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Special Issue on Statphys 26

Instabilities on a turbulent background

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Abstract. We present a review of several experimental results that concern the problem of hydrodynamic instabilities occurring in turbulent flows. The first experiment is related to the generation of a magnetic field by a turbulent flow of liquid sodium, i.e. the dynamo effect. We show how the bifurcation to the dynamo regime is affected by turbulent fluctuations. Above the dynamo threshold, we study the reversal dynamics of the magnetic field and present a model showing the respective contributions of deterministic and stochastic aspects. The second experiment is a nearly two-dimensional turbulent flow forced in a confined domain. We study a sequence of bifurcations that involve a large scale flow that develops over a turbulent background and displays random reversals. We emphasize both the similarities and differences with the magnetic reversal problem.

Keywords: nonlinear dynamics, stochastic processes

Contents

1.	Introduction	2
2.	Bifurcation in the presence of multiplicative noise: a canonical model	4
3.	Generation of a magnetic field by a turbulent flow of liquid metal 3.1. Introduction: the dynamo effect. 3.2. The VKS experiment 1 3.3. Role of the iron impellers 1 3.4. Dynamics of the magnetic field generated by a von Kármán flow of liquid sodium 1 3.5. Reversals of the magnetic field 1 3.5.1. Modes and symmetries. 1 3.5.2. A simple case. 1 3.5.3. Effect of fluctuations.	8 8 .0 .0 .0 .2 .4 .4 .5 .6
4.	3.6. Geomagnetic reversals 1 Two-dimensional turbulence forced in a confined domain 1 4.1. Introduction 1 4.2. Experimental set-up and dimensionless parameters 1	.8 .8 .9
	 4.3. Turbulent transitions at the edge of the regime with large scale flow reversals	$\frac{21}{23}$
5.	Conclusion 2	5
	Acknowledgments 2	6
	References 2	6

1. Introduction

Hydrodynamic instabilities have been studied for more than a century and are well documented experimentally as well as theoretically. During the past decades, the emphasis has been put on nonlinear aspects related to pattern formation, transition to chaos and turbulence. Bifurcation theory has been developed and provides a useful tool to study the stability of stationary and time or spatially periodic flows. Much less studies exist on instabilities that occur when a control parameter is varied within the turbulent regime and theoretical tools are lacking to handle these problems. However, we know several examples of transitions displayed by turbulent flows. The oldest example is provided by the drag crisis [1]. The mean drag of a sphere or a cylinder in a turbulent flow suddenly drops for a critical value of the Reynolds number Re of order 10⁵. This corresponds to a transition where the mean flow pattern changes, the wake becoming narrower. More recent experiments have been conducted on instabilities of turbulent wakes. It has been found that two different configurations for the mean flow can coexist and the wake can display random switches between them [2]. Another example is related to Rayleigh–Bénard convection, i.e. the flow generated by heating from below a horizontal layer of fluid. It has been observed that for a Rayleigh number of order 10^6 , i.e. roughly 1000 times larger than the critical Rayleigh number for the onset of convection, a large scale flow is generated with a horizontal extension equal to the length of the container [3]. A more recent example concerns the effect of rotation on turbulent convection: for a large enough Rayleigh number, it has been found that increasing the rotation rate does not affect the heat flux up to a critical rotation rate above which it starts increasing as if a supercritical bifurcation has occured [4]. Von Kármán swirling flows, i.e. flows generated in a cylindrical volume by the rotation of two co-axial disks, also display transitions within the turbulent regime. In the case of co-rotating disks, an axisymmetric mean flow with a strong axial vortex is observed. When the rotation rates are varied, this flow breaks axisymmetry, thus generating a roughly periodic modulation of the turbulent velocity field superimposed to turbulent fluctuations with $\text{Re} \sim 10^5$ [5]. In the case of disks counter-rotating at the same frequency, the forcing is symmetric with respect to a rotation of angle π about any radial axis in the mid-plane between the two disks (see section 3.5). It has been found that this symmetry can be broken through a bifurcation that occurs for Reynolds numbers in the range 10^5 to 10^6 [6, 7].

It is often believed that a strongly turbulent flow has the symmetries of the flow domain and forcing. This should be of course understood with a statistical meaning, i.e. for all the moments of the velocity field. The argument to justify this claim is that the system is likely to explore all the available phase space if the turbulent fluctuations are strong enough. The above examples show that this is not always true. When some control parameter is changed, for instance the Reynolds number, a strongly turbulent flow with the symmetries of the flow domain and forcing, can bifurcate to another turbulent regime and spontaneously break some of the symmetries. This transition sometimes occurs through the generation of a large scale mean flow, $\langle \mathbf{V} \rangle \neq 0$, as in the turbulent convection example quoted above ($\langle \cdot \rangle$ stands for the spatial average). Of course both signs of the large scale velocity are possible if the forcing does not impose a particular direction. If the two turbulent attractors related to $\pm \langle \mathbf{V} \rangle$ are disconnected in phase space, one solution is selected at the bifurcation and the symmetries are broken. Another bifurcation can occur when the control parameter is changed further and reconnect the two attractors related by the symmetry transformation. The mean flow then jumps, generally randomly in time, between the opposite values of $\langle \mathbf{V} \rangle$. These reversals of the large scale flow on a turbulent background statistically restore the symmetries of the flow domain and forcing.

The dynamo effect, i.e. the generation of a magnetic field by the flow of an electrically conducting fluid, provides a nice example in this context. The magnetic field is generated by an instability process related to electromagnetic induction. For any liquid metal, the instability threshold is reached when the flow is strongly turbulent (see section 3). We thus have an instability on a fully turbulent background. The broken symmetry at instability threshold is the $\mathbf{B} \to -\mathbf{B}$ symmetry of the equations of magnetohydrodynamics. This symmetry can be restored statistically if the system undergoes a secondary bifurcation to a regime that involves periodic or random reversals of the magnetic field. Although dynamo experiments involve technical problems such that the dynamo threshold is difficult to reach in the laboratory, this example is conceptually simpler than the purely hydrodynamic ones quoted above. Once it bifurcates, the magnetic field is easy to measure and, in the vicinity of threshold, it evolves on a slow time scale compared to the ones of the flow. In contrast, in purely hydrodynamical systems, it is very difficult to disentangle the bifurcating modes from the turbulent background.

Another class of transitions between different turbulent regimes concern shear flows. The laminar flow often becomes unstable through a subcritical bifurcation giving rise to localized patches of turbulence within the laminar flow. A homogeneous turbulent flow is observed only at higher Reynolds number after a sequence of bifurcations that depends on the geometry of the walls that confine the flow [8]. Although these transitions involve the dynamics of fronts between different states as usually observed in the vicinity of subcritical bifurcations in extended systems, a common feature with the transitions mentioned above is that the turbulent regimes either possess the symmetries of the experimental device or break some of them. For instance, in planar Couette flows, the turbulent regime occurs within periodic stripes at intermediate Reynolds numbers, i.e. breaks translational symmetry [9]. A similar phenomenon involving a transition between two different turbulent regimes, has been observed earlier in Taylor-Couette flow [10]. We will not consider this type of transitions here.

This paper is organized as follows: in section 2, we will start by reviewing a simple model of a bifurcation in the presence of noise. This model mimics a bifurcation that occurs on a turbulent regime. We will consider a pitchfork bifurcation with a noisy bifurcation parameter and show that the qualitative behavior of the system close to threshold strongly differs from the one further from threshold: in particular, anomalous scaling of the different moments and 1/f-type noise are observed close to threshold. We will review some experimental results of the VKS dynamo in section 3 and consider similarities and differences of the dynamo bifurcation with the pitchfork bifurcation with multiplicative noise. We will then consider reversals of the magnetic field and show that a model with a low dimensional dynamical system can describe most of the experimental observations. A purely hydrodynamical examples will be considered in section 4: a nearly two-dimensional flow generated by a spatially periodic forcing in a confined domain. We will study transitions between different turbulent regimes. Finally, concluding remarks will be given in section 5.

2. Bifurcation in the presence of multiplicative noise: a canonical model

We consider a canonical model of bifurcation subject to multiplicative noise. It consists in the evolution equation of a scalar x, which is a function only of time t and satisfies the Langevin equation

$$\dot{x} = \mu_0 x - x^3, \quad \mu_0 = \mu + \xi(t).$$
 (1)

where μ is the parameter that controls the instability and ξ , the noise term, represents random fluctuations with zero mean.

