Large amplitude spin waves in ultra-cold gases

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Abstract

We discuss the theory of spin waves in non-degenerate ultra-cold gases, and compare various methods which can be used to obtain appropriate kinetic equations. We then study non-hydrodynamic situations, where the amplitude of spin waves is sufficiently large to bring the system far from local equilibrium. The full position and momentum dependence of the distribution function must then be retained.

In the first part of the article, we compare two general methods which can be used to derive a kinetic equation for a dilute gas of atoms (bosons or fermions) with two internal states (treated as a pseudo-spin 1/2). The collisional methods are in the spirit of Boltzmann's original derivation of his kinetic equation where, at each point of space, the effects of all sorts of possible binary collisions are added. We discuss two different versions of collisional methods, the Yvon-Snider approach and the S matrix approach. The second method uses the notion of mean field, which modifies the drift term of the kinetic equation, in the line of the Landau theory of transport in quantum liquids. For a dilute cold gas, it turns out that all these derivations lead to the same drift terms in the transport equation, but differ in the precise expression of the collision integral and in higher order gradient terms.

In the second part of the article, the kinetic equation is applied to spin waves (or internal conversion) in trapped ultra-cold gases. Numerical simulations are used to illustrate the strongly non-hydrodynamic character of the spin waves recently observed with trapped ^{87}Rb atoms. The decay of the phenomenon, which takes place when the system relaxes back towards equilibrium, is also discussed, with a short comment on decoherence.

In two appendices we calculate the Wigner transform of the interaction term in the S matrix method, to first order in gradients; appendix I treats the case of spin-independent interactions, appendix II that of spin-dependent interactions.

1   Introduction

Recent experiments [1, 2] have renewed the interest in spin waves in dilute quantum gases, in conditions where they had not been observed before. A first novelty is that these waves were neither purely nuclear nor electronic, but involved two hyperfine atomic levels where the nuclear and electronic spin are coupled. This difference is actually minor: the exchange effects from which the waves originate are unaffected by the precise nature of the levels - in other words one can, without any loss of generality, assimilate any pair of atomic states to up and down states of a fictitious spin. The second, more important, difference is that the waves were observed with very large amplitudes, leading to an almost complete apparent segregation of

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the atoms in each internal state, and involving situations where the gas is very far from local equilibrium; we study such situations in the present article.

In degenerate liquids, the existence of spin waves has been known for many years [3, 4]. In dilute gases, which are non-degenerate, it took about twenty more years [5, 6] to realize that they should also sustain similar waves, for both bosonic and fermionic systems. This came as a surprise to some since, before, these waves had been mostly associated with the Landau formalism for Fermi liquids, and with the notion of "molecular field" resulting from exchange and interactions with nearest neighbors. In a liquid, a test particle remains trapped in a sort of local cavity and interacts constantly with several neighbors, so that the notion of mean field emerges naturally from the averaging over the effect of several neighbors. By contrast, in a gas, the motion of a particle is often described as free flights along straight lines, interrupted by short collisional processes, so that the physics involved is clearly very different. It nevertheless turns out that, when averaged over all possible collisions at each point of space, the final result of the exchange interactions in a gas and a liquid are very similar (see, for example, Ref. [7] for a general discussion). Soon after these theoretical predictions, experimental observations demonstrated the existence of spin waves in spin polarized Hydrogen gas [8], ^{3}He gas [9], and dilute ^{3}He-^{4}He solutions [10].

The physical difference between gases and liquids has its counterpart in the different theoretical approaches used to derive kinetic equations. Historically, the first derivation was that of Boltzmann, with his Stosszahlansatz (or molecular chaos Ansatz): collisions are considered as point processes (with no duration and no spatial extent) taking place between particles which are completely uncorrelated before collision. As a consequence, one can add at each point of space the effect of all possible binary collisions between uncorrelated particles, as if one was adding the effect of many "beam to beam collision experiments" in atomic physics. This provides the famous Boltzmann equation, with a relatively complicated collision term on the right hand side, while the drift term on the left hand side corresponds merely to a completely free flight of particles between collisions. In the approach introduced later by Landau for the study of degenerate Fermi liquids [11], quasi-particles are never free, since their motion is constantly guided by a mean field created by the neighbor particles. Therefore interactions modify the drift term while, on the other hand, the collision term is generally treated phenomenologically, by a simple relaxation approximation. Both point of view have their advantages. Boltzmann's point of view does not require the use of a pseudo-potential and therefore allows a more microscopic treatment of collisions, for instance a more precise inclusion of lateral collisions including their full angular end energy dependence. On the other hand, it does not encompass quantum degenerate systems, does not introduce the powerful notion of quasi-particles, and generally speaking remains limited to dilute interacting gases.

