Auto-encoder

$$\varepsilon^2 \equiv \text{Squared Error} = \left\langle \left(\vec{v} - h(\vec{v})\, \vec{m}' \right)^2 \right\rangle_{\vec{v}}$$

with $h(\vec{v}) = \vec{m} \cdot \vec{v}$

$$= \sum_i \left\langle \left(v_i - \left(\sum_j m_j v_j \right) m'_i \right)^2 \right\rangle_{\vec{v}}$$

I assume $\langle v_i \rangle = 0$, $\langle v_i v_j \rangle = \Gamma_{ij}$ $(= \Gamma_{ji})$

$$\varepsilon^2 = \sum_i \Gamma_{ii} - 2 \sum_{ij} m'_i \Gamma_{ij} m_j + \left(\sum_{jk} m_j m_k \Gamma_{jk} \right) \left(\sum_i m_i'^2 \right)$$

$$\frac{\partial \varepsilon^2}{\partial m_i} = -2 \sum_j \Gamma_{ji} m'_j + \left(\sum_j m'^2_j \right) \cdot 2 \sum_k m_k \Gamma_{ki} = 0$$

Vectorial notation:

$$\boxed{\Gamma \cdot \vec{m}' = |\vec{m}'|^2 \cdot \Gamma \vec{m}} \quad (1)$$

$$\frac{\partial \varepsilon^2}{\partial m'_i} = -2 \sum_j \Gamma_{ij} m_j + 2 m'_i \left(\sum_{jk} m_j m_k \Gamma_{jk} \right) = 0$$

$$\boxed{\Gamma \vec{m} = \left(\vec{m}^{+} \Gamma \vec{m} \right) \vec{m}'} \quad (2)$$

(1) and (2) $\Rightarrow$ $\Gamma \vec{m}' = \underbrace{|\vec{m}'|^2 \left(\vec{m}^{+} \Gamma \vec{m} \right)}_{\text{scalar}} \vec{m}'$

$\Rightarrow$ $\vec{m}'$ is an eigenvector $\vec{w}$ of Γ with eigenvalue $\lambda = |\vec{m}'|^2 (\vec{m}^{+} \Gamma \vec{m})$

$\underset{(1)}{\Longrightarrow}$ $\Gamma \vec{m} \underset{|\vec{m}'|^2 \neq 0}{=} (\vec{m}^{+} \Gamma \vec{m})\, \vec{w}$

$\underset{\Gamma \text{ has no } 0 \text{ eigenvalue and } \lambda \text{ non-degenerate}}{\Longrightarrow}$ $\vec{m} = \vec{w}$ (with $|\vec{w}|^2 = 1$)

Hence : $\lambda = |\vec{m}'|^2 \underbrace{\vec{m}^{\dagger} \Gamma \vec{m}}_{\lambda} \Rightarrow |\vec{m}'|^2 = 1$

What $(\lambda, \vec{w})$ is selected?

$$\varepsilon^2 = \sum_i \Gamma_{ii} - 2\,\vec{m}'^{\dagger} \Gamma \vec{m} + |\vec{m}'|^2 \left(\vec{m}^{\dagger} \Gamma \vec{m}\right)$$

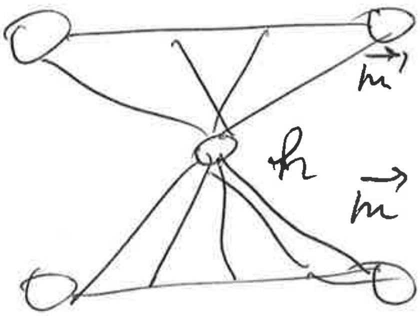
$$= \sum_\alpha \lambda_\alpha - 2\lambda + \lambda = \sum_\alpha \lambda_\alpha - \lambda$$

ε^2 is minimal if λ is maximal $\Longrightarrow$ PCA

Case of multiple ~~th~~ hidden units:

Denoising

$\vec{v}' \simeq \vec{v}$



$$\Longrightarrow \begin{cases} \vec{\cancel{m}} = \vec{w} \\ \vec{m}' = \dfrac{\lambda}{\lambda+\eta^2}\vec{w} \end{cases}$$

corrupted input $\vec{v} + \vec{\eta}$; $\langle \eta_i \rangle = 0$, $\langle \eta_i \eta_j \rangle = \eta^2 \delta_{ij}$

noise

"true" $\vec{v}$

$$\varepsilon^2 = \left\langle \left(\vec{v} - h(\vec{v}+\vec{\eta})\vec{m}'\right)^2 \right\rangle$$

$$= \sum_i \Gamma_{ii} - 2\sum_{ij} m_i' \Gamma_{ij} m_j + \left(\sum_{jk} m_j \left(\Gamma_{jk} + \eta^2 \delta_{jk}\right) m_k\right)\left(\sum_i m_i'^2\right)$$

$$\hookrightarrow \begin{cases} \Gamma \vec{m}' = |\vec{m}'|^2 \left(\Gamma + \eta^2 I\right) \vec{m} \\ \Gamma \vec{m} = \left(\vec{m}^{\dagger}\left(\Gamma + \eta^2 I\right)\vec{m}\right) \vec{m}' \end{cases}$$

We look for solution: $\begin{cases} \vec{m} = \alpha \vec{w} \\ \vec{m}' = \alpha' \vec{w} \end{cases}$

Then: $\begin{cases} \alpha^{*} \lambda = \alpha'^{2} (\lambda + \eta^{2}) \alpha \\ \alpha \lambda = \alpha^{2} (\lambda + \eta^{2}) \alpha' \end{cases}$ i.e. $\boxed{\alpha \alpha' = \frac{\lambda}{\lambda + \eta^{2}}}$