The magnetic field generated by the dynamo instability and certain chemical reactions are systems for which such multiplicative noises are involved. More generally, close to the onset of any instability that breaks an existing $x \to -x$ symmetry, equation (1) is the evolution equation of the unstable mode amplitude if the departure from onset fluctuates for instance because of loose experimental control.

In the absence of noise, the solution x = 0 becomes unstable for $\mu > 0$ and at long time x tends towards $\pm \sqrt{\mu}$.

To model the noise term, we assume that ξ is a Gaussian delta-correlated noise, i.e. $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$. Because the noise acts multiplicatively in the evolution equation, one needs to define precisely the meaning of the equation. From now on, in the case that ξ is white noise, we use the Stratanovich interpretation [11]. Had we used the Ito interpretation, we should replace in the following μ by $\mu - D$. Note that x(t) cannot cross 0 and never changes sign. Without loss of generality, we consider x(0) > 0 and therefore x(t) > 0.

The probability distribution function (PDF) of x, P(x), satisfies the Fokker-Planck equation. For $\mu > 0$ its steady solution is given by

$$P(x) = \mathcal{N}x^{\frac{\mu}{D} - 1} e^{\frac{-x^2}{2D}}$$
(2)

where the normalization constant \mathcal{N} is set by the condition $\int P(x) dx = 1$ [12]. For $\mu < 0$, it is $P(x) = \delta(x)$. The moments can be calculated and are non zero for positive μ . Due to the singularity of P at x = 0, we obtain $\langle x^m \rangle \propto \mu$ in the small μ -limit. For what concerns the most probable value, it is zero for $0 \leq \mu \leq D$ and is equal to $(\mu - D)^{1/2}$ for $D \leq \mu$.

The behavior of the moments (all linear in μ) is an effect of the multiplicative noise on the dynamics of x close to the onset. Indeed, the time series alternates between phases where the value of x is large and nonlinearities are important (on-phases) and phases where x fluctuates close to zero (off-phases) [13–15]. This behavior is called on-off intermittency and an example of time series is displayed in figure 1.

This phenomenon was first identified in the study of coupled chaotic dynamical systems. At strong coupling, the solutions follow the same trajectories and the distance between two solutions vanishes. At zero coupling, the trajectories are independent and the distance between trajectories fluctuates around a value that is set by the structure of the chaotic attractor. At intermediate coupling, one observes phases where the trajectories are the same and the distance between them vanishes, and phases of burst into a state of desynchronised trajectories, with large distance. Close to the onset of appearance of this intermittent behavior, the evolution equation of the distance between trajectories has the same linear term as equation (1) and was first studied in [13, 14]. The name 'on-off intermittency' was given later in the context of the dynamics of an unstable mode close with its onset in the presence of multiplicative fluctuations [15]. This behavior is associated to several properties that we will discuss in the following. When these properties are controlled by the off-phases, they are generic in the sense that they do not depend on the nonlinear term.

During an off-phase, x is very small, the nonlinear term can be neglected and the time series satisfy



Figure 1. Time series of x(t) solution of equation (1) in the regime of on-off intermittency. The top figure uses a log scale for x. The horizontal line is an arbitrary threshold, corresponding to $x = 10^{-2}$, so that if x is below this threshold, it appears as nearly zero in the bottom figure that uses a linear scale. The numerical solutions are obtained using the Stratanovich interpretation of equation (1) and the associated temporal discretization.

$$\log(x(t)) = \log(x(0)) + \mu t + \int_0^t \zeta(t') dt'.$$
(3)

Averaging over the realizations of the noise, the stochastic term disappears and we conclude that $\langle \log(x) \rangle$ tends toward $-\infty$ if μ is negative and towards $+\infty$ if μ is positive. $\mu = 0$ thus corresponds to the onset of instability of x = 0. For small and positive μ , $\log(x)$ is a random walk with a bias towards positive values. This is true only for small x. For large x, the nonlinear effects prevent the random walk from reaching too large positive values. This is displayed in figure 1. The origin of the on-off behavior is easily understood: the phases where $\log(x)$ is smaller than a given threshold (here arbitrarily taken to $x = 10^{-2}$) appear as nearly zero for x.

In addition to this prediction that results from the analogy with a biased random walk, we can make the following predictions for small drift. For $\mu = 0$ and small x, if a steady distribution were to exist for $\log(x)$, it should be uniform as there is no preferred direction for the evolution in the off-phase. Then the distribution of x should be $P(x) \propto x^{-1}$ which is not normalizable. It can thus not be the equilibrium distribution; at long time, x tends to zero and the steady distribution is $P(x) \propto \delta(x)$. For positive μ and small x, we have $P(x) \propto x^{\mu/D-1}$; for large x, in the on-phase, the distribution displays a cut-off due to the nonlinear effects. The divergent behavior of P at small xresults from the off-phases, during which x reaches nearly vanishing values.

The duration of the off-phases is also an important property that can be understood from the analogy with a biased random walk [16]. Indeed, during an off-phase, x remains below a given threshold, say x_t , until the random walk reaches again x_t . The duration of the off-phase is thus the same as the one of the return time of a weakly biased Brownian motion. The distribution of this quantity is a well-known property



Figure 2. Power spectrum density of x solution of equation (1): (cyan), $\mu = 1$, the spectrum is a Lorentzian; (red), $\mu = 0.01$; (black), $\mu = 0.004$; (magenta) $\mu = 0.002$ and (blue) $\mu = 0.001$. At high frequency, the spectrum is nearly f^{-2} , it is flat at low frequency and when μ is decreased, a $f^{-1/2}$ regime appears as indicated by the thick black line. For the smaller values of μ the crossover frequency to the flat behavior is smaller than the smallest frequency displayed in the figure.

and is of the form $P(T_{\text{off}}) \propto T_{\text{off}}^{-3/2}$ for small T_{off} with a cut-off at large T_{off} of the form² $\exp(-\mu^2 T_{\text{off}}/D)$.

The moments of x display an unusual scaling: they are linear in the departure from onset $\langle x^n \rangle \propto \mu$, which is different from the deterministic behavior $\langle x^n \rangle \propto \mu^{n/2}$. Obviously this scaling law results from the form of the PDF P(x) and more precisely from its divergent behavior for small x. We can also understand it from the time series of x. At $\mu = 0$, the mean duration of the off-phase $\langle T_{\text{off}} \rangle$ diverges, like the mean return time of an unbiased Brownian motion. At weak bias μ , we have $\langle T_{\text{off}} \rangle \propto \mu^{-1}$. As displayed in figure 1, it is clear that the off-phases do not contribute to the moments. In the limit of small μ , the duration of the on-phases and the value achieved by the moments do not depend on μ . With these properties, we can estimate the moments as $\langle x^n \rangle \simeq \langle T_{\text{on}} \rangle x_{\text{on}}^n / (\langle T_{\text{on}} \rangle + \langle T_{\text{off}} \rangle)$ and the only singular term is $\langle T_{\text{off}} \rangle$ that diverges as μ^{-1} , so that we recover the scaling of the moments that are linear in μ .

The power spectrum density of x, $S_x(f)$, displays also a rich behavior [13, 14]. At large μ , it is a Lorentzian function. Indeed, x fluctuates around some non zero mean value, say X. The dynamical equation contains the terms $\dot{x} = X\zeta(t) + \dots$ so that the noise acts as an additive term from which the f^{-2} behavior at large f is easily recovered.

At small μ , the spectrum displays three regimes (see figure 2). At large frequencies, for $f \gg D$, we recover a f^{-2} behavior similar to the former case. At very low frequencies for $f \ll \mu^2/D$, the spectrum is flat. In between these two regimes, it displays an unusual $f^{-1/2}$ power law. This behavior is related to the distribution of the off-phases $P(T_{\text{off}}) \propto T_{\text{off}}^{-3/2}$. We start by calculating $\langle T_{\text{off}} \rangle_T$ the duration of the off-phases for a time series of duration T. Because the distribution of T_{off} is large, this is estimated by restricting the PDF

² We note that the PDF is not normalizable at short T_{off} . This is related to a property of the Brownian motion due to the non-smoothness of the trajectory: when it reaches a given value, it crosses this value an infinite number of times. For what concerns the on-off intermittent regime in a real system, the noise term has a finite (possibly) short correlation time, say τ_c . The PDF then has a cut-off for time shorter than τ_c and is therefore normalizable.

to values smaller than T, so that $\langle T_{\text{off}} \rangle_T \simeq \int^T T_{\text{off}} P(T_{\text{off}}) dT_{\text{off}} \propto T^{1/2}$. The number of off-phases during T is $N(T) = T/\langle T_{\text{off}} \rangle_T \propto T^{1/2}$, which is also the number of on-phases. As their duration does not diverge, the duration spent in the on-phases behaves as $T^{1/2}$ and the probability to be in the on-phase as $T^{1/2}/T = T^{-1/2}$. As only the on-phases give a contribution to the autocorrelation function $C(t) = \langle x(t)x(0) \rangle$, we obtain $C(t) \propto t^{-1/2}$. Using Wiener–Khintchin theorem, we obtain $S_x(f) = \int C(t) \exp(i2\pi ft) dt \propto f^{-1/2}$. The power-law behavior of the spectrum is thus due to the self similar behavior of the time series and more precisely of the arrival of the on-phases. We can also note that the distribution of T_{off} has a cut-off at $T_c = D/\mu^2$, which sets the crossover frequency in the spectrum between the $f^{-1/2}$ and the flat, f^0 behavior.