The connection between the two points of view is provided by collisions in the forward direction. In two body collision theory, it is well known that wave interference in the forward direction is responsible for particle absorption (optical theorem). In addition, more physics may be involved in this forward interference effect: for spinless particles, retardation effects are introduced (see for instance [12]); for particles with spin, in addition, identical spin rotation effects (ISRE) due to particle indistinguishability [13, 6] also take place. In many-body transport theory, both these effects have their equivalents. The cumulated retardation effects during collisions give rise to an average force, or equivalently to a scalar mean field (see for instance [14] and references therein); in the kinetic equation, this corresponds to terms which are quadratic in the distribution function, as the collision term, but contain gradients (spatial gradients of the density as well as momentum gradients of the particle distribution); in other

words, one gets zero sound Landau type mean field terms which are naturally grouped with the drift term on the l.h.s. of the kinetic equation. As for the identical spin rotation effects, they average to a spin molecular field which is the equivalent of the molecular field considered by Silin [3] and Leggett [4]; the ISRE is already present to zero order gradient expansion, as opposed to the scalar retardation terms [13, 15], which explains why the dispersion relation of spin waves is different from the usual sound wave dispersion. A detailed study of forward scattering in a binary collision can therefore lead to a microscopic understanding of the origin of the Landau mean field (lateral scattering introduces collisional damping). From a practical point of view, it remains true that the mean field treatment of the drift term, necessarily associated with the use of a pseudo-potential, is often more compact and elegant than a detailed study of the collisional properties in the forward direction; it is nevertheless interesting to check precisely to what extent they are equivalent in various situations.

Our purpose in this article is twofold. First, for particles with spins, the equivalence in question has been verified in the literature only to zero order in the gradient expansion, in other words only for the local terms appearing in the kinetic equation. Here we wish to derive the kinetic equation to first (non-local) order in the gradient expansion. Moreover, we will consider the situation where the scattering may depend on the relative spin orientations (even if this dependence was relatively weak in the experiments of the JILA group [1, 2]). We will therefore examine in detail the properties of collisional interference in the forward direction in this more general case; the details of the calculations are given in two appendices (gradient expansion of the interaction term within the S-matrix collision Ansatz). Second, we also wish to study situation where the gas is far from local equilibrium. In fact, most of the work on spin waves in the literature deals with hydrodynamic situations where, at every point of space, the gas is close to local equilibrium, so that a simple hydrodynamic expression of the spin current can be used; of course, this reduces the number of variables, but at the price of an approximation which is not necessarily justified. In fact, in the experiments in question, the gas sometimes evolves very far from an hydrodynamic regime, which is not so surprising since the spin waves have a large amplitude and lead to an almost complete segregation of the two spin species. For a study of spin waves in the collisionless regime (mean-free-path large compared to characteristic lengths), but still close to equilibrium (linear regime), see Ref. [16].

2 Kinetic equation

We study an ensemble of identical atoms with two internal levels, obeying either Bose or Fermi statistics. As mentioned in the introduction we can, without any loss of generality, assimilate these two levels to two spin levels, whatever their real physical origin is. For instance, hyperfine atomic structure states, even in a situation of intermediate magnetic decoupling, are possible; the only important thing is that the quantum states describing the internal variables should be orthogonal. We now derive a kinetic equation for a dilute gas of such atoms.

2.1 Collisional methods; Yvon-Snider and Lhuillier-Laloë equations

In classical statistical mechanics, Boltzmann's intuitive method for deriving a kinetic equation is based on the study of individual collisions; one adds the effect of all possible binary collisions taking part at each point of space on the single particle distribution in phase-space $f_1(\mathbf{r}, \mathbf{p})$. Another point of view starts from the infinite hierarchy of BBGKY equations and closes its

first equation, which relates f_1 to the two particle distribution function f_2, by studying the evolution of f_2 during a binary collision and expressing it as some functional of f_1 (see for instance Ref. [17, 18]).

