Phase transitions and search processes in computer science:

a dynamical analysis of the backtrack resolution of random 3-Satisfiability problems.

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StatPhys 2001, Cancun

3-SAT and its phase transition

$$\boxed{(\ w \quad \text{or} \quad \text{NOT}\ x \quad \text{or} \quad y\ )}$$

and

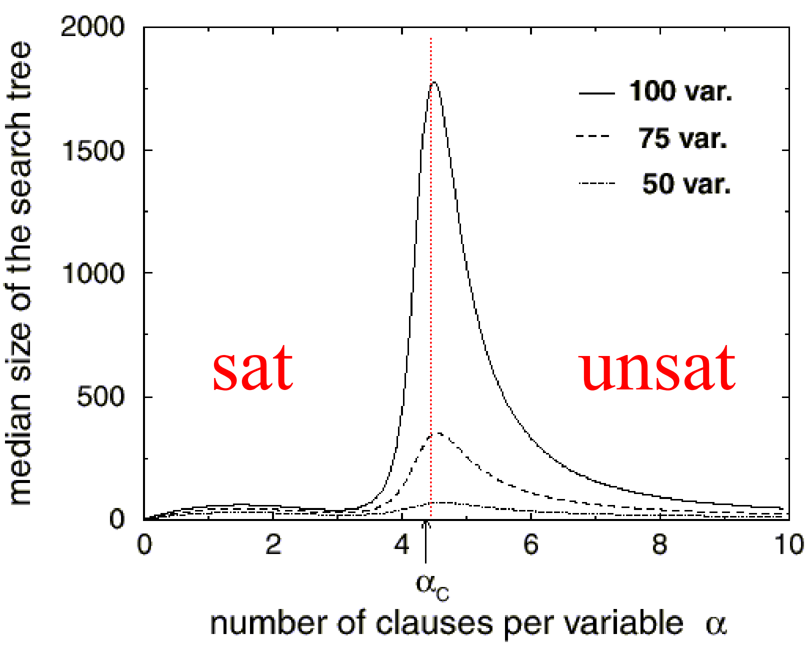
$$\boxed{(\ \text{NOT}\ w \quad \text{or} \quad x \quad \text{or} \quad z\ )}$$

and

$$\boxed{(\ x \quad \text{or} \quad y \quad \text{or} \quad \text{NOT}\ z\ )}$$

?

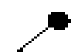
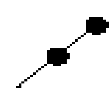
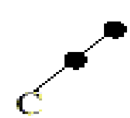
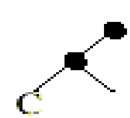
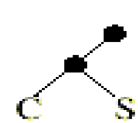
$$\alpha = \frac{\text{nb. of constraints}}{\text{nb. of variables}} \rightarrow \boxed{\alpha_C \approx 4.3}$$



phase transition !