We point out that there is a relation between the exponent of the power law of the distribution of the off-phase (say β so that $P(T_{\text{off}}) \propto T_{\text{off}}^{-\beta}$, with $\beta = 3/2$) and the exponent of the spectrum (say α , with $S_x(f) \propto f^{-\alpha}$, with $\alpha = 1/2$). Indeed, we have $\alpha = \beta - 1$ for $1 < \beta < 2$. As we will see in the following, this is one of the general laws that relate exponents of distribution of rare events and exponents of spectrum (see section 3).

3. Generation of a magnetic field by a turbulent flow of liquid metal

3.1. Introduction: the dynamo effect

It is strongly believed that many planetary and stellar magnetic fields are generated by a dynamo effect, i.e. an instability mechanism that results from electromagnetic induction by the flow of an electrically conducting fluid [17]. Maxwell's equations together with Ohm's law give the governing equation of the magnetic field, $\mathbf{B}(\mathbf{r}, \mathbf{t})$. In the approximation of magnetohydrodynamics (MHD), it takes the form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B},\tag{4}$$

where μ_0 is the magnetic permeability of vacuum and σ is the electrical conductivity. The last term on the right hand side of (4) represents ohmic dissipation, and the first one, electromagnetic induction due to the velocity field $\mathbf{v}(\mathbf{r}, \mathbf{t})$. $\mathbf{B} = 0$ is an obvious solution of (4), and for $\mathbf{v} = 0$, any perturbation of $\mathbf{B}(\mathbf{r}, \mathbf{t})$ (respectively of current density $\mathbf{j}(\mathbf{r}, \mathbf{t})$) decays to zero due to ohmic diffusion. $\mathbf{B} = 0$ can be an unstable solution if the induction term compensates ohmic dissipation. The ratio of these two terms defines the magnetic Reynolds number, $R_m = \mu_0 \sigma V L$, where V is the typical velocity amplitude and L the typical length scale of the flow. If $\mathbf{v}(\mathbf{r}, \mathbf{t})$ has an appropriate geometry, perturbations of magnetic field grow when R_m becomes larger than a critical value R_m^c (in the range 10–1000 for most studied examples). Magnetic energy is generated from part of the mechanical work used to drive the flow.

In order to describe the saturation of the magnetic field above the dynamo threshold R_m^c , we need to take into account its back reaction on the velocity field. $\mathbf{v}(\mathbf{r}, \mathbf{t})$ is governed by the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \left(\frac{p}{\rho} + \frac{B^2}{2\mu_0 \rho} \right) + \nu \nabla^2 \mathbf{v} + \frac{1}{\mu_0 \rho} (\mathbf{B} \cdot \nabla) \mathbf{B},\tag{5}$$

that we have restricted to the case of an incompressible flow ($\nabla \cdot \mathbf{v} = 0$). ν is the kinematic viscosity and ρ is the fluid density. In the MHD approximation, the Lorentz force, $\mathbf{j} \times \mathbf{B}$, can be split into the two terms involving **B** in (5). If the modification of the flow under the action of the growing magnetic field weakens the dynamo capability of the flow, the dynamo bifurcation is supercritical, i.e. the magnetic field grows continuously from zero when R_m is increased above R_m^c .

Assuming that the set of parameters defined so far fully characterizes the problem, we should have another independent dimensionless parameter besides R_m . We can choose either the kinetic Reynolds number, $Re = VL/\nu$, or the magnetic Prandtl number, $P_m = R_m/\text{Re} = \mu_0 \sigma \nu$. Then, dimensional analysis implies that we have, $R_m^c = f(P_m)$, for the dynamo threshold and $\langle B^2 \rangle = \mu_0 \rho V^2 g(R_m, P_m)$, for the mean magnetic energy generated above the dynamo threshold. f and g are arbitrary functions at this stage. Their dependence on P_m (or equivalently on Re) can be related to the effect of turbulence on the dynamo threshold and saturation. In many realistic situations, more parameters should be taken into account. For instance, f and g also depend on the choice of boundary conditions (for instance their electrical conductivity or their magnetic permeability, etc). In the context of planetary or stellar dynamos, the effect of the rotation rate Ω should be also taken into account via the Rossby, $Ro = V/\Omega L$ or Ekmann number $E = \nu / \Omega L^2$.

For planetary or stellar dynamos, as well as for any laboratory experiment performed with a liquid metal, we have $P_m < 10^{-5}$, the largest value being reached using liquid sodium. This has strong consequences on the dynamo bifurcation and makes the problem both difficult and interesting. P_m being the ratio of the diffusive time scales of the magnetic field and the velocity field, no direct simulation solving the full MHD equations can handle such small values. In addition, small P_m implies that the flow is strongly turbulent when the dynamo threshold is reached (Re ~ $R_m^c/P_m > 10^6$). The velocity field being fully turbulent when the dynamo threshold is reached for any experiment performed with a liquid metal, we can use the Reynolds decomposition and write $\mathbf{v}(\mathbf{r}, \mathbf{t}) = \mathbf{V}(\mathbf{r}) + \tilde{\mathbf{v}}(\mathbf{r}, \mathbf{t})$ where $\mathbf{V}(\mathbf{r})$ is the mean flow and $\tilde{\mathbf{v}}(\mathbf{r}, \mathbf{t})$ are the turbulent fluctuations. The over-bar stands for a temporal average in experiments. Therefore, both the mean flow $\mathbf{V}(\mathbf{r})$ and the fluctuations $\tilde{\mathbf{v}}(\mathbf{r}, \mathbf{t})$ are involved in the induction term of (4) and one has to understand their respective effects on the dynamo process. Dynamo experiments thus provide a way to study an instability problem from a fully turbulent state. Several interesting questions arise: does the generated magnetic field involve a mean large scale component, as observed in planetary or stellar dynamos? What are the behaviors of f and g when $P_m \to 0$? Are they constant with respect to P_m in this limit, thus giving $R_m^c = \text{constant}$ and $\langle B^2 \rangle \propto \mu_0 \rho V^2 g(R_m)$, i.e. $\langle B^2 \rangle \propto [\rho/(\mu_0 \sigma^2 L^2)] g(R_m)$ close to threshold [18]? What is the effect of turbulent fluctuations on the bifurcation? Is $g(R_m) \propto R_m - R_m^c$ as for a usual supercritical bifurcation close to threshold, or should we expect a behavior involving an anomalous exponent [19]? What is the effect of turbulent fluctuations on the dynamics of the magnetic field?

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3.2. The VKS experiment

Three successful fluid dynamo experiments have been performed so far: the Karlsruhe experiment [20], the Riga experiment [21] and the VKS experiment [22]. The VKS experiment differs from the two others as follows: the Karlsruhe and Riga experiments have been designed by geometrically constraining a mean flow $\overline{\mathbf{V}}(\mathbf{r})$ known for its efficient dynamo action, the Roberts' flow (respectively the Ponomarenko flow) for the Karlsruhe (respectively Riga) experiment. Thanks to geometrical constraints, turbulent fluctuations were roughly an order of magnitude smaller than the mean flow. The experimentally observed dynamo threshold as well as the geometry of the mean magnetic field, have been found in good agreement with a linear stability analysis based only the mean flow, i.e. discarding the fluctuations.

