SATURATION OF A PONOMARENKO TYPE FLUID DYNAMO

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1. Introduction

The kinematic dynamo problem is rather well understood in the case of laminar flows [1]. Several simple but clever examples have been found in the past [2, 3, 4, 5] and more realistic geometries can be easily studied numerically [6]. However, most flows of liquid metal are fully turbulent before reaching the dynamo threshold: indeed, the magnetic Prandtl number, $Pm = \mu_0 \sigma \nu$, where μ_0 is the magnetic permeability of vacuum, σ is the electric conductivity and ν is the kinematic viscosity, is smaller than 10^{-5} for all liquid metals. Since the dynamo action requires a large enough magnetic Reynolds number, $Rm = \mu_0 \sigma L U$, where U is the fluid characteristic velocity and L is the characteristic scale, one expects to observe the dynamo effect when the kinetic Reynolds number, $Re = UL/\nu$, is larger that 10^6. The kinematic dynamo problem with a turbulent flow is much more difficult to solve. A theoretical approach exists only when the magnetic neutral modes grow at large scale. It has been shown that the role of turbulent fluctuations may be twofold: on one hand, they decrease the effective electrical conductivity and thus inhibits dynamo action by increasing Joule dissipation. On the other hand, they may generate a large scale magnetic field through the "alpha effect" or higher order similar effects [7]. Consequently, it is not known whether turbulent fluctuations inhibits or help dynamo action. More precisely, for a given configuration of the moving solid boundaries generating the flow, the behavior of the critical magnetic Reynolds number Rm_c for the dynamo threshold, as a function of the flow Reynolds number Re (respectively Pm) in the limit of large Re (respectively small Pm), is not known.

Another important open question concerns the prediction of the saturation level of the amplitude of the magnetic field. This problem has been considered several times in the past, but with very unrealistic values of the fluid parameters that cannot be achieved in laboratory experiments. Phenomenological descriptions or perturbative calculations of the saturation of the "alpha effect" have been performed[9, 10, 11, 12, 13] for a large scale growing magnetic field with the assumption Pm of order one or very large. The same assumptions (scale separation and large Pm) have been used in models of dynamically consistent convective dynamos [14, 15, 18, 16]. The only case considered so far without the assumption of scale separation concerns the saturation of a Ponomarenko type fluid dynamo[19]. However, the study has been performed in the limit of large Rm for which a lot of magnetic modes are strongly unstable. Or goal here is to study the saturation of the the first unstable magnetic mode in the vicinity of the bifurcation threshold Rm_c.

2. A fluid in solid body rotation and translation up to the dynamo threshold

We use the following simple idea in order to be able to study the saturation of the magnetic field analytically: we consider the simplest possible flow, i. e. a fluid in solid body rotation and translation. This is the only way to avoid turbulence at dynamo onset. This may look unrealistic but an experimental configuration is possible. Consider for example the Herzenberg dynamo [2]: it consists of 2 or 3 rotating solid spheres embeded in a static medium of the same conductivity with which they are in perfect electrical contact. A slightly different version of the Herzenberg dynamo was operated experimentally by Lowes and Wilkinson using two cylinders instead of spheres [8]. Now, assume that one of the cylinders is hollow and filled with liquid metal. The flow will remain in solid body rotation up to the dynamo threshold. Above threshold the Lorentz force will slow down the fluid and modify the flow, thus leading to saturation of the magnetic field.

We study here an even simpler configuration found by Ponomarenko[4]: It consists of a cylinder of radius R, in solid body rotation at angular velocity ω, and translation along its axis at speed V, embeded in an infinite static medium of the same conductivity with which it is in perfect electrical contact. In the same way as above, we consider that the cylinder is hollow and filled with a liquid metal with the same electrical conductivity. The kinematic dynamo problem is thus the same as the one studied by Ponomarenko. However, above the dynamo threshold the flow is modified by the Lorentz force and we show that this saturates the growth of the magnetic field.

