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Saturation of the magnetic field above the dynamo threshold

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Abstract. We show that the saturation level of the magnetic field generated by a fluid dynamo just above threshold, can be estimated using simple dimensional arguments when the flow is turbulent. We then discuss other scalings by considering the structure of the weakly nonlinear problem above the dynamo threshold. Finally, we compare these predictions with recent experimental results.

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1 Introduction

The kinematic dynamo problem is rather well understood in the case of laminar flows [1]. Several simple but clever examples have been considered in the past [2–5] and have led to the experimental realization of various types of homogeneous dynamos: the two rotor-dynamo operated by Lowes and Wilkinson [6] following the configuration proposed by Herzenberg [2], the "Riga dynamo" [7] using a Ponomarenko type flow [5], and the "Karlsruhe dynamo" [8] that consists of a constrained flow that mimics a periodic flow proposed by Roberts [4]. In the last two experiments, the saturation level of the magnetic field, due to the back reaction of the Lorentz force on the flow, has been measured. Our goal is to find how this saturation level scales with the fluid parameters. We first recall that an experimental dynamo generally bifurcates from a strongly turbulent flow and operate close to threshold; then, we show that, rather surprisingly, dimensional analysis together with reasonable arguments, allow to predict the scaling of the saturated magnetic field above the dynamo threshold. We then understand this scaling, and also the "laminar one" that would be observed if the fluid magnetic Prandtl number were larger that one, by considering the structure of the perturbative analysis describing the saturation of the magnetic field above the dynamo threshold. Finally, we compare our predictions with the measurements performed in the recent experiments and also comment on the case of a rapidly rotating fluid.

2 Gouverning equations and dimensional arguments

The equations governing the magnetic and velocity fields, $\mathbf{B}(\mathbf{r},t)$, $\mathbf{v}(\mathbf{r},t)$, in the MHD approximation are (1)

$$\nabla\cdot\mathbf{B}=0, \tag{1}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B})+\frac{1}{\mu_0\sigma}\Delta\mathbf{B}, \tag{2}$$

$$\nabla\cdot\mathbf{v}=0, \tag{3}$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla\left(p+\frac{B^2}{2\mu_0}\right)+\nu\Delta\mathbf{v}+\frac{1}{\mu_0\rho}(\mathbf{B}\cdot\nabla)\mathbf{B}, \tag{4}$$

where $p(\mathbf{r},t)$ is the pressure field, μ_0 is the magnetic permeability of vacuum, σ is the fluid electric conductivity, ν is its kinematic viscosity and ρ is its density. With a characteristic velocity V of the solid boundaries driving the fluid motion, and a characteristic integral scale L of the flow, we can define two independent dimensionless numbers, the magnetic Reynolds number, $R_\mathrm{m}=\mu_0\sigma LV$, and the magnetic Prandtl number, $P_\mathrm{m}=\mu_0\sigma\nu$. We have thus, $R_\mathrm{m}=ReP_\mathrm{m}$, where Re is the kinetic Reynolds number of the flow. For most known fluid dynamos, the dynamo threshold R_mc is roughly in the range 10–100. For liquid metals, $P_\mathrm{m}<10^{-5}$, thus the kinetic Reynolds number at dynamo onset is larger than 10^6 and, consequently, the flow is strongly turbulent. The power needed to maintain this flow thus scales like $P\propto\rho L^2V^3$ and we have

$$R_\mathrm{m}\propto\mu_0\sigma\left(\frac{PL}{\rho}\right)^{1/3}. \tag{5}$$

This formula has simple consequences: first, taking liquid sodium (the liquid metal with the highest electric conductivity), $\mu_0\sigma\approx 10\ \mathrm{m^{-2}s}$, $\rho\approx 10^3\ \mathrm{kg\,m^{-3}}$, and with

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a typical lengthscale $L \approx 1$ m, we get $P \approx R_{\mathrm{m}}^3$; thus a mechanical power larger than 100 kW is needed to reach a dynamo threshold of the order of 50. Second, it is unlikely to be able to operate experimental dynamos at R_{m} large compared with R_{mc}. Indeed, it costs almost 10 times more power to reach $2R_{\mathrm{mc}}$ from the dynamo threshold. In conclusion, most experimental dynamos should have the following characteristics:

(i) they bifurcate from a strongly turbulent flow regime,
(ii) they operate in the vicinity of their bifurcation threshold.