The VKS experiment consists of a von Kármán swirling flow of liquid sodium. It is generated in a cylinder by the motion of two coaxial counter-rotating disks fitted with eight blades as shown in figure 3 (left). The mean flow has the following characteristics: the fluid is ejected radially outward by the disks; this drives an axial flow toward the disks along their axis and a recirculation in the opposite direction along the cylinder lateral boundary. In the case of counter-rotating impellers, the presence of a strong axial shear of azimuthal velocity in the mid-plane between the impellers generates a high level of turbulent fluctuations, roughly of the same order as the mean flow. It is thus unlikely that the fluctuations $\tilde{\mathbf{v}}$ can be neglected compared to $\overline{\mathbf{V}}$ in (4). It has been indeed observed that when the disks counter-rotate with the same frequency, $F_1 = F_2$, a stationary magnetic field is generated with a dominant axial dipolar component, $\mathbf{B}_{\mathbf{P}}$, together with a related azimuthal component \mathbf{B}_{θ} , as displayed in figure 3 (left) [22]. Such an axisymmetric mean field cannot be generated by the mean flow alone, $\mathbf{V}(r, x)$, that would give a non axisymmetric magnetic field according to Cowling theorem [17], and also as observed in numerical modeling performed using $\overline{\mathbf{V}}(r,x)$ alone [23]. Non axisymmetric fluctuations $\tilde{\mathbf{v}}(r, \theta, x)$ thus play an essential role. As explained in [19], a possible mechanism is of $\alpha - \omega$ type, the α -effect being related to the helical motion of the radially expelled fluid between two successive blades of the impellers, and the ω -effect resulting from differential rotation due to counter-rotation of the impellers. This mechanism has been confirmed by kinematic dynamo simulations: it has been shown that when strong enough vortices along the blades are added to an axisymmetric mean flow, the generated dynamo is no longer an equatorial dipole but is dominated by an axial dipole as observed in the experiment [24]. Following the $\alpha - \omega$ mechanism described in [19], helical fluctuations have been modeled using mean-field MHD [25]. An axisymmetric dynamo has been recovered but it has been found that it requires unrealistically large values of the α -effect to explain the VKS observations (see below the role of the iron impellers).

In conclusion, the VKS dynamo is not generated by the mean flow alone in contrast to Karlsruhe and Riga experiments, and non-axisymmetric turbulent fluctuations play an essential role in the dynamo process.

3.3. Role of the iron impellers

The VKS dynamo has been observed so far only when impellers made of soft iron have been used. More precisely, one iron impeller is enough provided it rotates fast enough.





Figure 3. Left: experimental set-up showing the location of the Hall probes. Right: the dipolar mode, that corresponds to the dominant mean magnetic field generated when the propellers are counter-rotating at the same speed.

Both the disk and the blades should be made of iron. This of course does not mean that iron impellers are necessary to generate a dynamo. They are required in our experiment in order to reach the dynamo threshold R_m^c with the available motor power. In some configurations using impellers made of stainless steel, we have been able to predict an out of reach dynamo threshold by measuring the decay time of a transient magnetic field.

Impellers made of iron first modify the boundary conditions for the magnetic field. It is therefore not surprising that this changes the dynamo threshold. Indeed, numerical simulations have shown that magnetic boundary conditions corresponding to the high permeability limit significantly decrease the dynamo threshold both for an axial or equatorial dipolar mode [24, 26]. This shift in threshold does not fully explains the experimental results since an experiment performed with two impellers made of stainless steel within an iron cylinder did not reach the dynamo threshold. The estimated threshold using decay time measurements gave R_m^c of order 80 instead of 30 in the case of two iron impellers.

Numerical simulations using mean-field MHD with boundary conditions that mimic iron impellers of VKS experiment have shown that the required magnitude of the α effect for dynamo threshold decreases when the magnetic permeability of the impellers increases [27]. Below dynamo onset, ferromagnetic impellers lead to an increased decay time of the axisymmetric mode [28]. It has been claimed that impellers of high magnetic permeability are important 'to promote axisymmetric modes'. This is true only at low kinetic Reynolds number Re. When Re increases and the flow becomes turbulent, an axial dipolar dynamo is favoured compared to an equatorial dipole even without ferromagnetic boundary conditions [29] (see below). It has been also argued [27] that the periodic modulation of the magnetic permeability in the azimuthal direction resulting from presence of the blades, could generate a different dynamo mechanism in which the poloidal and toroidal field components are coupled through the boundary conditions. This is indeed a possible mechanism but it leads to a dynamo threshold orders of magnitude larger than the one observed in the VKS experiment [30, 31].

An analytical model using mean-field MHD allows to understand the physical mechanism explaining the decrease of dynamo threshold that results from the presence of an iron disk [32]. It should be first noticed that the differential rotation generating the ω -effect in the VKS experiment has opposite signs in the bulk and behind the disks. Assuming that the poloidal field does not change sign across the disk, the azimuthal field generated by the ω -effect should change sign, and therefore should vanish on the disk. The other alternative is that the poloidal field vanishes on the disk, the azimuthal field keeping the same sign on both sides of the disk. In both cases, if the component of the field that vanishes remains small close to the disk, the dynamo efficiency that requires the presence of both components, will decrease. It has been shown in [32] that increasing the magnetic permeability of a disk results in a more abrupt change of sign of the axial field. Therefore the axial field is large on both sides of the disk, thus leading to a configuration with a good dynamo efficiency. It has been also shown that there is an optimum value of the magnetic permeability for maximum dynamo efficiency, i.e. minimum dynamo threshold. The low dynamo threshold observed when both the disks and the blades are ferromagnetic can be understood along the same line of thought: the easy magnetisation direction is azimuthal in the disk and along the blades in the blades. The ferromagnetic disk (resp. blades) leads to a large toroidal (resp. poloidal) component of the magnetic field in the vicinity of the impeller. Therefore both the ω and α -effect are large in the same region close to the impellers, thus providing a high dynamo efficiency.

Direct numerical simulations taking into account the boundary conditions related to iron impellers have been recently performed [33]. Although the kinetic Reynolds number Re is orders of magnitude smaller than in the VKS experiment, interesting features have been observed. First, it has been confirmed that increasing Re in order to reach a fluctuating flow favours the axial dipolar dynamo mode. Second, it has been shown that when $\mu_r = 50$, the threshold of this mode decreases from $R_m^c = 130$ to 90 when Re in increased from 500 to 1500. Finally, when μ_r is decreased from 50 to 5 for Re = 1500, the same dynamo mode with a dominant axial dipole is observed. Only its threshold is increased. It would be interesting to check whether this mode is also observed for $\mu_r = 1$ or if one needs a larger value of Re. These results give confidence that the VKS dynamo (Re ~ 510⁶) would involve a dominant axial dipole with non ferromagnetic impellers, the effect of μ_r being just to shift the dynamo threshold without changing the geometry of the growing magnetic field.

3.4. Dynamics of the magnetic field generated by a von Kármán flow of liquid sodium

The magnetic field generated by impellers counter-rotating at the same speed involves strong fluctuations. The intensity and spectrum of these fluctuations depend on the location of the measurement point [22]. At low frequency 1/f-type spectra have been reported, i.e. power laws of the form $1/f^{\alpha}$ with $0 < \alpha < 2$. Measurements of the azimuthal component of the magnetic field, performed in the mid-plane for a rotation frequency of the impellers F = 20 Hz, are displayed in figure 4 together with the power spectrum which exhibits a $f^{-\alpha}$ power law with $\alpha \simeq 0.5$ for 1 < f < 15 Hz, i.e. below the inertial frequency range. For f > 20 Hz, the power-spectrum scales as $f^{-11/3}$, due to the passive stretching of the magnetic field by the small-scale turbulent fluctuations.

As observed in figure 4 (left), the magnetic field displays bursts with amplitudes up to eight times the average value. We first low-pass filter the times series of the magnetic field below 200 Hz. We then define a two-states signal s(t) by phases of weak and large





Figure 4. Left: time series B(t) of the azimuthal magnetic field, fluctuating around its mean value (dashed line). The grey line corresponds to the bursting process s(t) extracted from B(t). Right: power spectrum of B(t) displaying a $f^{-0.5}$ behavior at low frequency (dashed line).



Figure 5. Left: power spectrum of B(t) (black line) and s(t) (grey line) compared to the law $f^{-0.5}$ (dashed line). Right: distribution of the waiting time τ between bursts, with the law $\tau^{-2.5}$ (dashed line).

amplitudes. Among the possible criteria to define a burst, we consider a threshold equal to twice the average value of the magnetic field, such that above (resp. below) it, the system is in the high (resp. low) amplitude state. The resulting two-states signal s(t) is displayed in figure 4 (left) in grey. The power-spectrum of s(t) (in grey) is compared to the one of the magnetic field (in black) in figure 5 (left). At low frequency, both power spectra follow the same power-law. The distribution $P(\tau)$ of waiting times τ between bursts is displayed in figure 5 (right). For the range of duration $5.10^{-2} < \tau < 25.10^{-2}$ s, $P(\tau)$ follows a power-law $\tau^{-\beta}$ with $\beta \simeq 2.5$.