The same lines can be followed in quantum mechanics, but $f_1(\mathbf{r}, \mathbf{p})$ has to be replaced by the reduced single-particle density operator $\hat{\rho}_1$. Since we assume here that the particles have two internal states, which we treat as spin states, the single-particle density operator $\hat{\rho}_1$ acts in the product space of orbital and spin variables of one particle. The operatorial kinetic equation has the form:

$$\frac{d\hat{\rho}_1}{dt} + \frac{1}{i\hbar} \left[\hat{\rho}_1, \hat{H}_1\right]_- \simeq \left.\frac{d}{dt}\right|_{coll} \hat{\rho}_1 \tag{1}$$

where:

$$\hat{H}_1 = \frac{\mathbf{p}^2}{2m}\hat{1} + \hat{V}^{ext} \tag{2}$$

is the single-particle Hamiltonian, $\mathbf{p}$ the momentum of the particle, m its mass, $\hat{1}$ the unity operator in spin space and $\hat{V}^{ext}$ the operator describing the external forces acting on the particle, which may be spin dependent. The r.h.s. of eq. (1) takes into account the effects of binary interactions between the particles; it contains the result of the collisional approximation mentioned above, and will be discussed in more detail below. The equation is only valid for a dilute gas ($n^{-1/3} \gg a$, where n is the number density of the gas and a the scattering length) and on time/length scales much greater than the duration of a collision/scattering length.

The next step is to introduce the Wigner transform [19, 20] $\hat{\rho}_W(\mathbf{r}, \mathbf{p})$ of $\hat{\rho}_1$ with respect to orbital variables:

$$\hat{\rho}_W(\mathbf{r}, \mathbf{p}) \equiv (2\pi\hbar)^{-3} \int d^3\mathbf{r}' \, e^{i\mathbf{p}\cdot\mathbf{r}'/\hbar} \, \langle\mathbf{r} - \frac{\mathbf{r}'}{2}|\hat{\rho}_1|\mathbf{r} + \frac{\mathbf{r}'}{2}\rangle \tag{3}$$

It is a classical function of position and momentum, but still an operator - or a 2×2 matrix - in spin space. In the following, we will treat the orbital degrees of freedom semi-classically, but not the spin degrees of freedom. For the sake of simplicity, from now on we will drop the index W, assuming that the dependence on $\mathbf{r}$ and $\mathbf{p}$ is sufficient to signal a Wigner distribution. One then performs a Wigner transform of equation (1), which involves taking the Wigner transform of products of operators. This is possible by using Groenewold's formula [21, 20], which provides the result as a infinite series of gradient expansion: the first term is merely the product of the Wigner transforms of the two operators, followed by product of gradients with respect to $\mathbf{r}$ and $\mathbf{p}$, followed by higher order gradients, etc. The Wigner transform of the l.h.s. of equation (1) then gives:

$$\partial_t\hat{\rho}(\mathbf{r}, \mathbf{p}) + \frac{\mathbf{p}}{m} \cdot \nabla_\mathbf{r}\hat{\rho}(\mathbf{r}, \mathbf{p}) + \frac{1}{i\hbar}[\hat{\rho}(\mathbf{r}, \mathbf{p}), \widehat{V}^{ext}(\mathbf{r})]_-$$
$$- \frac{1}{2}[\nabla_\mathbf{p}\hat{\rho}(\mathbf{r}, \mathbf{p}), \cdot\nabla_\mathbf{r}\widehat{V}^{ext}(\mathbf{r})]_+ + ... \tag{4}$$

The first two terms are exact, whereas the next two are only the first orders terms in a gradient expansion involving the external potential. The commutator is obtained to zero order of the gradient expansion while the anticommutator occurs only to first order, followed by higher order terms symbolized by the dots. The anticommutator corresponds to the classical force term (describing the effect of a trap, for example), which we wish to retain in our calculations; as a consequence, we have include up to first order gradient expansion in the Wigner transform of the r.h.s. of equation (1) .

The non-trivial part in the derivation for a kinetic equation begins when one gives a precise expression to the formal r.h.s. of equation (1). This is the subject of many studies in the literature and even the subject of books; here, we limit ourselves to a simplified discussion of two different approaches which are appropriates for the study of spin waves. We consider only collisions taking place in elastic channels, where the number of atoms in each internal spin state is conserved, and begin with some simple considerations on elementary collision theory. For each channel, at very low temperatures, collisions occur in the extreme quantum regime where the typical s-wave scattering length a is small compared to the de Broglie thermal wavelength λ_T. In this case, collisions are well described by the isotropic s-wave, and the T matrix can be approximated by:

$$T_k(\hat{k}, \hat{k}') = -\frac{\hbar^2}{2\pi^2 m} f_k(\theta) = \frac{\hbar^2}{2\pi^2 m}(a - ika^2 + ..) \tag{5}$$

(terms of order a^2 should be retained in order to satisfy explicitly the unitarity of the S-matrix in the optical theorem); in this equation, m is the mass of the particles, $f_k(\theta)$ the scattering amplitude which, for s-wave scattering, is independent of the polar angle θ, and k the collision wave vector.