Using respectively, R, $\mu_0\sigma R^2$, $(\mu_0\sigma R)^{-1}$, $\rho(\mu_0\sigma R)^{-2}$ and $\sqrt{\mu_0\rho}/\mu_0\sigma R$, as units for length, time, velocity, pressure and magnetic field, the governing equation for the velocity field, $\vec{v}(\vec{r},t)$, and the magnetic field, $\vec{B}(\vec{r},t)$, are

$$\nabla \cdot \vec{B} = 0 \tag{1}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \Delta \vec{B} \tag{2}$$

$$\nabla \cdot \vec{v} = 0 \tag{3}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\nabla \tilde{P} + Pm\ \Delta \vec{v} + (\vec{B} \cdot \nabla)\vec{B} \tag{4}$$

where $\tilde{P}$ is the sum of the hydrodynamic and magnetic pressures and ρ is the fluid density. Pm is the magnetic Prandtl number, $Pm = \mu_0 \sigma \nu$, and the boundary conditions for $\vec{v}$ involves the magnetic Reynolds number, $Rm = \mu_0 \sigma R \sqrt{(V^2 + (R\omega)^2}$ and the Rossby number, $Rb = V/R\omega$.

Below the dynamo threshold, $\vec{B} = 0$ and solid body rotation is solution of equations (3, 4). The corresponding kinematic dynamo problem has been solved by Ponomarenko[4]. Using cylindrical coordinates, he considered unstable modes of the form

$$\vec{B}(\vec{r}, t) = \vec{b}(r) \exp i(m\theta + kz + \omega_0 t) + c.c., \tag{5}$$

where c. c. stands for complex conjugate. We get from (2) that $\vec{b_p}(r)$ is an eigenmode of the operator L, defined by

$$L\vec{b_p} = i\,(\omega_0 + \mu\Gamma(r))\,\vec{b_p} - \Delta\vec{b_p} + (boundary\ terms)\vec{b_p} = 0, \tag{6}$$

with $\mu = m\omega + kV$, $\Gamma(r) = 1$ if $r < 1$ and zero if $r > 1$, Δ results from the Laplacian operator applied to (5).

This formulation is of course equivalent to Ponomarenko's one and the boundary terms are the mathematical translation of the discontinuity in the derivative of $\vec{b_p}$ induced by the discontinuity of velocity at the boundary. The interest of this formulation will become clear in the nonlinear analysis.

The critical magnetic Reynolds number $Rm_c(Rb, m, k, \omega_0)$ reaches a minimum $Rm_c = 17.722$ for $Rb = 1.314$, $k = -.388$, $m = 1$, $\omega_0 = 0.410$. The growth rate above Rm_c is $\alpha = (0.0268 + 0.00174\ i)(Rm - Rm_c)$.

3. Nonlinear saturation of the growing magnetic field

If the magnetic field saturates at a small amplitude just above the dynamo threshold, we see from (4) that the velocity perturbation that results from the Lorentz force is proportional to the square of the field amplitude. We thus choose the scalings: $Rm = Rm_c\ (1 + \Lambda\epsilon)$, $\tilde{P} = (\tilde{P}_f + \epsilon\tilde{P}_1 + \epsilon^2\tilde{P}_2 + \cdots)$, $\vec{v} = \vec{v}_f + \epsilon\vec{v}_1 + \epsilon^2\vec{v}_2 + \cdots$, $\vec{v}_f = \vec{v}_0\ (1 + \Lambda\epsilon)$, $\tilde{P}_f = \tilde{P}_0\ (1 + \Lambda\epsilon)$, $\vec{B} = \sqrt{\epsilon}\,(\vec{B}_0 + \epsilon\vec{B}_1 + \epsilon^2\vec{B}_2 + ...)$, $T = \epsilon t$, where ϵ is a small parameter representing the distance from criticality and Λ is of order one. v_f (respectively $\tilde{P}_f$) is the velocity (respectively the pressure) in the absence of magnetic field and v_0 (respectively $\tilde{P}_0$), is the value of the corresponding field at onset. T is the slow time that describes the growth of the magnetic field.