This result shows that the low frequency spectrum is related to the PDF of the waiting time between successive bursts. The agreement is quantitative, the theoretical prediction being $\alpha = 3 - \beta$ for bursts when $2 < \beta < 3$ [34]. It has been shown on several examples of turbulent flows that the low frequency behavior can be related to the PDF of the waiting time between the occurrence of coherent structures [35, 36].

The equation for the amplitude of the bifurcating magnetic field in the VKS experiment cannot be derived from equations (4) and (5). Although turbulent fluctuations enter multiplicatively in equation (4), we do not expect that the form of this amplitude equation is the one of (1). We can however consider the similarities and differences between the two systems that undergo a pitchfork bifurcation in the presence of fluctuations. Both display strong fluctuations in the form of bursts close to threshold with 1/f-type noise spectra at low frequency. However, the dynamo experiment does not display a





Figure 6. Possible eigenmodes of the VKS experiment. The two disks counterrotate with frequency F_1 and F_2 . Left: magnetic dipolar mode. Right: magnetic quadrupolar mode. Poloidal (blue) and toroidal (red) components are sketched.

PDF of the amplitude of the magnetic field that diverges for $B \rightarrow 0$ close to threshold and correspondingly no anomalous scaling of the moments of the magnetic field. Possible explanation for the absence of on-off intermittency have been proposed [37].

3.5. Reversals of the magnetic field

3.5.1. Modes and symmetries. A striking feature of the VKS experiment is that time dependent magnetic fields are generated only when the impellers rotate at different frequencies [38, 39]. It has been shown in [40] that this is related to the broken invariance under \mathcal{R}_{π} when $F_1 \neq F_2$ (rotation of angle π around an axis in the mid-plane). Dipolar (resp. quadrupolar) modes of the magnetic field are displayed in figure 6 (left) (resp. right): a dipolar mode is changed into its opposite by \mathcal{R}_{π} , whereas a quadrupolar mode is unchanged. When the impellers counter-rotate at the same frequency, the system is invariant under \mathcal{R}_{π} . Thus dipolar and quadrupolar modes are not linearly coupled. They become coupled when the impellers rotate at different frequencies such that the \mathcal{R}_{π} symmetry is broken.

We assume that the magnetic field is the sum of a dipolar component with an amplitude D and a quadrupolar one, Q. This is justified when the thresholds of these two large-scale modes have similar values. We define A = D + iQ and we write an amplitude equation in the form of an expansion in powers of A and its complex conjugate \overline{A} . Taking into account the invariance $\mathbf{B} \to -\mathbf{B}$, i.e. $A \to -A$, we obtain

$$A = \mu A + \nu \bar{A} + \beta_1 A^3 + \beta_2 A^2 \bar{A} + \beta_3 A \bar{A}^2 + \beta_4 \bar{A}^3, \tag{6}$$

where we limit the expansion to the lowest order nonlinearities. In the general case, the coefficients are complex and depend on the experimental parameters.

We point out that equation (6) looks like the normal forms for strong resonances, i.e. for the complex amplitude of an oscillatory mode generated by a Hopf bifurcation in the presence of an external forcing. The \overline{A} (respectively \overline{A}^3) term results from a forcing around twice (respectively 4 times) the frequency of the oscillatory instability (see [41] for a study of the bifurcations and resulting dynamics). We also note that a model involving coupled dipolar and quadrupolar modes that bifurcate through Hopf bifurcations instead of stationary ones as considered here, has been proposed to understand some features of the dynamics of the magnetic field of the Sun [42].

We obtain additional constraints on the coefficients when the impellers rotate at the same frequency because D and Q change in different ways under the transformation $\mathcal{R}_{\pi}: D \to -D, Q \to Q$, thus $A \to -\overline{A}$. We conclude that, in the case of exact counterrotation, all the coefficients are real. Writing the equations for D and Q, we recover that the two modes are not linearly coupled as mentioned above. More generally, the real parts of the coefficients are even and the imaginary parts are odd functions of the frequency difference $f = F_1 - F_2$.

Let us denote the real (resp. imaginary) parts of the coefficients with subscript r (resp. *i*). In the counter-rotating case, the growth-rate of the dipole (resp. the quadrupole) is $\mu_r + \nu_r$ (resp. $\mu_r - \nu_r$). We take $\nu_r > 0$ such that the dipolar mode bifurcates first as observed in the experiment. The solution $\mathbf{B} = \mathbf{0}$ is unstable to a growing dipolar mode for $\mu_r > -\nu_r$. When μ_r is increased, the quadrupolar mode also becomes linearly unstable for $\mu_r > \nu_r$. More generally, the linear stability analysis of the solution A = 0 gives the dispersion relation for the growth rate s

$$s^{2} - 2\mu_{r}s + |\mu|^{2} - |\nu|^{2} = 0.$$
⁽⁷⁾

We have a stationary bifurcation for $|\mu| = |\nu|$ if $\mu_r < 0$, a Hopf bifurcation for $\mu_r = 0$ if $|\mu_i| > |\nu|$, and a codimension-two bifurcation for $\mu_r = 0$ and $\mu_i^2 = |\nu|^2$. Therefore, close enough to the counter-rotating case, a stationary bifurcation is predicted as observed in the experiment. A Hopf bifurcation from the $\mathbf{B} = \mathbf{0}$ solution is possible only if μ_i increases fast enough compared to ν_i when f is increased from zero. It is not clear that this occurs in the VKS experiment although in some parameter range, a time periodic dynamo bifurcates from a regime with a small magnetic field (see figure 2 in [47]). It is difficult to decide whether this regime is a weak dynamo or results from the amplication of the ambiant magnetic field by the flow.

3.5.2. A simple case. Writing $A = R \exp i\phi$, the stability of finite amplitude solutions can be studied in the phase approximation provided the amplitude R is slaved to the phase. We assume for simplicity that the nonlinear terms just saturate the amplitude without qualitatively changing the dynamics. This is the case if $A^2 \bar{A}$ is the dominant nonlinear term with β_2 real and negative. The imaginary part of the linear part of (6) gives

$$\phi = \mu_i - \nu_r \sin 2\phi + \nu_i \cos 2\phi. \tag{8}$$

The stationary solutions disappear via a saddle-node bifurcation when $\mu_i^2 = |\nu|^2$ and a limit cycle that corresponds to periodic reversals is generated [40]. The period at instability onset diverges as observed in the experiment when the oscillatory reversal regime is reached from a stationary dynamo with a large amplitude [47].

The saddle-node bifurcation occurs only if μ_i increases faster than ν_i when the \mathcal{R}_{π} symmetry is externally broken. If $|\nu_i| > |\mu_i|$, the solutions remain stationary. A broken symmetry that induces $|\nu_i|$ much larger than $|\mu_i|$ has been found as a mechanism for hemispherical dynamos [43]. Such a dynamo regime with the magnetic field localized close to one impeller, has been observed in the VKS experiment [44]. We emphasize that the location of the saddle-node bifurcation is modified when the other nonlinear terms are taken into account, which can favour or not the reversal regime.



Figure 7. A generic saddle-node bifurcation in a system with the $\mathbf{B} \rightarrow -\mathbf{B}$ invariance: below threshold, fluctuations can drive the system against its deterministic dynamics (phase a). If the effect of fluctuations is large enough, this generates a reversal (phases b and c). Otherwise, an excursion occurs (phase a').

The above phase approximation breaks down in the vicinity of the codimension-two point. Another type of bifurcation from stationary solutions to a limit cycle takes place in that case [45, 46]. It is a subcritical Hopf bifurcation that can be easily discriminated from the previous scenario; the limit cycle appears with a finite period whereas the period diverges when it is generated through a saddle-node bifurcation. Stationary and oscillatory solutions coexist in some parameter range, thus this second scenario displays bistability. Both bifurcation types, saddle-node and subcritical Hopf, have been reported for the reversals of the magnetic field in the VKS experiment [47].

We have thus found a simple mechanism to explain how the dipolar modes, observed for counter-rotating impellers at the same frequency $F_1 = F_2$ in the VKS experiment, first evolve to stationary solutions that also involve a larger and larger quadrupolar component when the frequency difference $|F_1 - F_2|$ is increased. Then, for a critical value of $|F_1 - F_2|$, a limit cycle is generated at finite amplitude and vanishing frequency by a saddle-node bifurcation.