A first obvious remark is that the intensity of the spherical scattered wave is proportional to the square of a, while interference effects in the forward direction between the incoming plane wave and the spherical scattered wave are proportional to a itself. Generally speaking, the relative magnitude of forward and lateral scattering effects is of order λ_T/a, which is large at low energies. This is why, here, we concentrate on forward scattering effects and their relation to mean field corrections to the drift terms. Another remark is that, because waves interfere only if they are in phase, phases and i factors are important here. It turns out that, in quantum mechanics as well as in optics [22, 23], the summation over many scatterers in the forward direction introduces an additional i factor (which is actually the origin of the i factor in the optical theorem). Hence, in every scattering channel, equation (5) shows that no first order a interference effect occurs in the forward direction, the first contribution arising from the second term in ika^2; to first order in a, only the phase of the forward wave is changed, not its intensity. But, if several scattering channels are open, the situation is different, because these different phase shifts in the forward scattered waves can introduce spin rotation. Actually, this is precisely the origin of the first order identical spin rotation effect in the forward direction [13, 6].

2.1.1 Spin-independent interactions

We now briefly summarize two approaches that have been developed in the context of particles with spin: the Yvon-Snider approximation, which emphasizes more the relation to the BBGKY hierarchy, and closes the infinite set of equations with an approximation suitable for a binary collision; the Lhuillier-Laloë (or S matrix) approach, which is closer to the initial approach of Boltzmann and makes use of the S collision matrix. We begin with the simplest case, when the interactions are independent of spin.

Yvon-Snider An interesting method is that originally proposed by Yvon [24], then independently by Snider [25]; it provides a result which, in the literature, is often called the Waldmann-Snider equation; the main idea is to express the collision term as the trace over a collision partner (particle 2) of a commutator containing the binary interaction potential V_{12}

and a product of single particle operators modified by the unitary transformation associated with the Møller collision operator $\Omega \equiv \Omega^{(+)}$:

$$\left. \frac{d}{dt} \right|_{coll} \hat{\rho}_1(1) = (i\hbar)^{-1} \, Tr_2 \left\{ \left[V_{12} \, , \, \Omega \, \hat{\rho}_1(1)\hat{\rho}_1(2) \, \Omega^\dagger \right]_- \right\} \tag{6}$$

This equation is valid for dilute gases of distinguishable particles; for non-degenerate gases ($n\lambda_T^3 \ll 1$, where λ_T is the thermal wavelength of the particles) of bosons or fermions, it generalizes into:

$$\left. \frac{d}{dt} \right|_{coll} \hat{\rho}_1(1) = (i\hbar)^{-1} \, Tr_2 \left\{ \frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}} \left[V_{12} \, , \, \Omega \, \hat{\rho}_1(1)\hat{\rho}_1(2) \, \Omega^\dagger \right]_- \frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}} \right\} \tag{7}$$

where $\hat{P}_{ex.}$ is the exchange operator between particles 1 and 2, and where:

$$\epsilon = 1 \text{ for bosons, } \epsilon = -1 \text{ for fermions} \tag{8}$$

Wigner transforms can then be applied to these expressions to complete the derivation of the closed kinetic equation for $\hat{\rho}(\mathbf{r}, \mathbf{p})$. For instance, expanding (6) in a Groenewold gradient expansion of the products provides, to zero order in gradients, the straightforward Boltzmann expression for the collision term:

$$\left. \frac{d}{dt} \right|_{coll} \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) = - \int d^3 q \frac{q}{m} \left[\sigma_T(k) \, \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) \, f(\mathbf{r}_1, \mathbf{p}_2) \right.$$
$$\left. - \int d^2\hat{k}' \, \sigma_k(\theta) \, \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1') \, f(\mathbf{r}_1, \mathbf{p}_2') \right] \tag{9}$$

where $f(\mathbf{r}, \mathbf{p})$ is the spin trace of $\hat{\rho}(\mathbf{r}, \mathbf{p})$:

$$f(\mathbf{r}, \mathbf{p}) = Tr_S \left\{ \hat{\rho}(\mathbf{r}, \mathbf{p}) \right\} \tag{10}$$

and where $\hat{k}$ is the unit vector in the direction of $\mathbf{k}$ and θ the angle between $\mathbf{k}$ and $\mathbf{k}'$. Here we use the usual notation in Boltzmann theory:

$$\begin{array}{cc} \mathbf{q} = 2\hbar\mathbf{k} & \mathbf{p}_2 = \mathbf{p}_1 - \mathbf{q} \\ \mathbf{p}_1' = \mathbf{p} - (\mathbf{q}/2) + \hbar k\hat{k}' & \mathbf{p}_2' = \mathbf{p} - (\mathbf{q}/2) - \hbar k\hat{k}' \end{array} \tag{11}$$

as well as the definition of the differential and total cross sections:

$$\sigma_k(\theta) = (4\pi^4 m^2/\hbar^4) \, |T(\mathbf{k}, \mathbf{k}')|^2 \, ; \quad \sigma_T(k) = \int d^2\hat{k}' \, \sigma_k(\theta) \tag{12}$$

Higher order terms will introduce scalar mean field terms, which we do not write here explicitly for concision; they can be found in the appendix of ref. [14].

Now, if we introduce spin and statistics, we have to use (7) instead of (6). To zero order in gradients, this introduces the additional exchange terms to the right hand side of (9):

$$-\frac{\epsilon}{2} \int d^3 q \, \frac{q}{m} \left\{ i\tau_{fwd.}^{ex.} (k) \, [\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1), \widehat{\rho}(\mathbf{r}_1, \mathbf{p}_2)]_- \right.$$
$$- \int d^2\hat{k}' \, i\tau_k^{ex.}(\theta) \, [\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1'), \, \widehat{\rho}(\mathbf{r}_1, \mathbf{p}_2')]_- +$$
$$\left. + \int d^2\hat{k}' \, \sigma_k^{ex.}(\theta) \left([\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1), \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2)]_+ - [\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1'), \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2')]_+ \right) \right\} \tag{13}$$

where the "generalized cross sections" are defined by:

$$\tau_{fwd.}^{ex.}(k) = (-8\pi^3 m/\hbar^2 k) \ \mathrm{Re}\left\{T(-\mathbf{k}, \mathbf{k})\right\} \tag{14}$$

and:

$$\sigma_k^{ex.}(\theta) - i\tau_k^{ex.}(\theta) \ = (4\pi^4 m^2/\hbar^4) \ T(-\mathbf{k}, \mathbf{k}') \ T^*(\mathbf{k}, \mathbf{k}') \tag{15}$$

In (13), the two terms in the first two lines correspond respectively to the ISRE in the forward and lateral direction; the two terms in the third line correspond to exchange effects changing the values of the total and lateral cross sections, as discussed in more detail in [13]. For simplicity, we do not write the first order gradient terms, but more details on the limit of validity of the Yvon-Snider equation and of the various terms which it contains can be found in [26], [15] or [14] (in particular the Appendix of this last reference). Inserting the low energy limit (5) is trivial; we write the explicit results in the next section.

Lhuillier-Laloë Another point of view, more directly in the spirit of the initial Ansatz of Boltzmann, was employed by Lhuillier and Laloë (LL) [13]; it is based on the use of the S collision matrix. When collisions are treated only as "closed processes" (one ignores particles "in the middle of a collision"), it is indeed possible to use the S matrix to relate exactly the single particle operators after and before collision. One gets the following expression for the single particle density operator after collision $\hat{\rho}_1'$ as a function of the same operator $\hat{\rho}_1$ before collision:

$$\hat{\rho}_1'(1) = Tr_2\left\{\frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}} \ \hat{S} \ \hat{\rho}_1(1) \ \hat{\rho}_1(2) \ \hat{S}^\dagger \ \frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}}\right\} \tag{16}$$

(the trace acts over orbital and spin variables of the collision partner 2) where the S matrix is related to the collision T matrix by:

$$\langle 1 : \mathbf{k}_f; 2 : -\mathbf{k}_f \mid \hat{S} \mid 1 : \mathbf{k}_i; 2 : -\mathbf{k}_i\rangle = \delta(\mathbf{k}_f - \mathbf{k}_i) - i\frac{\pi m}{\hbar^2 k_i} \ \delta(k_f - k_i) \ T_{k_i} \tag{17}$$

($\hbar k$ is the relative momentum of the two particles). One then approximates the rate of change of $\hat{\rho}_1$ by the variation $(\hat{\rho}_1' - \hat{\rho}_1)/\Delta t$ during a short time interval [1] Δt, which is however much longer than the duration of a collision, to obtain:

$$\left.\frac{d}{dt}\right|_{coll}\hat{\rho}_1(1) \simeq \frac{1}{\Delta t}Tr_2\left\{\frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}}\left[\hat{S} \ \hat{\rho}_1(1) \ \hat{\rho}_1(2) \ \hat{S}^\dagger - \hat{\rho}_1(1) \ \hat{\rho}_1(2)\right]\frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}}\right\} \tag{18}$$