At first order we get

$$L\vec{B}_0 = 0, \tag{7}$$

$$\frac{\partial \vec{v}_f}{\partial t} + (\vec{v}_f.\nabla)\vec{v}_f = -\nabla \tilde{P}_f + Pm\ \Delta \vec{v}_f, \tag{8}$$

L being the operator defined by equation (5). We thus have

$$\vec{B}_0(t,T) = A(T)\vec{B}_p + c.c. = A(T)\vec{b}_p \exp i\left(m\theta + kz + \omega_0 t\right) + c.c., \tag{9}$$

where $\vec{b}_p$ is Ponomarenko's eigenmode. $\vec{v}_f$ represents solid-body rotation and translation and is thus solution of Navier-Stokes equation without magnetic field.

At second order we get

$$L\vec{B}_1 = -\frac{\partial \vec{B}_0}{\partial T} + \Lambda \nabla \times (\vec{v}_0 \times \vec{B}_0) + \nabla \times (\vec{v}_1 \times \vec{B}_0) \tag{10}$$

$$\frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_0.\nabla)\vec{v}_1 + (\vec{v}_1.\nabla)\vec{v}_0 = -\nabla \tilde{P}_1 + Pm\ \Delta \vec{v}_1 + (\vec{B}_0.\nabla)\vec{B}_0 \tag{11}$$

From equation (11), we can calculate the perturbation in velocity induced by the magnetic field (see appendix B).

Using the solvability condition for equation (10), we get the amplitude equation for $A(T)$. Let $\vec{C}$ be in the kernel of $L^\dagger$ the adjoint operator of L (see appendix A). We have

$$\left\langle \vec{C}|\vec{B}_p \right\rangle \frac{dA}{dT} = \Lambda\ \left\langle \vec{C}|\nabla \times (\vec{v}_0 \times \vec{B}_0) \right\rangle + \left\langle \vec{C}|\nabla \times (\vec{v}_1 \times \vec{B}_0) \right\rangle, \tag{12}$$

which is of the form

$$\frac{dA}{dt} = \alpha A + \beta |A|^2 A. \tag{13}$$

We thus find the normal form of a Hopf bifurcation. Although this is obvious from symmetry considerations, we note that the calculation of the coefficients requires the solvability condition which cannot be easily guessed as in most examples of nonlinear oscillators or pattern forming instabilities.

The first term on the right hand side of equation (12) gives the linear growth rate

$$\alpha = (0.0268 + 0.00175\, i)(Rm - Rm_c), \tag{14}$$

in very good agreement with Ponomarenko's stability analysis.

The second term on the right hand side of equation (12) traces back to the magnetic retroaction on the velocity field. Pm being very small for all liquid metals, we approximately have (see appendix B)

$$\beta \approx \beta_0 = \frac{1}{Pm}(-0.0034 - 0.0015\, i). \tag{15}$$

Thus, the bifurcation is supercritical ($Re(\beta_0) < 0$) and the amplitude saturates. This gives for the magnetic field in the M.K.S.A. unit system

$$\vec{B}_{sat} \approx \frac{2.81}{R}\sqrt{\frac{\rho\nu}{\sigma}}\sqrt{Rm - Rm_c}\, Re(\vec{B}_p). \tag{16}$$

We have for the velocity perturbation,

$$\vec{v}_{sat} \approx \frac{7.88}{\mu_0 \sigma R}(R_m - Rm_c)\, \vec{v}_1 \tag{17}$$

where $\vec{v}_1$ is the zero frequency component of the solution of equation (11) (see figure 1).

It may look surprising that the field saturates at a larger value when the viscosity is large whereas the velocity perturbation does not depend on the viscosity. This is due to the fact that the Lorentz force is balanced by the viscous term in the equation for the velocity perturbation (11).

For a turbulent flow, we expect a different balance between $\vec{v}_1$ and $\vec{B}_0$. Indeed, the saturated field amplitude should not depend any more on the kinematic viscosity in the large Re limit. Dimensional analysis then gives

$$B_{sat} \propto \sqrt{\frac{\rho}{\mu_0 \sigma^2 R^2}}\sqrt{\frac{Rm - Rm_c}{Rm_c}}, \tag{18}$$

which is larger than the above laminar scaling by a factor $Pm^{-1/2}$. The later scaling is likely to be appropriate for the "Karlsrhue" and "Riga" experiments (see these proceedings) and gives a magnetic field of the order of 100 gauss 10% above threshold.