Although (i) makes almost impossible any realistic analytical calculation or direct numerical simulation, we will show that the above two characteristics allow an estimation of the nonlinearly saturated magnetic field above R_{mc} using dimensional analysis. Our goal is thus to find the expression of B as a function of ρ, ν, μ_0, σ, L and V. We have three independent parameters, R_{m}, P_{m} and for instance the square of the Lundquist number, $B^2\mu_0(\sigma L)^2/\rho$, thus we have in general

$$\frac{B^2\mu_0(\sigma L)^2}{\rho} = f(R_{\mathrm{m}}, P_{\mathrm{m}}). \tag{6}$$

Different scaling laws have been found for B using mean-field electrodynamics [9–11] or weakly nonlinear perturbations close to the dynamo threshold for laminar flows [10,12–17]. We will show that the two above assumptions can be used to find the scaling law for the realistic situation of a turbulent dynamo close to threshold: (i) implies that the momentum is mostly transported by turbulent fluctuations. Consequently, using the basic assumption of fully developed turbulence, we can neglect the kinematic viscosity, thus P_{m}. (ii) implies that the dependence of B^2 in R_{m} is proportional to $R_{\mathrm{m}} - R_{\mathrm{mc}}$, as expected for a supercritical bifurcation close to threshold. In other words, V is not a free parameter anymore, but should take approximately the value corresponding to the dynamo threshold. Thus, (i) and (ii) reduce the number of parameters from 6 to 4, and the saturated value of the magnetic field can be obtained using dimensional analysis

$$B^2 \propto \frac{\rho}{\mu_0(\sigma L)^2}\,(R_{\mathrm{m}} - R_{\mathrm{mc}}). \tag{7}$$

There is no paradox in the fact that the saturated magnetic field is inversely proportional to the square electric conductivity and to the square of the typical lengthscale of the flow. This does not mean that one should have σ and L small in order to observe large values of B since $R_{\mathrm{m}} = R_{\mathrm{mc}}$ will be then achieved for a larger flow velocity. Using the typical velocity V_{c} at dynamo threshold, we can write (7) in the form, $B^2/\mu_0\rho V_{\mathrm{c}}^2 \propto (R_{\mathrm{m}} - R_{\mathrm{mc}})/R_{\mathrm{mc}}^2$, which shows that the system is very far from equipartition of energy in the vicinity of the dynamo threshold. We emphasize also, that the interaction parameter, *i.e.* the ratio of the Lorentz force to the pressure force driving the flow, is much smaller than one.

3 Laminar versus turbulent scalings

We can understand the above scaling together with the very different one obtained if $P_{\mathrm{m}} \gg 1$, by looking at the structure of the perturbative analysis describing the saturation of the magnetic field above the dynamo threshold. This calculation is tractable only for $P_{\mathrm{m}} \gg 1$ in general such that the dynamo bifurcates from a laminar flow. For $P_{\mathrm{m}} \ll 1$, a lot of hydrodynamic bifurcations occur first and the flow becomes turbulent before the dynamo threshold. The structure of the weakly nonlinear analysis above threshold is as follows: the dynamo bifurcates from a flow field $\mathbf{v}_{\mathrm{c}}$ at $R_{\mathrm{m}} = R_{\mathrm{mc}}$. We write (2) in the form

$$\mathcal{L} \cdot \mathbf{B}_0 = 0, \tag{8}$$

where $\mathbf{B}_0$ is the neutral mode at threshold and $\mathcal{L}$ is a linear operator that depends on the bifurcation structure (stationary or Hopf bifurcation). Slightly above threshold, we have

$$\mathbf{v} = \mathbf{v}_{\mathrm{f}} + \epsilon\mathbf{v}_1 + \cdots, \tag{9}$$

where $\mathbf{v}_{\mathrm{f}} = \mathbf{v}_{\mathrm{c}} + \epsilon\mathbf{v}_{\mathrm{p}} + \cdots$, with $\epsilon = (R_{\mathrm{m}} - R_{\mathrm{mc}})/R_{\mathrm{mc}} \ll 1$. $\epsilon\mathbf{v}_{\mathrm{p}}$ is the order ϵ velocity correction due to the driving of the fluid slightly above threshold. $\epsilon\mathbf{v}_1$ is the leading order flow distortion by the Lorentz force. We have for $\mathbf{B}$

$$\mathbf{B} = \sqrt{\epsilon}\,(\mathbf{B}_0 + \epsilon\mathbf{B}_1 + \cdots). \tag{10}$$