One should note that the S matrix Ansatz of equation (18) is only valid for a non-degenerate gas where:

$$Tr_2\left\{\frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}}\hat{\rho}_1(1) \ \hat{\rho}_1(2)\frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}}\right\} = \hat{\rho}_1(1) + \epsilon\hat{\rho}_1(1)^2 \simeq \hat{\rho}_1(1) \tag{19}$$

As above, the kinetic equation is obtained from equation (18) by performing a Wigner transform followed by a gradient expansion. In Appendix I we first calculate the zero order gradient expansion, which is given by equation (76) ; it is easy to see that it coincides exactly

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[1]We will see that Δt is a time which emerges naturally from the calculation through the occurrence of square of delta functions of the energy (see Appendix I).

with the sum of (9) and (13), which shows that the Yvon-Snider and S matrix approximations are exactly equivalent to this order. The first order gradient terms are also given in the same Appendix; here, we only give their low energy limit by using equation (5), which provides the following kinetic equation:

$$\partial_t \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) + \frac{\mathbf{p}_1}{m} \cdot \nabla_{\mathbf{r}} \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) + \frac{1}{i\hbar} \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1), \widehat{V}^{ext}(\mathbf{r}_1) + \epsilon \, g\hat{n}(\mathbf{r}_1) \right]_-$$

$$- \frac{1}{2} \left[\nabla_{\mathbf{p}} \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1), \cdot \nabla_{\mathbf{r}} \left(\widehat{V}^{ext}(\mathbf{r}_1) + gn(\mathbf{r}_1)\hat{1} + \epsilon \, g\hat{n}(\mathbf{r}_1) \right) \right]_+ = I_{coll}[\hat{\rho}] \tag{20}$$

where $g \equiv 4\pi\hbar^2 a/m$ and $\hat{n}(\mathbf{r}_1)$ is the local spin density operator integrated over velocities:

$$\hat{n}(\mathbf{r}) = \int d^3p \ \hat{\rho}(\mathbf{r}, \mathbf{p}) \tag{21}$$

In this equation, as usual, forward scattering terms linear in the coupling constant $g \propto a$ have been included in the l.h.s. The local term of the collision integral, which regroups terms proportional to a^2, is given by:

$$I_{coll}[\hat{\rho}] = - \int d^3q \, \frac{q}{m} \int d^2\hat{k}' \, a^2 \Big(\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) f(\mathbf{r}_1, \mathbf{p}_2) - \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1') f(\mathbf{r}_1, \mathbf{p}_2')$$

$$+ \frac{\epsilon}{2} \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1), \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2) \right]_+ - \frac{\epsilon}{2} \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1'), \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2') \right]_+ \Big) \tag{22}$$

where $\mathbf{p}_2$, $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are defined in (11). In addition to the anticommutator in the l.h.s. of (20), obtained to first order in the gradient expansion, we recover exactly the low-energy limit of the LL equation [13], with in the l.h.s. the ISRE in the forward direction (the ISRE in the lateral directions disappears in the low energy limit); the second line of (22) provides the statistical terms introduced by particle indistinguishability into the total and differential collision cross section.

2.1.2 Spin-dependent interactions

We now study the gas with spin-dependent interactions and, for a moment, consider the particles as distinguishable. In the limit of slow elastic collisions, we need only consider four s-wave scattering lengths: a_{11}, a_{22}, $a_{12}^{(d)}$ and $a_{12}^{(t)}$. The two first describe collisions processes between particles that are in the same internal state; when they are in othogonal internal states, two different processes occur for distinguishable particles: a direct collision process without energy transfer described by $a_{12}^{(d)}$, and a transfer collision process described by $a_{12}^{(t)}$ - similar processes occur in the theory of spin exchange collisions [27]. The T matrix is now a 4×4 matrix in spin space; to lowest order in the scattering lengths, its matrix elements with distinguishable particles are given by:

$$\langle \, 1 : \gamma \, ; 2 : \delta | \ \hat{T}_k \ | \, 1 : \alpha \, ; 2 : \beta \rangle = \frac{\hbar^2}{2\pi^2 m} \begin{pmatrix} \widetilde{a}_{11}(k) & 0 & 0 & 0 \\ 0 & \widetilde{a}_{22}(k) & 0 & 0 \\ 0 & 0 & \widetilde{a}_{12}^{(d)}(k) & \widetilde{a}_{12}^{(t)}(k) \\ 0 & 0 & \widetilde{a}_{12}^{(t)}(k) & \widetilde{a}_{12}^{(d)}(k) \end{pmatrix} \tag{23}$$

where α, β, γ and δ are the internal state indexes taking values $\{1, 2\}$ and where:

$$\widetilde{a}_{\alpha\beta}(k) = a_{\alpha\beta} \left[1 - ika_{\alpha\beta} \right] \tag{24}$$

Equation (23) replaces (5) when the interactions depend on spin.