Mitchell et al. ‘92
Crawford, Auton ‘93
Gent, Walsh ‘94

How to solve 3-Sat ?
The Davis-Putnam algorithm

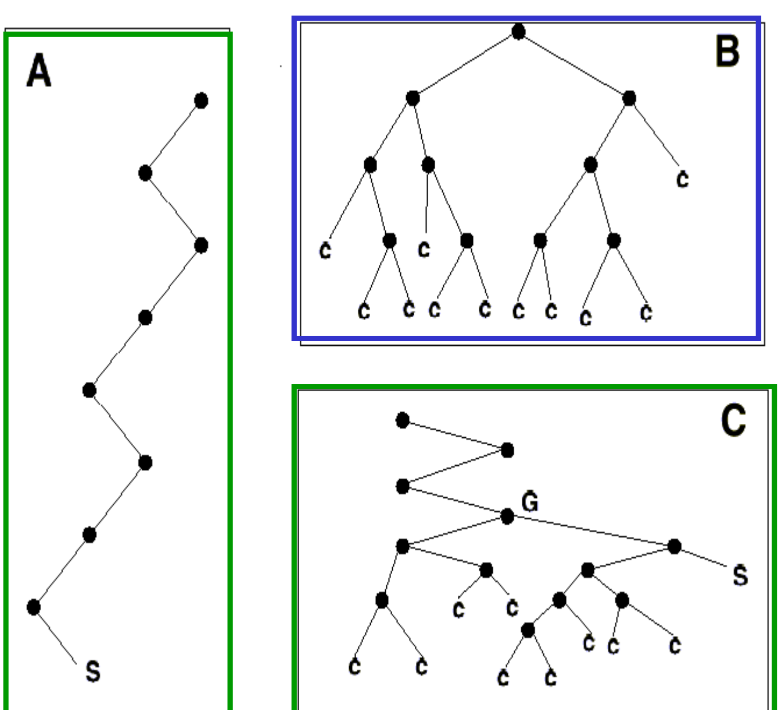
step	clauses	search tree
0	$w \vee \bar{x} \vee y$ $\bar{w} \vee x \vee z$ $\bar{w} \vee \bar{x} \vee \bar{y}$ $\bar{w} \vee \bar{x} \vee y$ $x \vee y \vee \bar{z}$	
1	split : $w = T$	
2	$x \vee z$ $\bar{x} \vee \bar{y}$ $\bar{x} \vee y$ $x \vee y \vee \bar{z}$	
3	split : $x = T$	
4	$\bar{y}$ y	
5	propagation : $y = F$, $y = T$ contradiction	
6	backtracking to stage 1 : $x = F$	
7	z $y \vee \bar{z}$	
8	propagation : $z = T$, $y = T$ solution : $w = T, x = F, y = T, z = T$	

Backtrack algorithm, search tree and heuristic

Davis-Putnam algorithm = heuristic + backtracking

- Unit-Clause rule: pick variable in 1-clause if any, or any unset variable.
- Generalized unit-clause rule: any variable in shortest clause.
- Majority rule: most frequent variable in 3-clauses

Chao, Franco ‘86, ‘90



A satisfiable (easy)

B unsatisfiable (hard)

C satisfiable (hard)

