

Scaling laws of turbulent dynamos

Comportements asymptotiques des dynamos turbulentes

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Abstract

We consider magnetic fields generated by homogeneous isotropic and parity invariant turbulent flows. We show that simple scaling laws for dynamo threshold, magnetic energy and Ohmic dissipation can be obtained depending on the value of the magnetic Prandtl number.

keywords : dynamo ; turbulence ; magnetic field

Version française abrégée Il est à présent admis que les champs magnétiques des étoiles voire même des galaxies sont engendrés par l'écoulement de fluides conducteurs de l'électricité [1,2,3]. Ceux-ci impliquent des nombres de Reynolds cinétique, Re , et magnétique, R_m , très élevés ($Re = VL/\nu$, $R_m = \mu_0\sigma VL$, où V est l'écart-type des fluctuations de vitesse, L , l'échelle intégrale de l'écoulement, ν , la viscosité cinématique du fluide, σ , sa conductivité électrique et μ_0 , la perméabilité magnétique). Aucune expérience de laboratoire ou simulation numérique directe des équations de la magnétohydrodynamique, ne permet l'étude du problème dans des régimes de paramètres, Re et R_m , d'intérêt astrophysique. Il est donc utile de considérer des hypothèses plausibles afin de pousser plus loin l'analyse dimensionnelle qui, à partir des paramètres V , L , ν , σ , μ_0 et de la densité du fluide ρ , prédit pour le seuil de l'effet dynamo et la densité moyenne d'énergie magnétique, $B^2/2\mu_0$, saturée non linéairement au-delà du seuil,

$$R_m^c = f(Re), \quad (1)$$

$$\frac{B^2}{\mu_0} = \rho V^2 g(R_m, Re). \quad (2)$$

Dans le cas d'un écoulement turbulent homogène isotrope, donc de vitesse moyenne nulle, et invariant par symétrie plane, donc sans hélicité, les résultats des simulations numériques les plus performantes réalisées à ce jour montrent que R_m^c augmente continuellement en fonction de Re [5]. Schekochihin et al. proposent que deux scénarios extrêmes, schématisés dans la figure 1, seront susceptibles d'être observés lorsque les ordinateurs auront acquis la puissance requise pour effectuer des calculs à Re plus élevé : (i) une saturation $R_m^c \rightarrow \text{constante}$, ou alors (ii) une croissance de la forme $R_m^c \propto Re$. D'autres simulations numériques directes, réalisées avec des écoulements turbulents possédant un champ de vitesse moyen de géométrie fixée $\nabla(\mathbf{r}) \neq 0$, semblent suivre le scénario (i) [13].

Lorsque le nombre de Prandtl magnétique, $P_m = R_m/Re = \mu_0\sigma\nu$, est faible, $P_m \ll 1$, l'échelle de dissipation Joule du champ magnétique, $l_\sigma = LR_m^{-3/4}$, est grande par rapport à l'échelle de Kolmogorov $l_K = LRe^{-3/4}$. Le champ magnétique se développe donc à une échelle suffisamment grande pour ne pas être affecté par la viscosité cinématique. Cette hypothèse, couramment effectuée en turbulence, permet de conclure en faveur du scénario (i). En effet, si ν n'est pas pris en compte, l'analyse dimensionnelle impose $R_m^c \rightarrow \text{constante}$. Il n'est donc pas surprenant que les modélisations numériques des grandes échelles, qui ne résolvent pas les échelles dissipatives, donnent ce résultat. Le scénario (i) sera donc toujours observé à P_m suffisamment faible sous réserve bien sûr que l'on ait dynamo.

