretation of the importance of the zero frequency component of the noise is the following. Let τ_d be the characteristic time of the deterministic evolution, which is here around unity. Since we studied a regime in which τ_d, $\langle T_e\rangle$ for the escape problem and ω^{-1} for the stochastic resonance are larger than the correlation time of the noise, we are studying long time evolutions of the system. For both problems, the cumulative effect of the noise is important and it plays a role if its realization is such that $Y = \int_0^T \zeta(t')\mathrm{d}t'$ is large. This variable is Gaussian with zero mean and its standard deviation for large time T is $\langle Y^2 \rangle \simeq 2ST$. Therefore persistent realizations of the noise are more probable if its spectrum at zero frequency is large. Because the escape problem and the stochastic resonance are sensitive to these persistent behavior of the noise, both phenomena are controlled by the zero-frequency component of the noise. It is essentially the same argument that we introduced in the case of on-off intermittency [5]. Therein, the departure from onset, say a, is associated to a time a^{-1}. At criticality, this time diverges and it selects the zero-frequency component of the noise as the control parameter for on-off intermittency.

Our result is linked to a separation between the noise correlation time and the other characteristic times of the system. Out of this regime, different behaviors are expected. For example, for the problem of escape from a potential, if the frequency of the harmonic noise goes close to that of the potential, the exit time can be strongly reduced [12]; for an O-U noise, the stochastic resonance can be suppressed when the characteristic time of the noise gets close to that of the potential [16], at even larger noise correlation time, resonant crossing can be driven by tuning the modulation frequency [17]. Besides, noise in physical systems might be even more complicated than harmonic noise with possibly many extrema in the spectrum. Nevertheless, we think that our result applies in most situations since a noise usually models fluctuations at high frequencies (respectively, short correlation time) compared to the frequencies (respectively, characteristic time) of the system. Thus, whatever the noise spectra, if a long time evolution of the system is studied then it will be controlled by the zero frequency component of the noise.

* * *

This work has benefited from many discussions with N. Mordant and S. Fauve.

REFERENCES

[1] Horsthemke W. and Lefever R., *Noise-Induced Transitions* (Springer-Verlag) 1984.

[2] Pétrélis F. and Aumaître S., *Eur. Phys. J. B*, **34** (2003) 281.

[3] Doering C. R. and Horsthemke W., *J. Stat. Phys.*, **38** (1985) 763.

[4] Hänggi P., *Chem. Phys.*, **180** (1994) 157.

[5] Aumaître S., Pétrélis F. and Mallick K., *Phys. Rev. Lett.*, **95** (2005) 064101; Aumaître S., Mallick K. and Pétrélis F., *J. Stat. Phys.*, **123** (2006) 909.

[6] van Kampen N. G., *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam) 1992.

[7] Hänggi P., Talkner P. and Borkovec M., *Rev. Mod. Phys.*, **62** (1990) 251.

[8] Benzi R., Sutera A. and Vulpiani A., *J. Phys. A*, **14** (1981) L453.

[9] Fauve S. and Heslot F., *Phys. Lett. A*, **97** (1983) 5.

[10] Gammaitoni L., Hänggi P., Jung P. and Marchesoni F., *Rev. Mod. Phys.*, **70** (1998) 223.

[11] Sawford B. L., *Phys. Fluids A*, **3** (1991) 1577.

[12] Schimansky-Geier L. and Zülicke Ch., *Z. Phys. B*, **79** (1990) 451.

[13] van Kampen N. G., *Phys. Rep.*, **24** (1976) 173.

[14] Jung P. and Hänggi P., *Phys. Rev. A*, **44** (1991) 8032.

[15] Hänggi P., Marchesoni F. and Grigolini P., *Z. Phys. B*, **56** (1984) 333.

[16] Hänggi P., Jung P., Zerbe C. and Moss F., *J. Stat. Phys.*, **70** (1993) 25.

[17] Gammaitoni L., Marchesoni F., Menichella-Saetta E. and Santucci S., *Phys. Rev. Lett.*, **71** (1993) 3625.