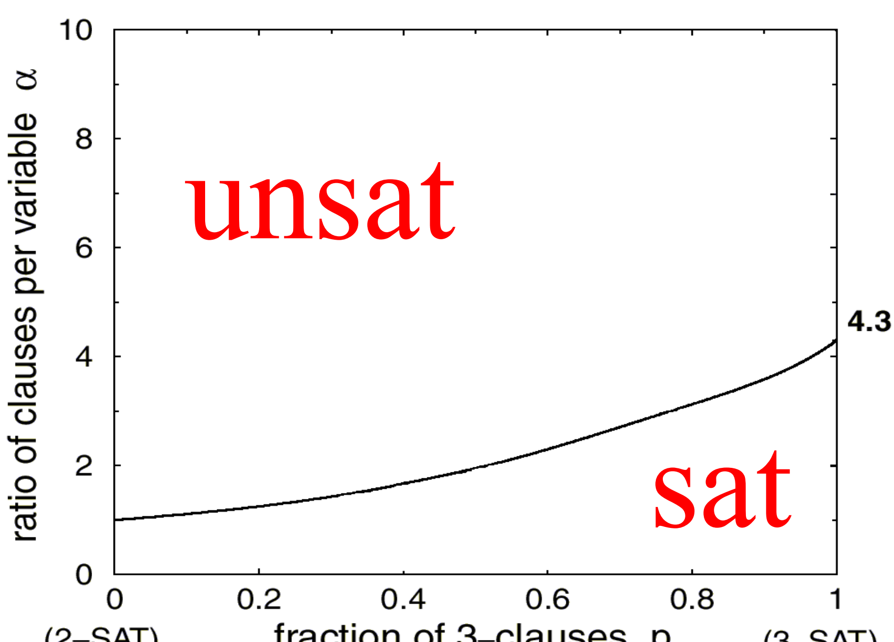
Dynamical flow, and the 2+p-SAT problem

constraints with 3 variables $\rightarrow$ α

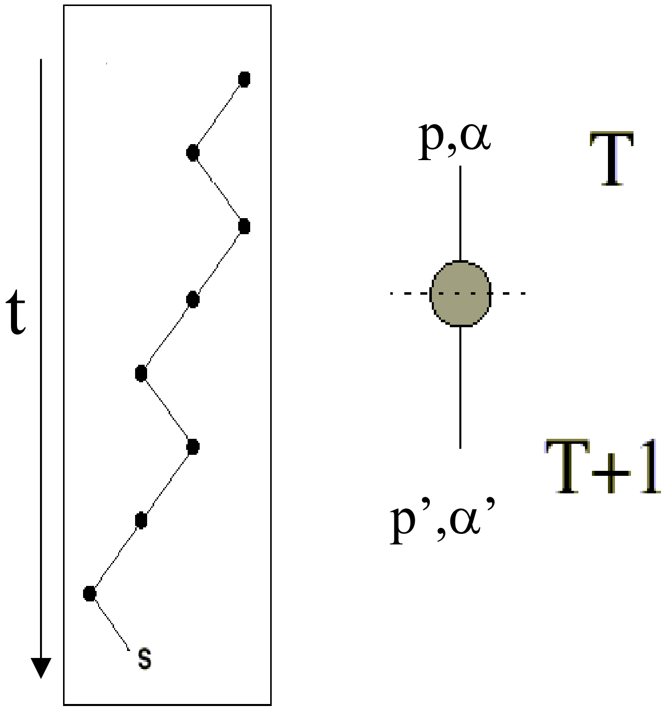
dynamics of the algorithm

constraints with 3 variables
constraints with 2 variables $\rightarrow$ α , p

phase diagram of the 2+p-SAT model



Satisfiable and easy instances
$\alpha < 3.003$



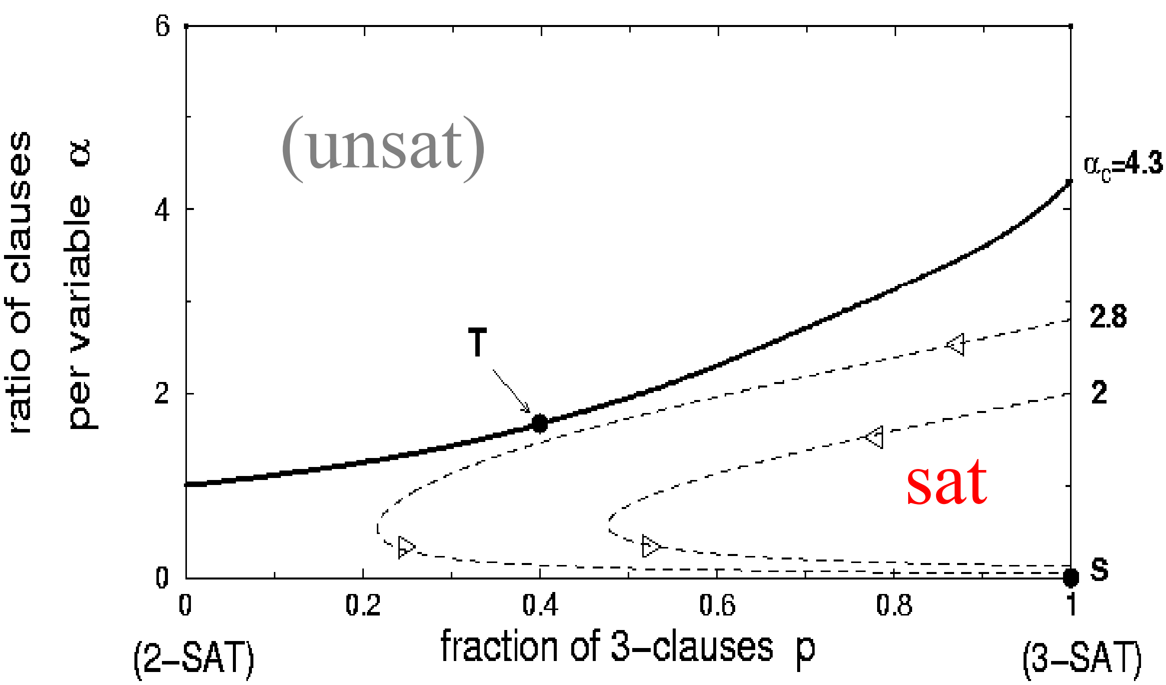