Il est cependant utile d'analyser le scénario (ii) d'autant plus que, comme nous pouvons le remarquer, il correspond à la prédiction faite par Batchelor en 1950 [4]. En se basant sur une analogie entre l'équation de l'induction et celle de la vorticit , Batchelor avait estim  que le seuil d'une dynamo engendr e par un  coulement turbulent devait correspondre   P_m d'ordre unit , soit $R_m^c \propto Re$. M me si nous savons aujourd'hui que l'analyse de Batchelor est discutable, il est int ressant de d terminer sous quelle hypoth se minimale sa pr diction est correcte. Supposons donc que nous nous limitons aux modes instables de champ magn tique, suffisamment localis s au sein de l' coulement afin de ne pas  tre affect s par les conditions aux limites. Il est alors possible de ne pas prendre en compte l' chelle spatiale L , et l'analyse dimensionnelle impose pour le seuil, $P_m = \text{constante}$, soit le scénario (ii) $R_m^c \propto Re$.

Les sc narios consid r s ci-dessus conduisent aussi   des pr dictions diff rentes pour la densit  d' nergie magn tique engendr e par effet dynamo. Le scenario (i) qui consiste   ne pas prendre en compte ν revient   n gliger la d pendance en Re de $g(R_m, Re)$ dans (2). Au voisinage du seuil, V est d termin  par $R_m^c \sim \mu_0\sigma VL$ et $g(R_m) \propto R_m - R_m^c$ dans le cas d'une bifurcation supercritique. Il en r sulte [12]

$$B^2 \propto \frac{\rho}{\mu_0(\sigma L)^2}(R_m - R_m^c). \quad (3)$$

Loin du seuil pour $P_m \ll 1$, $Re \gg R_m \gg R_m^c$, on peut supposer que B ne d pend plus de σ   condition que le champ magn tique se d veloppe   une  chelle plus grande que l_σ . Il en r sulte alors l' quipartition entre  nergie magn tique et cin tique, $B^2/\mu_0 \propto \rho V^2$, tel que suppos  initialement par Biermann et Schl ter [15].

Un r sultat compl tement diff rent est obtenu dans le sc nario (ii). Il convient de consid rer les param tres du probl me sous la forme  quivalente, B , $\epsilon = V^3/L$, L , ν , σ , μ_0 et ρ . En effet, le champ magn tique   petite  chelle est aliment  par la puissance par unit  de masse ϵ qui cascade depuis l' chelle int grale, et il est donc important de conserver ce param tre m me si l'on ne prend pas en compte explicitement L . L'analyse dimensionnelle conduit alors  

$$\frac{B^2}{\mu_0} = \rho\sqrt{\nu\epsilon}h(P_m) = \frac{\rho V^2}{\sqrt{Re}}h(P_m), \quad (4)$$

qui, pour $P_m \sim 1$, n'est autre que le r sultat obtenu par Batchelor en supposant que la saturation correspond   l' quipartition entre l' nergie magn tique et l' nergie cin tique   l' chelle de Kolmogorov.

Revenons au cas $P_m \ll 1$ qui correspond aux  coulements de m taux liquides et plasmas   l'origine du champ magn tique des plan tes et des  toiles ($P_m < 10^{-5}$). Dans ce cas, le champ magn tique se d veloppe   des  chelles a priori comprises entre L et l_σ avec $l_\sigma \gg l_K$ et il en r sulte que R_m^c ne d pend pas de variations de P_m (ou de Re) et que $B^2/\mu_0 = \rho V^2 g(R_m)$ (sc nario (i)). Int ressons nous   la puissance dissip e par effet Joule par une telle dynamo. Il faut   cet effet d terminer   quelles  chelles se d veloppe le champ magn tique. Utilisons pour cela un argument   la Kolmogorov en supposant que dans la zone inertielle, c'est   dire pour les nombres d'onde k tels que $kl_\sigma \ll 1 \ll kL$, la puissance spectrale $|\hat{B}(k)|^2$ est ind pendante de L , σ and ν . Il en r sulte

$$|\hat{B}|^2 \propto \mu_0\rho\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}. \quad (5)$$