There is of course never energy equipartition close to the dynamo threshold since the kinetic energy is finite whereas the magnetic energy tends to zero. In our example, we have for Joule dissipation, $P_j \propto B_{sat}^2 \propto \frac{\rho\nu}{\sigma}(R_m - Rm_c)$ whereas for viscous dissipation, $P_\nu \propto v_{sat}^2 \propto (Rm - Rm_c)^2$. Thus, close to threshold, most of the mechanical power is used to create the magnetic field. This is due to the nature of the basic flow. In turbulent flows, it would be interesting to check whether Joule dissipation scales like the additional viscous dissipation that results from the perturbed velocity field above dynamo threshold.

Appendix A: the adjoint problem

For $\vec{B}_a$ of the form given in (5), we define the scalar product, $\left\langle \vec{B}_a | \vec{B}_b \right\rangle = \int_0^\infty \vec{b}_a^*(r) \cdot \vec{b}_b(r)\ r dr$, where $\vec{b}_a^*$ is the complex conjugate of $\vec{b}_a$. With this definition and for $\vec{C} = \vec{c} \exp i(m\theta + kz + \omega_0 t)$, we have

$$L^\dagger \vec{c} = -i(\omega_0 + \mu\Gamma(r))\vec{c} - \Delta\vec{c} + (boundary\ terms)^\dagger \vec{c} \tag{19}$$

Except for the boundary terms, $L^\dagger$ is obtained from L with the transformations: $\omega_0 \to -\omega_0$, $\mu \to -\mu$, i. e. by changing the signs of all velocities. However, the boundary terms dramatically change the form of the eigenvectors of $L^\dagger$: indeed, the eigenvector $\vec{c}$ has no component in the z direction and is not even divergenceless. Thus, the adjoint problem of a kinematic dynamo problem may be not a dynamo problem, as already observed by Roberts [20].

Appendix B: calculation of the velocity perturbation

We calculate the velocity perturbation $\vec{v}_1$ by solving equation (11). This is a linear second order equation for $\vec{v}_1$, with the forcing term $(\vec{B}_0.\nabla)\vec{B}_0$. The response $\vec{v}_1$ involves a zero frequency component and two oscillatory components at frequencies: $\pm 2(m\theta + kz + \omega_0 t)$. The non-zero contributions to the scalar product $\left\langle \vec{C} | \nabla \times (\vec{v}_1 \times \vec{B}_0) \right\rangle$ come from the zero frequency and the second harmonic components of $\vec{v}_1$. We call β_0 (respectively β_d) their contribution to the value of the coefficient β in (13).

We first calculate the response at zero frequency. Since we have to consider velocity and pressure fields that are only functions of r, the resolution is easy. Equation (3) implies $v_{1r} = 0$ and the equations for the other components of $\vec{v}_1$ are decoupled. The boundary conditions are, the non-slip condition, $\vec{v}_1(1) = 0$, and, in order to keep the velocity field smooth, $v_{1r}(0) = 0$, $v_{1\theta}(0) = 0$, $v'_{1z}(0) = 0$. We solve the equations for $v_{1\theta}(r)$ and $v_{1z}(r)$ with Mathematica (see figure 1). Note that $\vec{v}_1$ and thus β_0 are inversely proportional to Pm because the right hand side of (11) does not contribute to the zero frequency response.

The calculation of the harmonic two response is more complicated because two components of the velocity field v_{1r}, $v_{1\theta}$ and the pressure are

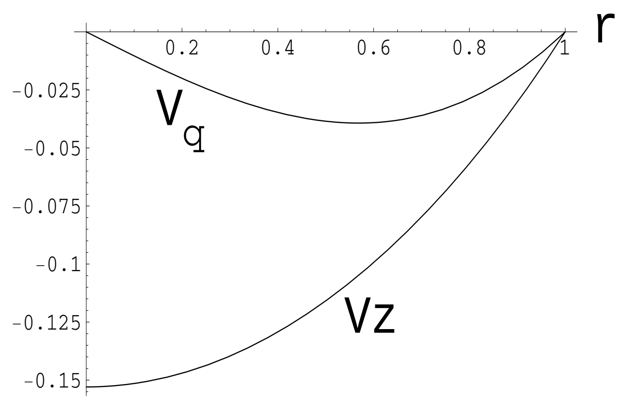

Figure 1. Zero frequency velocity perturbation for $Pm = 1$.