$$p(0) = 1\ ,\ \alpha(0) = \alpha_0$$

$$\frac{dp}{dt} = F_p(p, \alpha, t)$$

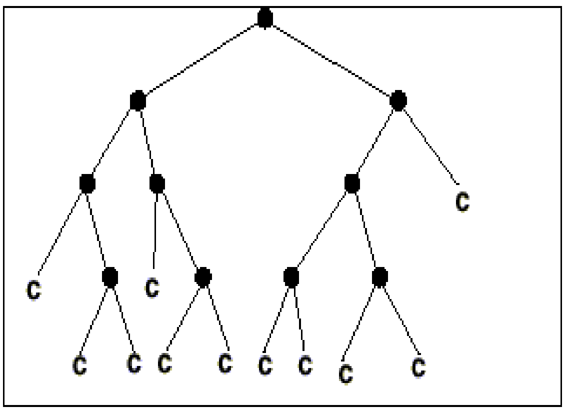
$$\frac{d\alpha}{dt} = F_\alpha(p, \alpha, t)$$

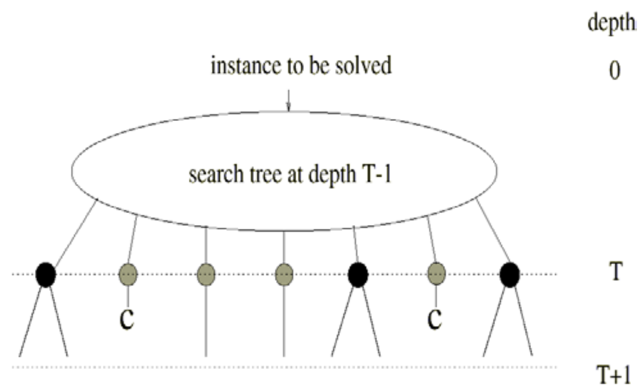
(ODE)

Chao, Franco '90; Frieze, Suen '96



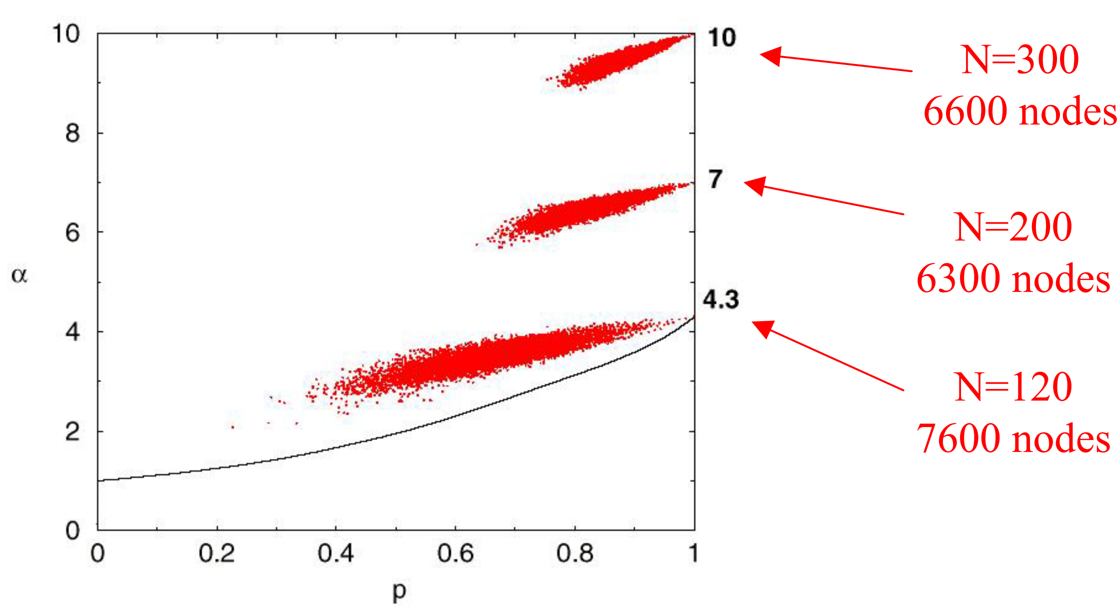
Unsatisfiable, hard instances $\alpha > 4.3$





DP induces a non Markovian evolution of the search tree ….