Ceci n'est pas la seule possibilit  parmi les nombreuses pr dictions relatives au spectre de la turbulence magn to-hydrodynamique, mais dans le cas pr sent, c'est probablement la plus simple. L'int gration sur k redonne l' quipartition $B^2/\mu_0 \propto \rho V^2$. La contribution dominante   l'effet Joule provient de l' chelle l_σ . Nous obtenons

$$\frac{\mathbf{j}^2}{\sigma} = \frac{1}{\sigma} \int |\hat{j}|^2 dk \propto \frac{1}{\mu_0^2\sigma} \int k^2 |\hat{B}|^2 dk \propto \frac{\rho}{\mu_0\sigma} \epsilon^{\frac{2}{3}} l_\sigma^{-\frac{4}{3}} \propto \rho \frac{V^3}{L}, \quad (6)$$

o  $\mathbf{j}$ est le vecteur densit  de courant. Nous constatons donc que la dissipation Joule est du m me ordre que la puissance totale disponible. Remarquons qu'il en serait de m me pour une dynamo de Batchelor suivant le sc nario (ii) pour $P_m \sim 1$, car bien que la densit  d' nergie soit plus faible, l' chelle caract ristique du champ magn tique l'est  galement.

1. Introduction

It is now believed that magnetic fields of stars and possibly galaxies are generated by the motion of electrically conducting fluids through the dynamo process [1,2,3]. These flows involve huge kinetic, Re , and magnetic, R_m , Reynolds numbers ($Re = VL/\nu$, $R_m = \mu_0\sigma VL$, where V is the *rms* velocity amplitude, L is the integral length scale, ν is the kinematic viscosity of the fluid, σ is its electrical conductivity and μ_0 is the magnetic permeability). No laboratory experiments, neither direct numerical simulations are possible in the range of Re and R_m involved in astrophysical flows. It is thus interesting to try to guess scaling laws for the magnetic field using some simple hypothesis. We consider here the minimum set of parameters, V , L , ν , μ_0 , σ and ρ , the fluid density. We note that discarding global rotation makes our results certainly invalid for many astrophysical objects but not all of them. Rotation is indeed not assumed important for the galaxies which do not display a large scale coherent magnetic field [1,2,3]. Calling B its *rms* value, dimensional analysis gives

$$R_m^c = f(Re), \quad (7)$$

for the dynamo threshold, and

$$\frac{B^2}{\mu_0} = \rho V^2 g(R_m, Re), \quad (8)$$

for the mean magnetic energy density in the nonlinearly saturated regime. Our aim is to determine f and g in various regions of the parameter space (R_m, Re) , assuming that turbulence is homogeneous, isotropic and parity invariant (thus with no mean flow and no mean magnetic field generation through an alpha effect). As already mentioned, this may look like an academic exercise compared to most natural dynamos. It is however not more academic that the concept of homogeneous and isotropic turbulence with respect to real turbulent flows. We thus expect that our simple arguments may shed some light on open problems concerning the effect of turbulence on the dynamo threshold and on the dynamic equilibrium between magnetic and kinetic energy.

The dependence of the dynamo threshold $R_m^c = f(Re)$ in the limit of large Re is still an open problem, even in the case of a homogeneous isotropic and parity invariant turbulent flow. Note that parity invariance prevents the generation of a large scale magnetic field via an alpha effect type mechanism and isotropy implies zero mean flow. Recent direct numerical simulations show that R_m^c keeps increasing with Re at the highest possible resolution without any indication of a possible saturation [5]. Schekochihin et al. thus propose that two limit scenarios, sketched in figure 1, could be observed when computers will be able to reach higher Re : (i) saturation, $R_m^c \rightarrow \text{constant}$, or (ii) increasing threshold in the form $R_m^c \propto Re$.