Kinetic equation The derivation of the kinetic equation then proceeds along the same lines as in the previous section: we use (16) again to symmetrize the density operator and take quantum statistics into account, but now insert (23) instead of (17) in it, calculate the partial trace, and obtain an interaction term for the single particle density operator; we then Wigner transform the result and use a gradient expansion limited to first order. This calculation is similar to that of Appendix I for spin-independent interactions, but of course more complicated; it provides more general expressions for the mean field terms and the collision integrals. For conciseness, we limit ourselves to the mean field terms, which will actually be sufficient for our calculations below. In this particular case, the terms can actually be obtained more conveniently by an operatorial method, different from that of Appendix I, as shown in Appendix II. We only reproduce the final kinetic equation:

$$\partial_t \hat{\rho}(\mathbf{r}, \mathbf{p}) + \frac{\mathbf{p}}{m} \cdot \nabla_\mathbf{r} \hat{\rho}(\mathbf{r}, \mathbf{p}) + \frac{1}{i\hbar} [\hat{\rho}(\mathbf{r}, \mathbf{p}), \widehat{U}(\mathbf{r})]_-$$
$$- \frac{1}{2} [\nabla_\mathbf{p} \hat{\rho}(\mathbf{r}, \mathbf{p}), \cdot \nabla_\mathbf{r} \widehat{U}(\mathbf{r})]_+ = I_{coll}[\hat{\rho}] \tag{25}$$

where the effective single-particle potential $\widehat{U}$ is:

$$\widehat{U} = \widehat{V}^{ext} + \begin{pmatrix} (1+\epsilon)g_{22}n_2 + g_{12}n_1 & \epsilon g_{12}n_{21} \\ \epsilon g_{12}n_{12} & (1+\epsilon)g_{11}n_1 + g_{12}n_2 \end{pmatrix} \tag{26}$$

with $g_{\alpha\beta} \equiv 4\pi\hbar^2 a_{\alpha\beta}/m$, where $a_{12} \equiv a_{12}^{(d)} + \epsilon a_{12}^{(t)}$; it turns out that only this combination of the direct and transfer coupling constants enters the calculation (for more details, see Appendix II).

This kinetic equation resembles the Landau-Silin equation [3, 28] for a normal Fermi liquid in a magnetic field. The second term in the l.h.s. of (25) is the usual free drift term; the anticommutator is a force term including both the effect of an external potential and of the mean field; the commutator is a spin precession term containing effective magnetic fields from different origins (differential Zeeman effect, spin mean field, etc.).

Evolution of the density and spin density in phase space The structure of the distribution function, which is still a matrix in the space of internal variables, is more explicit if we decompose $\hat{\rho}$ and $\hat{U}$ in the basis formed by Pauli matrices $\hat{\boldsymbol{\sigma}}$ and the unit 2×2 matrix:

$$\hat{\rho}(\mathbf{r}, \mathbf{p}, t) = \frac{1}{2} \left(f(\mathbf{r}, \mathbf{p}, t)\hat{1} + \mathbf{M}(\mathbf{r}, \mathbf{p}, t) \cdot \hat{\boldsymbol{\sigma}} \right)$$
$$\hat{U}(\mathbf{r}, t) = U_0(\mathbf{r}, t)\hat{1} + \mathbf{U}(\mathbf{r}, t) \cdot \hat{\boldsymbol{\sigma}} \tag{27}$$

When written in terms of the phase-space density f and spin density $\mathbf{M}$, equation (25) becomes:

$$\partial_t f + \frac{\mathbf{p}}{m} \cdot \nabla_\mathbf{r} f - \nabla_\mathbf{r} U_0 \cdot \nabla_\mathbf{p} f - \nabla_r \mathbf{U} \cdot \nabla_\mathbf{p} \mathbf{M} = I_{coll}^{(f)}[f, \mathbf{M}]$$
$$\partial_t \mathbf{M} + \frac{\mathbf{p}}{m} \cdot \nabla_\mathbf{r} \mathbf{M} - \nabla_\mathbf{r} U_0 \cdot \nabla_\mathbf{p} \mathbf{M} - \nabla_\mathbf{r} \mathbf{U} \cdot \nabla_\mathbf{p} f - \frac{2\mathbf{U}}{\hbar} \times \mathbf{M} = I_{coll}^{(M)}[f, \mathbf{M}] \tag{28}$$