Imaginary, and parallel building up of the search tree

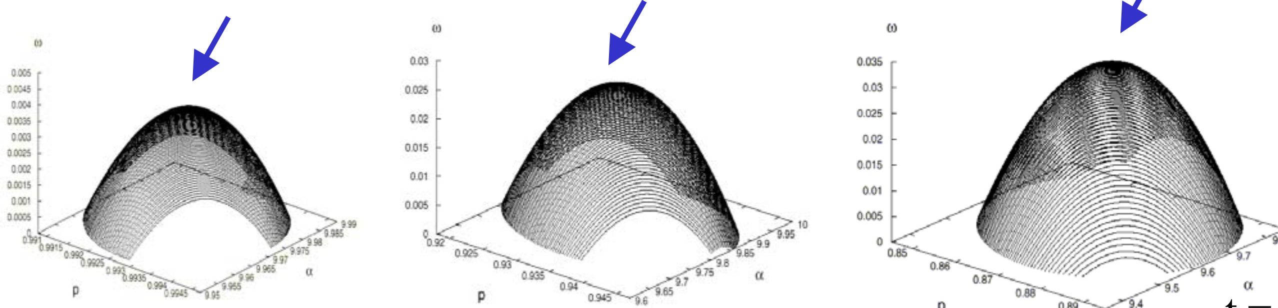


Growth of the search tree

one branch $\longrightarrow$ many branches

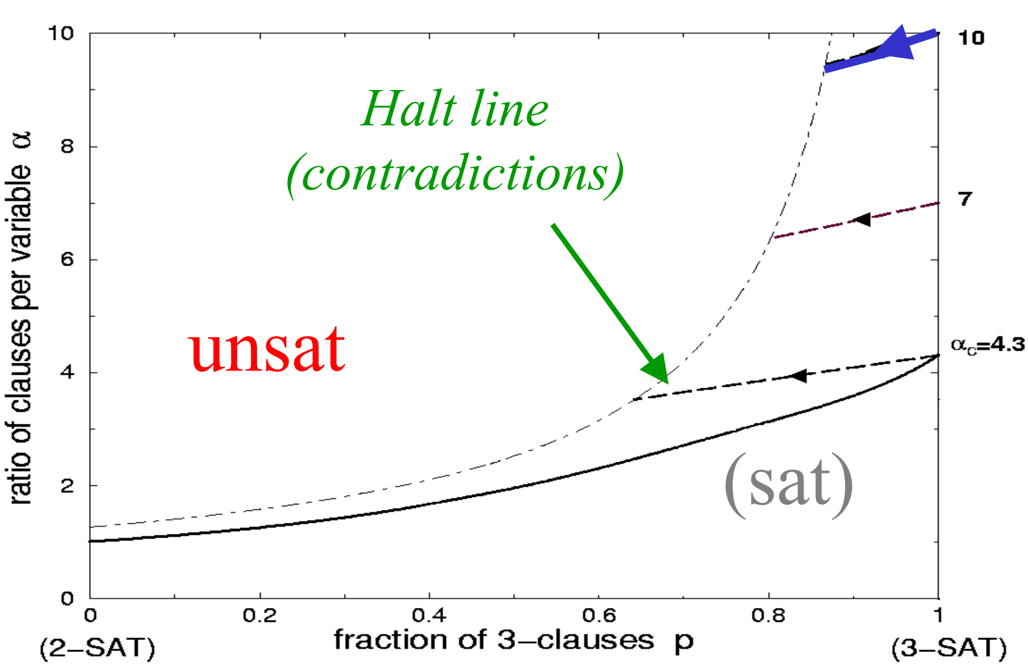
$p(t)$, $\alpha(t)$ $\qquad$ $\omega\,(p,\alpha,t)$

$\alpha_0 = 10$



t = 0.01 $\qquad$ t = 0.05 $\qquad$ t = 0.09

$$\frac{\partial \omega}{\partial t} = \mathcal{H}\left[p, \alpha, \frac{\partial \omega}{\partial p}, \frac{\partial \omega}{\partial \alpha}, t \right] \qquad \text{(PDE)}$$