3.5.3. Effect of fluctuations. The effect of hydrodynamic fluctuations on reversals can be easily modeled by adding some noisy component to the coefficients of equation (6). We consider the scenario of reversals generated through a saddle-node bifurcation. Before the bifurcation, the solutions of equation (8) correspond to mixed dipolar-quadrupolar modes. The stable (resp. unstable) ones originate from $\pm D$ (resp. $\pm Q$) when f=0. These solutions are labeled $\pm B_s$ and $\pm B_u$ in figure 7. When a saddle-node bifurcation occurs for a larger value of f, the stable and unstable solutions collide by pairs and disappear. A limit cycle is generated that connects the collision point, B_c , with its opposite.

This provides an elementary mechanism for field reversals. First, in the absence of fluctuations, the limit cycle generated at the saddle-node bifurcation connects $\pm B_c$. This corresponds to periodic reversals. Slightly above the bifurcation threshold, the system





Figure 8. Left: reversals of the magnetic field modelled by (6) with an extra noise term that takes into account the effect of the turbulent fluctuations. Right: zoom on reversals and excursions.

spends most of the time close to the two states of opposite polarity $\pm B_c$. Second, in the presence of fluctuations, random reversals can be obtained slightly below the saddlenode bifurcation. B_u being very close to B_s , even a fluctuation of small intensity can drive the system to B_u from which it can be attracted by $-B_s$, thus generating a reversal. Adding a noisy component to the coefficients of equation (6), we obtain random reversals displayed in figure 8 (left). The system spends most of the time close to the stable fixed points $\pm B_s$. We observe in figure 8 (right) that a reversal consists of two phases. In the first phase, the system evolves from the stable point B_s to the unstable point B_u (in the phase space sketched in figure 7). The deterministic part of the dynamics acts against this evolution and the fluctuations are the motor of the dynamics. This phase is thus slow. In the second phase, the system evolves from B_u to $-B_s$, the deterministic part of the dynamics drives the system and this phase is faster.

The behavior of the system close to B_s depends on the local flow in phase space. Close to the saddle-node bifurcation, the position of B_s and B_u defines the slow direction of the dynamics. If a component of B_u is smaller than the corresponding one of B_s , that component displays an overshoot at the end of a reversal. In the opposite case, that component will increase at the beginning of a reversal. For instance, in the phase space sketched in figure 7, the component D decreases at the end of a reversal and the signal displays an overshoot. The component Q increases just before a reversal.

For some fluctuations, the second phase does not connect B_u to $-B_s$ but to B_s . It is an aborted reversal or an excursion in the context of the geodynamo. Note that during the initial phase, a reversal and an excursion are identical. In the second phase, the approaches to the fixed point differ because the trajectory that links B_u and B_s is different form the trajectory that links B_u and $-B_s$. In the case of figure 7, the dipole displays an overshoot at the end of a reversal and reaches smaller values during an excursion (see figure 8 right). By contrast the quadrupole exceeds its quasi-stationary value at the beginning of a reversal and reaches larger values during an excursion.

The dipolar and quadrupolar components of the magnetic field have been measured in the VKS experiment [49]. The results have been found in good agreement with the predictions of the present model. When the dipole reverses, part of the magnetic energy

is transferred to the quadrupole. Reversals begin with a slow decay of the dipole followed by its fast recovery with the opposite polarity together with an overshoot. The decay rates of the dipolar and quadrupolar modes have been also measured in the VKS experiment [50]. It has been confirmed that reversals of the magnetic field are observed only when their bifurcation thresholds are close enough. Finally, a direct numerical simulation of the dynamo generated by a flow driven in a sphere by two counter-rotating co-axial propellers [51] has reproduced the main dynamical features of the VKS experiment. Reversals of the axial dipole occur only when the propellers are rotated at different rates. When the magnetic Prandtl number is small enough, they involve an axial quadrupole and the dynamics of the dipolar and quadrupolar modes during a reversal is similar to the one observed in the experiment.

3.6. Geomagnetic reversals

We have proposed a scenario for reversals of the magnetic field generated by dynamo action in the VKS experiment. When the impellers are counter-rotated at different frequencies, the flow breaks the invariance by rotation \mathcal{R}_{π} , and thus couples modes with dipolar and quadrupolar symmetries. This coupling drives the system close to a saddle-node bifurcation, such that even non-coherent turbulent fluctuations can generate a reversal. The scenario offers a simple and unified explanation for reversals of a vector field. In particular, it explains many intriguing features of the reversals of Earth magnetic field [48]. In that case, dipolar and quadrupolar modes are coupled when the flow in the core breaks the equatorial symmetry. The most significant output of the model is that it predicts specific characteristics observed both in VKS experiment as well as in palaeomagnetic records. It also explains recent numerical simulations of the geodynamo that have pointed out the importance of hydrodynamic symmetry breaking during reversals [52, 53]. A lot of other models, proposed to describe the reversals of the magnetic field of the Earth, are reviewed in [54].

4. Two-dimensional turbulence forced in a confined domain

4.1. Introduction

Experiments on nearly two-dimensional flows generated in a thin layer of fluid by a spatially periodic forcing, often referred to as Kolmogorov flows, have been first performed to study the generation of large scale flows [55, 56] and the properties of two-dimensional turbulence [57–60]. These flows have been modeled using the twodimensional Navier-Stokes equation with an additional term describing fluid friction on the bottom boundary of the fluid layer. Thus, besides the Reynolds number, Re, a second dimensionless parameter, Rh, characterizes the ratio of the inertial term to fluid friction (see section 4.2 for a detailed description).

In the case of two-dimensional confined turbulent flows, it has been predicted [61] and observed [57, 58] that the inverse cascade of energy either leads to an homogeneous turbulent flow displaying a wide range of wave numbers k with a $k^{-5/3}$ scaling law, or to a condensate regime that results from an accumulation of kinetic energy in the mode

of the fluid layer with the lowest wavenumber. The transition between the two regimes depends on the importance of large scale friction, i.e. on the value of Rh. In the presence of friction, the inverse cascade stops at a length scale l_I . When Rh increases, l_I reaches the size of the fluid layer L and energy accumulates at the lowest wave number, thus leading to a dominant large scale circulation.

We will study here the transitions that occur within the turbulent regime when Rh is increased with $Re \gg Rh$, starting from a turbulent state with a Gaussian large scale velocity observed for intermediate values of $Rh \sim 10$. We first observe that the probability density function (PDF) of the large scale velocity changes from Gaussian to bimodal when Rh is increased. Above a critical value Rh_a , we show that the PDF can be fitted by the superposition of two symmetric Gaussians with a separation between their mean values increasing like $\sqrt{Rh - Rh_a}$ and a nearly constant standard deviation. The bimodality becomes more and more pronounced as Rh is increased and is related to random reversals of the large scale circulation. The average waiting time between successive reversals becomes longer and longer and a condensed regime with no reversal is finally observed above $Rh = Rh_c$. Therefore, the regime with random reversals of the large scale circulation is located in parameter space between the condensed state and the turbulent regime with Gaussian large scale velocity. This scenario has been observed both in experiments [62] as well as in numerical simulations of the two-dimensional Navier-Stokes equation with large scale friction [63].

4.2. Experimental set-up and dimensionless parameters

A thin layer of liquid metal (Galinstan) of thickness h = 2 cm, is contained in a square cell of length L = 12 cm and is submitted to a uniform vertical magnetic field up to $B_0 \simeq 0.1$ T. A DC current I (0–200 A) is injected at the bottom of the cell through an array of 2×4 electrodes (diameter d = 8 mm flush to the bottom of the fluid layer). Other experiments have also been performed using mercury and an array of 6×6 electrodes to drive the flow. A sequence of transitions similar to the one described here for Galinstan, has been observed. The current density \mathbf{j} has a radial component above each electrode so that the associated Lorentz force density $\mathbf{f}_L = \mathbf{j} \times \mathbf{B}_0$ creates a local torque. This forcing therefore drives a periodic array of counter-rotating vortices (see [64] for a more detailed description of the experiment).

