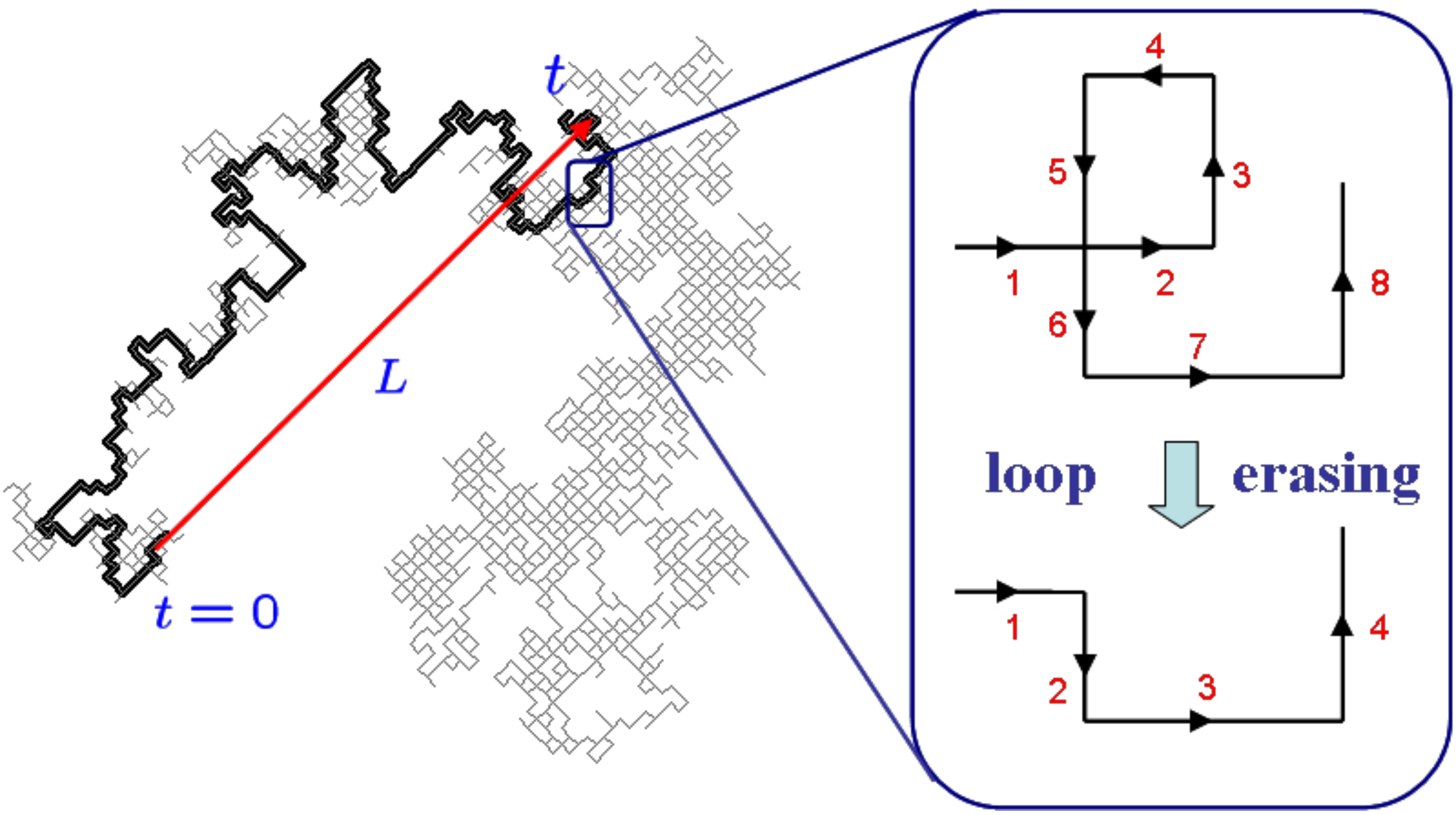
Research Highlight: Field theory for loop-erased random walks

The loop-erased random walk (LERW) was introduced by Lawler (1980) as a mathematically more tractable object than the self-avoiding walk (SAW). Nowadays, the LERW has numerous applications in combinatorics, self-organized criticality, conformal field theory and SLE. The SAW is defined as the uniform measure on all non-self-intersecting walks, and is well treatable by field-theoretic methods. The LERW is defined as the trajectory of a random walk in which any loop is erased as soon as it is formed:



A loop-erased random walk with its shadow, courtesy of Michel Bauer and Denis Bernard

The number of steps t (not counting the erased loops) it takes to reach the distance L scales as

$$t \sim L^{d_f}$$

where d_f is the fractal dimension of LERW. Contrary to the SAW, which, as shown by P.-G. de Gennes, is described by the $O(N)$ model at $N = 0$, up to now there was no field-theoretic approach to compute d_f in a dimensional expansion around the upper critical dimension $d_{uc} = 4$. We proposed a field-theoretic description for the LERW, based on the Functional Renormalization Group (FRG), a method developed to study disordered systems. We build on a long chain of mappings: depinning transition of a periodic elastic system in random media ⇒ sandpile models ⇒ uniform spanning trees ⇒ LERW. To second order in $\varepsilon = 4 - d$ we obtain

$$d_f = 2 - \frac{\varepsilon}{3} - \frac{\varepsilon^2}{9} + O(\varepsilon^3).$$

This is consistent with the exact value d_f=5/4 derived using conformal field theory in d = 2. In d = 3 the Padé approximation gives d_f = 1.614 ± 0.011, in fairly good agreement with the numerical estimation d_f = 1.6183 ± 0.0004. The FRG passes all the tests of presently known results for LERW. In particular, it reproduces the correct leading logarithmic corrections at the upper critical dimension and makes a prediction for the subleading logarithmic correction:

$$L^2 \sim t(\ln t)^{1/3}\left[1 - \frac{\ln\ln t}{3\ln t} + O\left(\frac{1}{\ln t}\right)\right],$$

which remains to be tested in numerical simulations.

A.A. Fedorenko, P. Le Doussal, and K.J. Wiese, arXiv:0803.2357