Comparison to numerical experiments

$$\text{size of the tree} = 2^{N\omega}$$

	Initial Ratio α_0	Experiments $\log_2 Q$	$\log_2 B$	Theory $\hat{\omega}$
unsat	20	0.0153 ± 0.0002	0.0151 ± 0.0001	0.0152
	15	0.0207 ± 0.0002	0.0206 ± 0.0001	0.0206
	10	0.0320 ± 0.0005	0.0317 ± 0.0002	0.0319
	7	0.0482 ± 0.0005	0.0477 ± 0.0005	0.0477
	4.3	0.089 ± 0.001	0.0895 ± 0.001	0.0875
sat	3.5	0.034 ± 0.003		0.035
	G	0.040 ± 0.002	0.041 ± 0.003	0.044
		(nodes)	(leaves)	

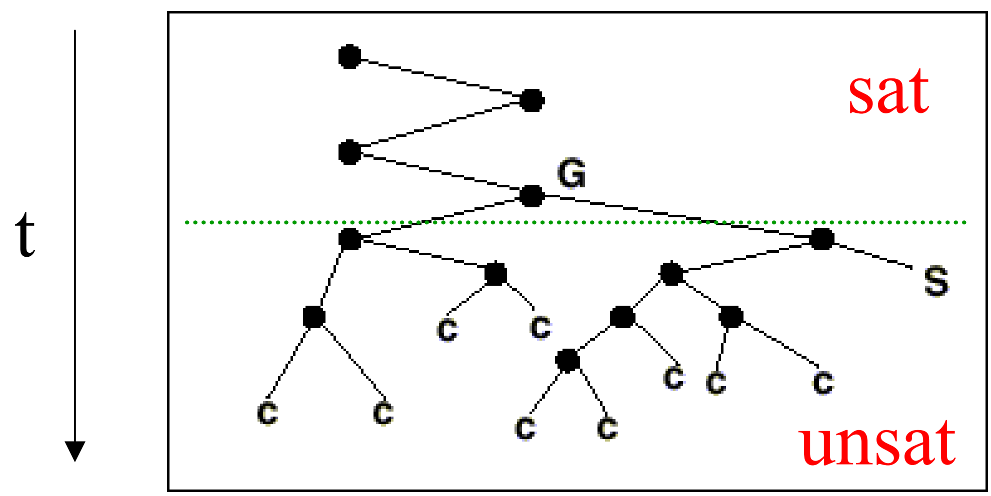
$$\omega = \frac{3+\sqrt{5}}{6\ln 2}\left[\ln\left(\frac{1+\sqrt{5}}{2}\right)\right]^2 \frac{1}{\alpha} \approx \frac{0.292}{\alpha}$$