We first compute $\mathbf{v}_1$ from (4) at order ϵ,

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_{\mathrm{c}} \cdot \nabla)\mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_{\mathrm{c}} = -\frac{1}{\rho}\nabla\left(p_1 + \frac{B_0^2}{2\mu_0}\right) + \nu\Delta\mathbf{v}_1 + \frac{1}{\mu_0\rho}(\mathbf{B}_0 \cdot \nabla)\mathbf{B}_0. \tag{11}$$

If $P_{\mathrm{m}} \gg 1$, the flow is laminar at the dynamo threshold, and the Lorentz force is mostly balanced by the modification of the viscous force, thus

$$v_1 \propto \frac{B_0^2 L}{\mu_0\rho\nu}\,\cdot \tag{12}$$

We get from (2) at order ϵ,

$$\mathcal{L} \cdot \mathbf{B}_1 = \frac{\partial \mathbf{B}_0}{\partial T} - \nabla \times (\mathbf{v}_{\mathrm{p}} \times \mathbf{B}_0) - \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0), \tag{13}$$

where $T = \epsilon t$ is the slow time scale of $\mathbf{B}_0$ slightly above threshold. The amplitude equation for $\mathbf{B}_0$ that governs the saturation of the magnetic field is obtained by applying the solvability condition to (13),

$$\left\langle \mathbf{C} | \frac{\partial \mathbf{B}_0}{\partial T} \right\rangle = \langle \mathbf{C} | \nabla \times (\mathbf{v}_{\mathrm{p}} \times \mathbf{B}_0) \rangle + \langle \mathbf{C} | \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) \rangle, \tag{14}$$

where $\mathbf{C}$ is the eigenvector of the adjoint problem [17]. The first term on the right hand side of (14) corresponds to the linear growth rate of the magnetic field whereas the second describes the nonlinear saturation due to the modified velocity field $\mathbf{v}_1$. For nonlinearly saturated solutions, we thus get $v_\mathrm{p} \propto v_1$. In the vicinity of threshold, $\mu_0 \sigma L(v_\mathrm{f} - v_\mathrm{c}) \propto R_\mathrm{m} - R_\mathrm{mc}$, and we obtain

$$B^2 \propto \frac{\rho\nu}{\sigma L^2}\,(R_\mathrm{m} - R_\mathrm{mc}). \qquad (15)$$

We call (15) the "the laminar scaling", characterized by the fact that $B \to 0$ if $\nu \to 0$ with all the other parameters fixed. We have derived this formula for a Ponomarenko type flow in [17]. The main difficulty is that, in general, the dynamo problem is not self-adjoint and that the adjoint problem is not a dynamo problem [18].

For $P_\mathrm{m} \ll 1$, we recover the "turbulent scaling" (7) if we balance the Lorentz force with the inertial terms in (11). We have $B_\mathrm{laminar} \propto B_\mathrm{turbulent}\, P_\mathrm{m}^{1/2}$, thus the two scalings strongly differ for experiments using liquid metals ($P_\mathrm{m} < 10^{-5}$).