A lot of work has been performed on the determination of R_m^c as a function of Re for turbulent dynamos in the limit of large Re (or small P_m). We recall that (ii) has been proposed by Batchelor in one of the first papers on turbulent dynamos [4]. A lot of analytical studies have been also performed, mostly following Kazantsev's model [6] in order to show that purely turbulent flows can generate a magnetic field. Kazantsev considered a random homogeneous and isotropic velocity field, δ -correlated in time and with a wave number spectrum of the form k^{-p} . He showed that for p large enough, generation of a homogeneous isotropic magnetic field with zero mean value, takes place. This is a nice model but its validity is questionable for realistic turbulent flows. However, Kazantsev's model has been extrapolated to large Re . Various predictions, $R_m^c \propto Re$ [7], $R_m^c \rightarrow \text{constant} \approx 400$ for velocity spectra with $3/2 < p < 3$ and no dynamo otherwise [8], or dynamo for all possible slopes of the velocity spectrum $1 < p < 3$ [9] have been found. These discrepancies show that extrapolation of Kazantsev's model to realistic turbulence cannot be rigorous. The calculation is possible only in the case of a δ -correlated velocity field in time, and $\delta(t - t')$, which has the dimension of the inverse of time, should then be replaced by a finite eddy turn-over time in order to describe large Re effects. As already noticed, its choice is crucial to determine the behavior of R_m^c versus Re .

A different problem about turbulent dynamos has been considered more recently. It concerns the effect of turbulent fluctuations on a dynamo generated by a mean flow. The problem is to estimate to which extent the dynamo threshold computed as if the mean flow were acting alone, is shifted by turbulent fluctuations. This question has been addressed only recently [10] and should not be confused with dynamo generated by random flows with zero mean. It has been shown that weak turbulent fluctuations do not shift the dynamo threshold of the mean flow at first order. In addition, in the case of small scale fluctuations, there is no shift at second order either, if the fluctuations have no helicity. This explains why the observed dynamo threshold in Karlsruhe and Riga experiments [11] has been found in good agreement with the one computed as if the mean flow were acting alone, i.e. neglecting turbulent fluctuations. Recent direct numerical simulations have shown that in the presence

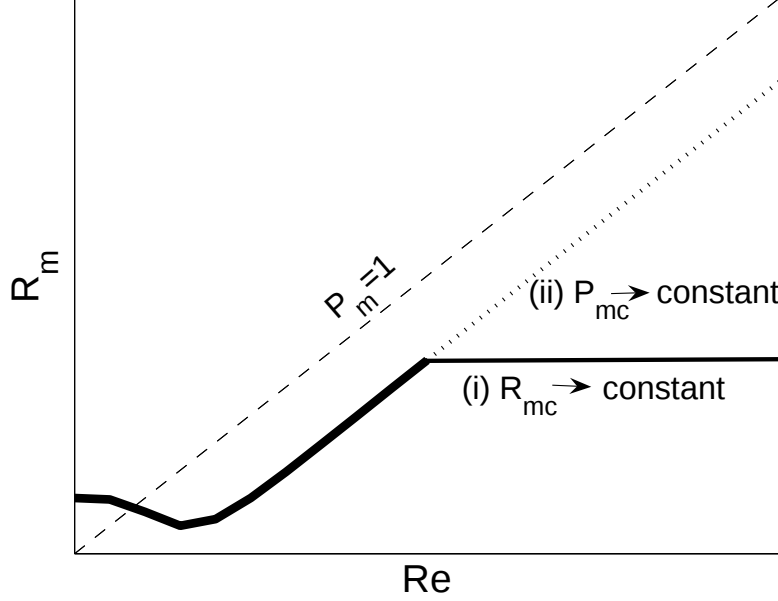


Figure 1. Dependence of the dynamo threshold R_m^c as a function of the Reynolds number Re . Scenario (i): R_m^c tends to a constant. Scenario (ii): R_m^c is proportional to Re .

of a prescribed mean flow, $\nabla(\mathbf{r}) \neq 0$, R_m^c increases with Re at moderate Re but then seems to saturate at larger Re , thus following scenario (i). For the same flows, numerical modeling of large scales, large eddy simulations (LES) for instance, gives $R_m^c \sim \text{constant}$ [13]. This last result follows from dimensional consideration as explained below, and has been also obtained for homogeneous isotropic turbulent non helical flows for which EDQNM closures have predicted $R_m^c \approx 30$ [14].