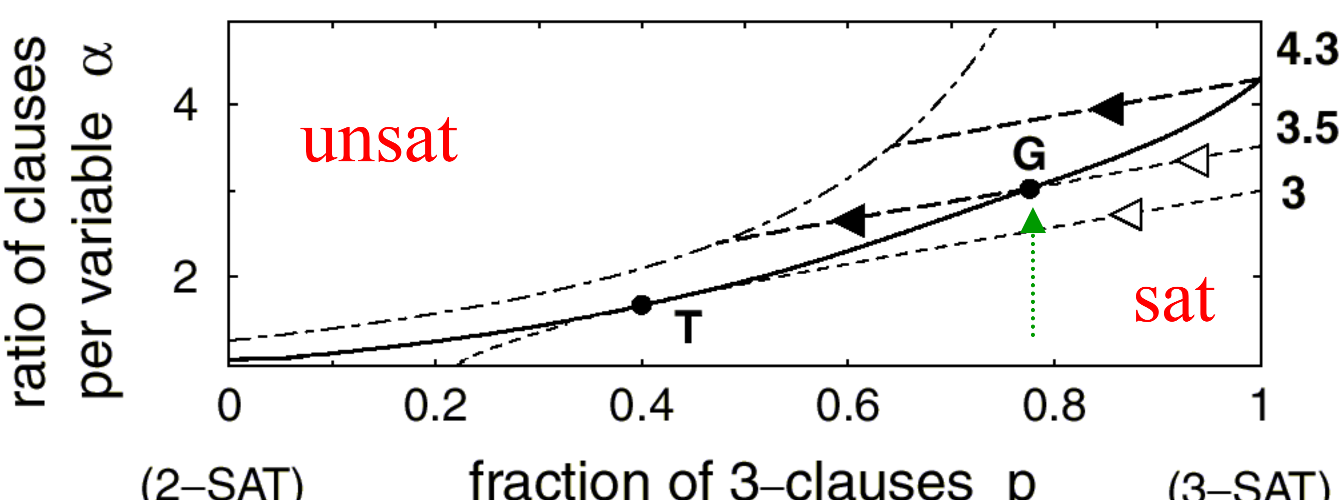
Beame, Karp, Pitassi, Saks ‘98

Satisfiable, hard problems

(that could made be easier?)

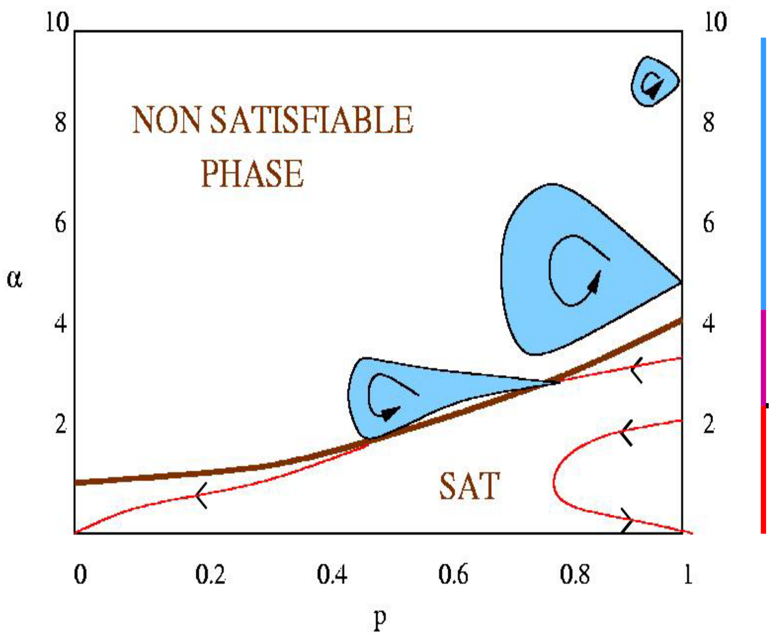
$3.003 < \alpha < 4.3$

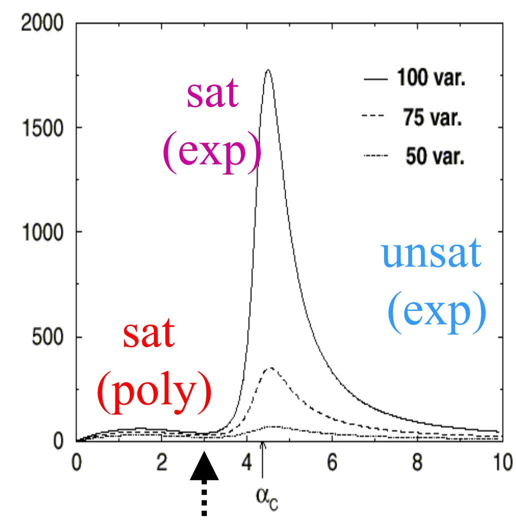




The complexity of 3-Sat solving is affected by the existence of a critical line for 2+p-Sat !

The polynomial/exponentiel crossover





“dynamical” transition
(depends on the heuristic)

$$\begin{aligned} UC &: \quad 2.667 \\ GUC &: \quad 3.003 \\ M &: \quad \approx 3 \end{aligned}$$

but

$$p_T = \frac{2}{5}, \alpha_T = \frac{5}{3}$$

T is largely heuristic independent ….
(and close to tricritical point!)

Conclusions

- Computational problems can be studied with statistical physics concepts and techniques

  (phase diagram, real-space renormalization, growth processes, ...)

- Algorithms induce interesting and unusual dynamics.

- General approach for decision or optimization problems solved by backtrack, or branch-and-bound algorithms.

➔ **Physical Review Letters 86, 1654 (2001)**

➔ **Cond-mat/0012191 , to appear in EPJ B**

➔ Nature 400, 133 (1999)