4 Discussion and concluding remarks

The Karlsruhe [8] and Riga [7] experiments have recently reported values of the saturated magnetic field of order 10 mT roughly 10% above threshold. Both experiments used liquid sodium ($\mu_0\sigma \approx 10\ \mathrm{m}^{-2}\,\mathrm{s}$, $\rho \approx 10^3\ \mathrm{kg\,m}^{-3}$). The inner diameter of the Riga experiment is $L = 0.25$ m. The spatial periodicity of the flow used in the Karlsruhe experiment is of the same order of magnitude, within a cylinder of radius 0.85 m and height 0.7 m. The presence of two lengthscales in the Karlsruhe experiment makes the comparison with our analysis more difficult, but we can easily compare the results of the Riga experiment with our "turbulent" (7) and "laminar" scalings (15), that predict a saturated field of order 10 mT (respectively 10 μT). Taking into account the qualitative nature of our analysis, we conclude that the "turbulent scaling" is in agreement with the experimental observations whereas the "laminar scaling" predicts a field that is orders of magnitude too small. The "turbulent scaling" also gives a correct order of magnitude for the Karlsruhe experiment if its spatial period is taken as the relevant lengthscale in (7). We thus note that the above experiments display a very interesting feature: turbulent fluctuations can be neglected when computing the dynamo threshold; indeed, the observed thresholds are in rather good agreement with the ones predicted by solving the kinematic dynamo problem for the mean flow alone. However, turbulence has a very strong effect on the value of the saturated magnetic field above the dynamo threshold.

It is also tempting to apply the above arguments to the geodynamo. Indeed, contrary to stellar or galactic magnetic fields that involve R_m very large compared to R_mc, it is likely that for the Earth, R_m is at most a few times R_mc. Several models have been considered in the past, all involving laminar flows with scale separation, and the saturated magnetic field has been computed using a weakly nonlinear analysis [12–15]. Although these models involve more dimensionless parameters than the system of equations (1–4) because of the convective driving of the flow, the rotation of the Earth and the different lengthscales of the basic laminar flow, most of them share the property of our "laminar scaling", *i.e.* $B \to 0$ if $\nu \to 0$ with all the other parameters fixed. The same is also true for dynamos generated by a flow driven by a periodic body force in space [10] or for Ponomarenko type dynamos, in the limit $Re \gg R_\mathrm{m} \gg 1$ [16] or close to the instability threshold [17]. As mentioned above, this occurs in a laminar flow as soon as the Lorentz force is balanced by the modification of the viscous force in (11). On the contrary, if the flow is turbulent, the momentum is dominantly transfered by turbulent fluctuations and it is more realistic to balance the Lorentz force with the inertial terms in (11); the "turbulent scaling" is then obtained. In the case of rapidly rotating fluids, the Coriolis term, $-2\boldsymbol{\Omega}\times\mathbf{v}$ should be taken into account in (11). If we get $\mathbf{v}_1$ by balancing the Lorentz force with the modification of the Coriolis force, we obtain for the saturated magnetic field in the vicinity of threshold

$$B^2 \propto \frac{\rho\Omega}{\sigma}\,(R_\mathrm{m} - R_\mathrm{mc}). \qquad (16)$$

In the case of the Earth core, $\sqrt{\rho\Omega/\sigma} \approx 10$ gauss, thus (16) gives a reasonable order of magnitude of the Earth field if $R_\mathrm{m} - R_\mathrm{mc}$ is small enough. The other scalings (15) and (7) give too small values of the field even if L is chosen small compared to the radius of the Earth core. In the context of the geodynamo, the prefactors in (15) and (16) are called the "weak field" (respectively "strong field") scalings [19]. Note however that the full expression (16) does not correspond to the "strong field" balance of the geodynamo $\sigma B^2 \approx \rho\Omega$. Our scaling (16) may be obtained in the vicinity of the dynamo threshold if one assumes that the Stokes force is negligible in (11) whereas it is believed that the "strong field" regime of the geodynamo occurs at finite amplitude *via* a subcritical bifurcation from the "weak field" one [19].

In conclusion, we emphasize that the correct evaluation of the dominant transport mechanism for momentum is essential to estimate the order of magnitude of the saturated magnetic field above the dynamo threshold. The reason is that it determines the flow distortion by the Lorentz force and thus the saturation mechanism of the field. A mean field laminar model of the flow may thus lead to a wrong estimate of the magnetic field although it sometimes correctly predicts the dynamo threshold. It would be interesting to test the validity of the scaling law (7) experimentally by varying the temperature of liquid sodium and thus its conductivity σ. Another fundamental experiment in the context of the geodynamo would be to observe the transition from (7) to (16) for a rapidly rotating flow.

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