In addition to h, L, B_0 and I, the relevant physical parameters are the fluid density $\rho = 6.44 \text{ kg} \cdot \text{m}^{-3}$, its viscosity $\nu = 3.72 \, 10^{-7} \, \text{m}^2 \cdot \text{s}^{-1}$, its electrical conductivity $\sigma = 3.46 \, 10^6 \, \text{S} \cdot \text{m}^{-1}$ and the magnetic permeability of vacuum μ_0 . Four independant dimensionless numbers can therefore be constructed. However two of them can be ignored since the magnetic field is strong and the flow speed is small. More precisely, in the limit of large Hartmann number Ha = $hB_0[\sigma/(\rho\nu)]^{1/2} \sim 10^2$ and small magnetic Reynolds number Rm = $\sigma\mu U_c L \sim 10^{-2}$ (with U_c the characteristic speed of the flow), the velocity field is nearly two dimensional, and its vertical average satisfies the two dimensional Navier-Stokes equation with an additional linear damping term $-\mathbf{v}/\tau_H$ with $\tau_H = h^2/(\nu \text{Ha})$ [65].

This quasi-2D flow depends on two dimensionless numbers, e.g. the Reynolds number $\text{Re} = U_c L/\nu$ and $\text{Rh} = U_c \tau_H/L$ which is the ratio of inertia to linear friction. For large Re and Rh, the characteristic velocity U_c is set by a balance between inertia and





Figure 9. Top: pictures of the flow as a function of Rh. From left to right: laminar flow, first bifurcation, chaotic flow, turbulent flow with moderate large scale flow, turbulent flow with strong large scale flow (condensate). Bottom: sketch of the PDF of large scale circulation. Symmetry breaking indicates that depending on the initial conditions one of the two states (i.e. one of the two peaks of the PDF) will be observed.

the Lorentz force per unit mass and reads $U_c = \sqrt{IB_0/(\rho h)}$. The ratio Re/Rh, independent of the injected current, is equal to $\text{Ha}(L/h)^2 \sim 10^4$. By changing I we vary Rh between 1 and 50. Since viscous dissipation becomes efficient at scales smaller than $l = L\sqrt{\text{Rh}/\text{Re}} \sim 10^{-3}$ m, dissipation at large scale is mainly due to the friction term. It follows from these order of magnitude estimates that Rh is the relevant control parameter for the large scale dynamics, which is well verified experimentally [64].

The large scale velocity is measured by the potential difference between a pair of electrodes in the external magnetic field [66]. One of the electrodes is located in the middle of the cell and the other one close to the lateral wall, 5 mm away from it. The flow induces an electromotive force $\Delta V = \int_{L/2} (\mathbf{u} \times \mathbf{B}_0) \cdot \mathbf{dl} \simeq \phi_L B_0 / h$ where L/2 is the distance between the two electrodes and ϕ_L is the flow rate between the center and the wall. From now on, we consider the spatially averaged velocity normalized by U_c , i.e. $V = 2\phi_L/(h L U_c)$, which is therefore the large scale velocity coarse-grained on size L/2. Local velocity measurements have been also performed using Doppler acoustic velocimetry. Their spatial average compares well with measurements made using the potential difference between two electrodes [64].

A summary of the evolution of the velocity field is presented in figure 9 together with pictures of the flow when Rh is increased. For $Rh \leq 1.55$, the forcing drives a laminar flow made of an array of 8 counter-rotating vortices. The large scale velocity vanishes as expected from the symmetries of the forcing. This linear response to the forcing becomes unstable when Rh is increased above 1.55: a stationary pitchfork bifurcation occurs first, for which the streamlines of one of the two pairs of diagonal



Figure 10. Time series of the large scale flow for Rh = 20.

vortices merge, leading to the appearance of a large scale circulation which breaks the planar symmetries of the forcing. This regime is followed by two successive Hopf bifurcations for Rh = 1.7 and Rh = 2. The resulting quasiperiodic regime becomes unstable for Rh = 2.6 and displays a transition to chaos via intermittency. This chaotic attractor merges with the symmetric one through a crisis and a statistically symmetric attractor tor is observed above $Rh \simeq 3.1$. For $Rh \sim 10$, a turbulent regime that involves random motion of interacting vortices of different sizes is reached. The planar symmetries are statistically restored and the PDF of the large scale velocity is Gaussian.

4.3. Turbulent transitions at the edge of the regime with large scale flow reversals

For Rh slightly below $Rh_a \simeq 11$, the amplitude of the large scale circulation fluctuates around zero. The next transition is related to the appearance of coherent states where the flow maintains its direction for long duration. Such events are visible in figure 10.

We want to quantify this transition using some kind of order parameter as done for bifurcations of stationary or time periodic flows. This is not straightforward as the basic state is already turbulent. We thus rely on statistical properties of the signal and first on the Kurtosis of the large scale velocity $(\langle V^4 \rangle / \langle V^2 \rangle^2)$. We observe that a transition occurs for $Rh_a \simeq 11$. The large scale velocity is Gaussian (K = 3) for $Rh \leq Rh_a$ and becomes non Gaussian above Rh_a . K decreases linearly from 3 to 2 when Rh is increased from 11 to 30 and then stays roughly constant when Rh is increased further.

To obtain a more precise description of the bifurcation, we have to consider the PDF of V. This can be technically difficult because any imperfection would bias the system even slightly toward one direction of rotation (i.e. one sign of V). Several analyses presented hereafter, in particular involving fits, are highly sensitive to any asymmetry of the system. For such analyses, we restrict to experiments for which the system is well equilibrated and the distributions are symmetrical.

As displayed in figure 11, the shape of the PDF changes with Rh. It is close to a Gaussian at Rh = 12, it is flatter at its center at Rh = 14 and, at Rh = 20, it is bimodal with a local minimum at V = 0. The evolution of the PDF traces back to the modification of the time series of V: the appearance of non zero temporary attractive states is responsible for the bimodal structure of the PDF.





Figure 11. PDF of the amplitude of the large scale velocity for (from left to right) Rh = 12, 14, 20. Symbols are experimental data. The blue continuous curve is equation (9), the sum of the two gaussians displayed with magenta dashed curves. The green dash-dotted curve is equation (10). For Rh = 12, the two gaussians are very close to each other, so that the whole PDF is nearly gaussian. It becomes flatter close to V = 0 at Rh = 14 and it is bimodal at Rh = 20.



Figure 12. Left: parameter $dX (\Box)$ and σ (°) of equation (9) as functions of Rh. Insert: same data, dX^2 as a function of Rh. Right: parameter a (°) and b (\Box) of equation (10) as functions of Rh.

The evolution of the shape of the PDF can be captured using the following model. We consider that it is the sum of two Gaussians of width σ and centered at $\pm dX$. We write

$$P[V] = (2\sqrt{2\pi}\sigma)^{-1}(\exp(-(V - dX)^2/(2\sigma^2)) + \exp(-(V + dX)^2/(2\sigma^2))).$$
(9)

For Rh below Rh_a a single Gaussian provides a good fit to the PDF, and this corresponds to dX = 0. For larger Rh, we extract dX and σ from best fits of the whole PDF. They are displayed in figure 12 (left). Note in particular that both the center and the tails of the PDF are well fitted by equation (9). The standard deviation σ of each PDF remains nearly constant, whereas its center, dX, increases with Rh. As displayed in the insert of figure 12 (left), this behavior is compatible with a power law $(\text{Rh} - \text{Rh}_a)^{1/2}$. As often when trying to extract critical exponents, we note that due to the error bars, our measurements do not exclude values of the exponent close to but different from 1/2.

The PDF becomes bimodal at a larger value $\operatorname{Rh}_b \simeq 17$. This corresponds to $dX = \sigma$, the Rayleigh criterion for separating two lines in an optical spectrum. This secondary transition of the PDF, associated with the appearance of bimodality, is similar to the

one of the free-energy in the context of second-order phase transitions in the model of Landau. Led by this analogy, in the vicinity of the transition, we model the PDF as

$$P[V] \propto \exp\left(aV^2 + bV^4\right). \tag{10}$$

We emphasize that this model is restricted to small values of V (it is not expected to model the tails of the PDF). Landau's assumption is that a varies linearly in the control parameter and changes sign at the transition while b remains roughly constant.

Extracting the values of a and b directly from the PDF results in large error bars and numerical values that strongly depend on the range over which the fit is achieved and on the possible asymmetry of the PDF. This is not the case with the following data treatment : equations (9) and (10) are expanded close to V = 0 and then a and bare expressed as functions of σ and dX. The obtained values of a and b are displayed in figure 12 (right). a increases linearly with Rh and changes sign in the vicinity of Rh_b. This corresponds to the change of concavity of the PDF at V = 0, and the appearance of two non-zero maxima. b is very small below Rh_b, in agreement with the Gaussian behavior of the PDF. It becomes negative when Rh increases. As displayed in figure 11, equation (10) is a good fit of the PDF only for small values of V: it does not describe the tails of the PDF.