2. Turbulent dynamo threshold

When the magnetic Prandtl number, $P_m = R_m/Re = \mu_0\sigma\nu$, is small, $P_m \ll 1$, the Ohmic dissipative scale, $l_\sigma = LR_m^{-3/4}$ is much larger than the Kolmogorov $l_K = LRe^{-3/4}$. Thus, if there is dynamo action, the magnetic field grows at scales much larger than l_K and does not depend on kinematic viscosity. This hypothesis is currently made for large scale quantities in turbulence and if correct, scenario (i) should be followed. If ν is discarded, $R_m^c = \text{constant}$ indeed follows from dimensional analysis. It is thus not surprising that numerical models that do not resolve viscous scales, all gives this result, although the value of the constant seems to be strongly dependent on the flow geometry and on the model. We conclude that if dynamo action is observed for $P_m \ll 1$, the dynamo threshold is

$$R_m^c \rightarrow \text{constant when } Re \rightarrow \infty. \quad (9)$$

However, we emphasize that no clear-cut demonstration of dynamo action by homogeneous isotropic and parity invariant turbulence exists for $P_m \ll 1$. Experimental demonstrations as well as direct numerical simulations all involve a mean flow and analytical methods extrapolated to $P_m \ll 1$ are questionable.

It may be instructive at this stage to recall the study on turbulent dynamos made more than half a century ago by Batchelor [4]. Using a questionable analogy between the induction and the vorticity equations, he claimed that the dynamo threshold corresponds to $P_m = 1$, i.e. $R_m^c \propto Re$, using our choice of dimensionless parameters (scenario (ii)).

It is now often claimed that Batchelor's criterion $P_m > 1$ for the growth of magnetic energy in turbulent flows is incorrect. However, the weaker criterion $P_m > \text{constant}$ (scenario (ii)) has not yet been invalidated

by direct numerical simulations or by an experimental demonstration without mean flow. It is thus of interest to determine the minimal hypothesis for which Batchelor's predictions for dynamo onset is obtained using dimensional arguments. To wit, assume that the dynamo eigenmodes develop at small scales such that the threshold does not depend on the integral scale L . Then, discarding L in our set of parameters, dimensional analysis gives at once $P_m = P_m^c$ = constant for the dynamo threshold, i. e.

$$R_m^c \propto Re. \quad (10)$$

It has been sometimes claimed that a non zero mean flow is necessary to get a dynamo following scenario (i). However, we note that even for a slow dynamo, i.e., growing on a diffusive time scale, the largest scales look stationary for a dynamo mode at wave length l_σ . For Kolmogorov turbulence, we indeed have, $\mu_0 \sigma l_\sigma^2 / (L/V) \propto R_m^{-1/2} \ll 1$. This remains true for a k^{-p} spectrum for $p < 3$.

3. Mean magnetic energy density

Dimensional arguments can be also used to determine scaling laws for the mean magnetic energy density. For $P_m \ll 1$ (scenario (i)), discarding ν gives

$$\frac{B^2}{\mu_0} = \rho V^2 g_0(R_m), \quad (11)$$

where g_0 is an arbitrary function. Close to threshold, the *rms* velocity V is given by $\mu_0 \sigma V L \sim R_m^c$. In the case of a supercritical bifurcation, $g_0(R_m) \propto R_m - R_m^c$, and we obtain [12]

$$B^2 \propto \frac{\rho}{\mu_0 (\sigma L)^2} (R_m - R_m^c). \quad (12)$$

Far from threshold, $Re \gg R_m \gg R_m^c$, one could assume that B no longer depends on σ provided that the magnetic field mostly grows at scales larger than l_σ . We then obtain equipartition between magnetic and kinetic energy densities,

$$B^2 / \mu_0 \propto \rho V^2, \quad (13)$$

as assumed by Biermann and Schlüter [15].