coupled. Taking the divergence of (11), we get

$$\Delta\tilde{P}_1 - 2\omega\left(\frac{v_{1\theta}}{r} + v'_{1\theta}\right) + 4im\omega\frac{v_{1r}}{r} - \nabla\cdot(\vec{B}_0.\nabla)\vec{B}_0 = 0. \qquad (20)$$

The boundary conditions for the pressure are $\tilde{P}(0) = 0$ and $\tilde{P}(\infty) = 0$. We solve (20) outside the cylinder where the velocity field is zero, and then solve (11) inside the cylinder using the continuity of the pressure at the boundary. The velocity and pressure fields are displayed in figure 2 for $Pm = .2$.

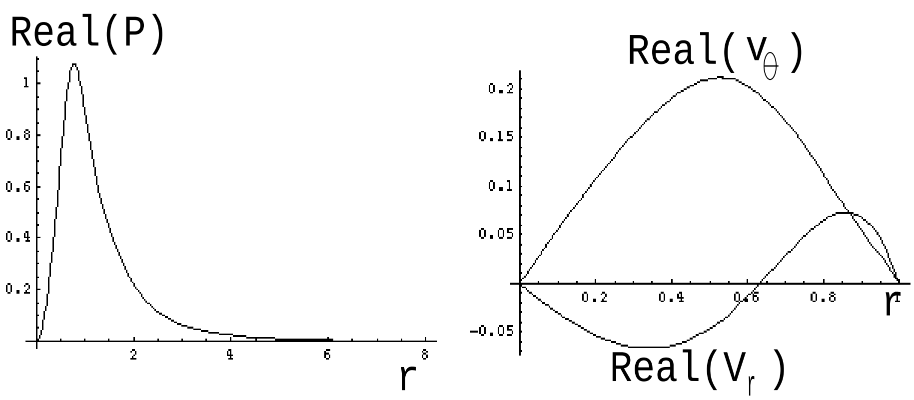

Figure 2. Real part of the complex amplitude of the harmonic-two response of the pressure and velocity for $Pm = 0.2$.

We observe that the harmonic-two response of the fluid velocity is locally enhanced on one part of the oscillation cycle, which may explain that $Re(\beta_d)$ is positive. Contrary to β_0, β_d is not inversely porportional to Pm. In order to compare it with β_0, we plot $Real(\beta_d)$ and $-Real(\beta_0)$ versus Pm in figure 3. For small Pm we observe that $-Real(\beta_0)$ is much larger than $Real(\beta_d)$. Thus the retroaction of the magnetic field on the amplitude of the unstable mode mostly results from the zero frequency response.

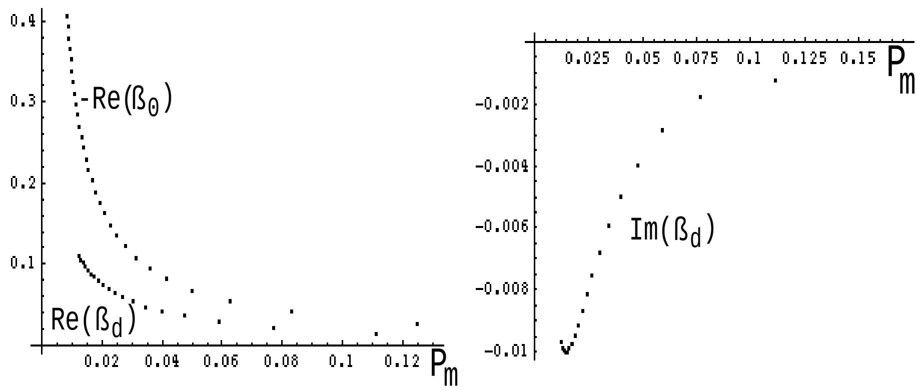

Figure 3. $-Re(\beta_0)$ and $Re(\beta_d)$ versus Pm and $Im(\beta_d)$ versus Pm.

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