Therefore, although equation (10) seems a more natural description for the transition of the PDF, it provides a less accurate fit than equation (9) that is also valid for the tails of the velocity distribution. Equation (10) would be obtained for a system described by a free-energy proportional to $-aV^2 - bV^4$ and subject to additive fluctuations. In contrast equation (9) is expected if V is the sum of a constant velocity $\pm dX$ and random fluctuations of constant energy.

When Rh is further increased, the value of the PDF close to V=0 decreases. This corresponds to the large scale circulation becoming more and more stable, and the reversals between these two directions of the flow becoming less and less frequent (figure 13). Ultimately, no reversals are observed on the maximum measurement time (set by the stability of the experiment). The obtained PDF becomes asymmetric and is peaked close to the sign of rotation that is selected initially.

4.4. Discussion

We have studied the different bifurcations that the large scale flow undergoes. At small Rh, a sequence of bifurcation drives the system from zero large scale circulation to a steady non-zero one, then to a time-periodic state followed by a quasi-periodic one. When increasing Rh further, the system becomes chaotic and explores successively positive and negative values of the large scale circulation.

Other bifurcations then occur over a fluctuating background. Bifurcations in that context are by far less documented than bifurcations occurring over a steady or timeperiodic state. Relevant quantities are statistical ones, such as moments or the PDF of the variable.

The PDF undergoes three bifurcations as Rh increases. First, it becomes non Gaussian, with a Kurtosis departing from 3. The whole PDF is then well described by the sum of two non-centered Gaussians. The distance between the center of the Gaussian increases while their standard deviation remains roughly constant, so that a



Figure 13. Reversals of the large scale velocity field in a layer of mercury driven by an array of 6×6 electrodes. Time series of the velocity V/V_c where $V_c = (BI/(\rho a))^{1/2} \simeq 4.28 \text{ cm} \cdot \text{s}^{-1}$ for Rh = 37.7 and Re = 45000. The two curves correspond to measurements located on opposite sides of the cell. The velocity is averaged between one centimeter and three centimeters away from the lateral wall.

second bifurcation occurs: the PDF becomes bimodal. We can describe this phenomenon using an analogy with Landau's theory: the behaviors of the parameters a and b follow Landau's assumption, which was not obvious since this theory fails in low dimension when spatial fluctuations are of importance. Here the unstable mode is at large scale so that spatial coupling is not relevant.

This evolution is associated in the time series with the appearance of long-lived coherent states during which the circulation does not change sign. The time series can then be described as random reversals between these two coherent states. We had studied in detail the spectral properties of these time series [35]. For Rh between 10 and 30, the time series display 1/f noise. This behavior results from the distributions of the duration between sign changes that are heavily tailed. We note that 1/f fluctuations occur for 10 < Rh < 30. They are therefore observed for Gaussian, non Gaussian and bimodal PDF.

At even larger Rh, the mean duration between sign changes diverges. When it is larger than the duration of stability of the experiment (several hours), the observed PDF, measured over this maximum duration, will be restricted to positive or negative values (depending on the initial condition). We have thus observed in this experiment, two different scenarios that describe the disappearance of a regime of reversals. At large Rh, reversals become less and less likely and eventually are no longer observed. The system remains stuck in one of the two states. At low Rh, reversals disappear because the time series are so fluctuating, that one cannot identify anymore the two states connected by reversals. We expect that these two possible ways to destroy or create reversals are generic and are observed in other contexts [54, 67].

Random transitions between two-dimensional turbulent flows with mean flows of different geometries have been also studied. In the framework of statistical mechanics of the two-dimensional Euler equation [68, 69], it has been shown that depending on the aspect ratio of the flow domain, the most probable flow can bifurcate, for instance from vortices to jets. Numerically integrating the two-dimensional Navier-Stokes equation with small viscosity ν , small friction coefficient $1/\tau_H$ and a noisy forcing proportional to $\sqrt{(1/\tau_H)}$, random transitions between the two mean flows have been observed [70].

Another method to capture the dynamics of random switching between different mean flows consists of integrating the Euler equation truncated at a maximum wave number. We recently showed that the transitions within the turbulent regime of Kolmogorov flows described above can be all captured using the truncated Euler equation [71]. The bifurcation parameter in this framework is related to the initial conditions. More precisely, it is a length scale given by the square-root of the energy to the enstrophy of initial conditions. Physically, this is the length scale l_0 at which the inverse cascade stops. l_0 is controlled by the large scale friction, i.e. Rh in the full Navier-Stokes equation. Increasing l_0 , we successively observe the turbulent regime with a Gaussian PDF, the transition to a bimodal velocity PDF related to reversals and finally the condensed state in agreement with the experiments or simulations of the two-dimensional Navier-Stokes equation when Rh is increased. In addition, the truncated Euler equation is a dynamical system that follows the Liouville theorem. It has been found that its microcanonical distribution correctly predicts the PDF of the large scale velocity and its transitions [71].

Let us finally mention that random reversals of a large scale flow have been recently studied in Rayleigh–Bénard convection in square or parallelepipedic containers [72–75]. They occur for different boundary conditions and in an extended range of Rayleigh Ra and Prandtl Pr numbers [73]. Reversals of the large scale flow involve interaction between dominant modes with different symmetries [72, 74, 75] as in the case of reversals of the magnetic field. It has been shown that, depending on their symmetries, the modes of the velocity and temperature fields either reverse or remained unchanged when the large scale flow reverses [74], some feature also observed for Kolmogorov flows. The transition from non reversing to reversing regime has not been studied in detail. However, it has been shown that reversals are observed for intermediate values of Ra and disappear both for small and large Ra [73]. This reminds reversals in Kolmogorov flows that are observed only for intermediate values of Rh. The mean waiting time between successive reversals also displays a similar trend: it increases by more than two orders of magnitude when Ra is increased. Therefore reversals appear with a finite mean frequency at low Ra and disappear with vanishing frequency at high Ra. It would be interesting to analyse these two transitions in order to check the similarities with reversals observed in Kolmogorov flows.

5. Conclusion

We have studied several experimental configurations displaying bifurcations on a strongly turbulent background that illustrate different possible behaviors. The dynamo threshold observed in the VKS experiment is qualitatively affected by the existence of non axisymmetric turbulent fluctuations. These fluctuations modify the geometry of the large scale magnetic field which is generated via the dynamo bifurcation. They also affect the dynamics of the magnetic field above threshold that involve random bursts and a related 1/f-type low frequency spectrum. Somewhat surprisingly, turbulent fluctuations play a simple role for reversals of the magnetic field. Their dynamics can be captured with a low dimensional dynamical system that describes the interaction

of dipolar and quadrupolar modes. The role of turbulent fluctuations is just to trigger random reversals slightly below the deterministic transition. The deterministic reversal trajectories in phase space are only weakly affected by turbulent fluctuations. The robust character of the low dimensional modeling certainly relies on the proximity of the bifurcation threshold of dipolar and quadrupolar modes and on the smallness of the magnetic Prandtl number that result in strongly different time scales for the magnetic and velocity fields. It has been indeed observed that the morphology of the reversals is changed when the magnetic Prandtl number is increased and becomes of order one [51]. They involve more modes because of the stronger coupling between the velocity and magnetic fields.

Bifurcations between different turbulent regimes have been also observed in twodimensional turbulence in a confined domain. This purely hydrodynamic problem is more complex than the magnetohydrodynamic one since it is less obvious to disentangle the bifurcating modes from the turbulent background. However, the transitions can still be defined by considering the symmetries broken or not by the turbulent state. We have observed a transition from a homogeneous turbulent flow with a Gaussian velocity PDF that has all the symmetries of the experimental set-up, to a flow with a bimodal PDF that involves a reversal dynamics of the large scale modes. In contrast to the magnetic case, this transition involves a lot of modes and no time scale separation (all the odd modes reverse when the large scale flow reverse [63]). Another transition is observed from the reversal regime to the condensate state which breaks the mirror symmetry of the driving device. In the vicinity of this transition, random reversals of the large scale flow become rare and we can expect that a low dimensional dynamical system involving a few large scale modes could describe the dynamics [67] in a similar way as for reversals of the magnetic field. However, in contrast to the magnetic field, it is not possible to identify a few modes which are close to some bifurcation threshold compared to the others that are damped and therefore can be eliminated. In other words, it is not clear how to disentangle the relevant modes from the others. Modeling these hydrodynamic transitions using the truncated Euler equation looks more promising and it seems worth using this method to study other turbulent flow configurations.

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