A completely different result is obtained in scenario (ii). Let us first recall that according to Batchelor's analogy between magnetic field and vorticity [4], the magnetic field should be generated mostly at the Kolmogorov scale, $l_K = L Re^{-3/4}$, where the velocity gradients are the strongest. He then assumed that saturation of the magnetic field takes place for $\langle B^2 \rangle / \mu_0 \propto \rho v_K^2 = \rho V^2 / \sqrt{Re}$, where v_K is the velocity increment at the Kolmogorov scale, $v_K^2 = \sqrt{\nu \epsilon}$. $\epsilon = V^3 / L$ is the power per unit mass, cascading from L to l_K in the Kolmogorov description of turbulence.

This can be easily understood. $\epsilon = V^3 / L$ being the power per unit mass available to feed the dynamo, it may be a wise choice to keep it, instead of V in our set of parameters, thus becoming $B, \rho, \epsilon, L, \nu, \mu_0$ and σ . Then, if we consider dynamo modes that do not depend on L , we obtain at once

$$\frac{B^2}{\mu_0} = \rho \sqrt{\nu \epsilon} h(P_m) = \frac{\rho V^2}{\sqrt{Re}} h(P_m) \quad (14)$$

for saturation, where $h(P_m)$ is an arbitrary function of P_m . Close to dynamo threshold, $P_m \approx P_m^c$, we have $h(P_m) \propto P_m - P_m^c$ if the bifurcation is supercritical. Only the prefactor $\rho V^2 / \sqrt{Re}$ of (14) is the kinetic energy at Kolmogorov scale, that was assumed to be in equipartition with magnetic energy in Batchelor's prediction. This class of dynamos being small scale ones, it is not surprising that the inertial range of turbulence screens the magnetic field from the influence of integral size, thus L can be forgotten. We emphasize that a necessary condition for Batchelor's scenario is that the magnetic field can grow below the Kolmogorov scale, i.e. its dissipative length l_σ should be smaller than l_K , thus $P_m > 1$.

There is obviously a strong discrepancy between (13) and (14). The prefactors in these two laws are the upper and lower limits of a continuous family of scalings that are obtained by balancing the magnetic energy with the kinetic energy at one particular length scale within the Kolmogorov spectrum. It is not known if one of them is selected by turbulent dynamos.

4. Ohmic losses

Ohmic losses due to currents generated by dynamo action give a lower bound to the power required to feed a dynamo. In order to evaluate them, it is crucial to know at which scales the magnetic field grows. Assuming that a dynamo is generated in the case $P_m \ll 1$ (scenario (i)), we want to give a possible guess for the power spectrum $|\hat{B}|^2$ of the magnetic field as a function of the wave number k and the parameters ρ , ϵ , L , ν , μ_0 and σ . Far from threshold, $Re \gg R_m \gg R_m^c$, the dissipative lengths are such that $l_K \ll l_\sigma \ll L$. For k in the inertial range, i.e. $kl_\sigma \ll 1 \ll kL$, we may use a Kolmogorov type argument and discard L , σ and ν . Then, only one dimensionless parameter is left, and not too surprisingly, we get

$$|\hat{B}|^2 \propto \mu_0 \rho \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}. \quad (15)$$

This is only one possibility among many others proposed for MHD turbulent spectra within the inertial range, but it is the simplest. Integrating over k obviously gives the equipartition law (13) for the magnetic energy. It is now interesting to evaluate Ohmic dissipation. Its dominant part comes from the current density at scale l_σ . We have

$$\frac{j^2}{\sigma} = \frac{1}{\sigma} \int |\hat{j}|^2 dk \propto \frac{1}{\mu_0^2 \sigma} \int k^2 |\hat{B}|^2 dk \propto \frac{\rho}{\mu_0 \sigma} \epsilon^{\frac{2}{3}} l_\sigma^{-\frac{4}{3}} \propto \rho \frac{V^3}{L}. \quad (16)$$

We thus find that Ohmic dissipation is proportional to the total available power which corresponds to some kind of optimum scaling law for Ohmic dissipation. Although, this does not give any indication that this regime is achieved, we note that the above scaling corresponds to the one found empirically from a set of numerical models [16]. Their approximate fit, $(B^2/\mu_0)/(j^2/\sigma) \propto L/V$, indeed results from equations (15, 16).

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