In these equations, U_0 plays the role of the overall trapping potential and $\mathbf{U}$ is the local effective magnetic field around which the spins rotate. The effective trapping potential U_0 is

given by:

$$U_0(\mathbf{r}, t) = \frac{V_2^{ext} + V_1^{ext}}{2} + \left[\frac{1+\epsilon}{2}(g_{22} + g_{11}) + g_{12} \right] \frac{n}{2}$$
$$+ \frac{1+\epsilon}{2}(g_{22} - g_{11}) \frac{m_\parallel}{2} \tag{29}$$

and the effective magnetic field is:

$$\mathbf{U}(\mathbf{r}, t) = \frac{\hbar\Omega(x,t)}{2}\mathbf{e}_\parallel + \epsilon\frac{g_{12}\mathbf{m}(x,t)}{2}$$
$$\hbar\Omega(\mathbf{r}, t) = V_2^{ext} - V_1^{ext} + \frac{1+\epsilon}{2}\left[(g_{22} - g_{11})n + (g_{22} + g_{11} - 2g_{12})m_\parallel \right] \tag{30}$$

where the local density and spin polarization density are defined as:

$$n = n_2 + n_1 \,;\, m_\parallel = n_2 - n_1$$
$$m_{\perp,1} = n_{21} + n_{12} \,;\, m_{\perp,2} = i(n_{21} - n_{12}) \tag{31}$$

The basis in spin space is denoted by $\{e_{\perp,1};\, e_{\perp,2};\, e_\parallel\}$. The two first vectors define the transverse plane, the last defines the longitudinal direction.

The collision integral in (25), i.e. the equivalent of eq. (22) when the interactions are spin-dependent; to simplify the calculations, and because we will mostly study experimental situations where lateral collisions do not play a dominant role, we limit ourselves to a simple relaxation time approximation:

$$I_{coll}^{(f)} \simeq -\frac{f(\mathbf{r}, \mathbf{p}, t) - f^{eq}(\mathbf{r}, \mathbf{p}, t)}{\tau};\, I_{coll}^{(M)} \simeq -\frac{\mathbf{M}(\mathbf{r}, \mathbf{p}, t) - \mathbf{M}^{eq}(\mathbf{r}, \mathbf{p}, t)}{\tau} \tag{32}$$

where f^{eq} (resp. $\mathbf{M}^{eq}$) is the local equilibrium (spin) density in phase-space, and τ is a relaxation time. Actually, one could allow for different relaxation times for f, $\mathbf{M}_\parallel$ and $\mathbf{M}_\perp$. Nevertheless, in the case of a non-degenerate gas with spin-independent interactions, it is known (Ref. [29] and references therein) that these relaxation times are equal; below we consider a non-degenerate gas of ^{87}Rb, for which the three scattering lengths a_{11}, a_{22} and a_{12} are very close [1], so that in practice we will we only need a single relaxation time τ given approximately by:

$$\tau \simeq \frac{1}{4\pi a_{12}^2\, n} \sqrt{\frac{m}{k_B T}} \tag{33}$$

Finally, the kinetic equations we will use below to discuss spin waves are:

$$\partial_t f + \frac{\mathbf{p}}{m} \cdot \nabla_\mathbf{r} f - \nabla_\mathbf{r} U_0 \cdot \nabla_\mathbf{p} f - \nabla_\mathbf{r}\mathbf{U} \cdot \nabla_\mathbf{p}\mathbf{M} \simeq -\frac{f - f^{eq}}{\tau}$$
$$\partial_t\mathbf{M} + \frac{\mathbf{p}}{m} \cdot \nabla_\mathbf{r}\mathbf{M} - \nabla_\mathbf{r} U_0 \cdot \nabla_\mathbf{p}\mathbf{M} - \nabla_\mathbf{r}\mathbf{U} \cdot \nabla_\mathbf{p} f - \frac{2\mathbf{U}}{\hbar} \times \mathbf{M} \simeq -\frac{\mathbf{M} - \mathbf{M}^{eq}}{\tau} \tag{34}$$

The main differences with the LL kinetic equation derived in Ref. [13] are that the full mean field is included (not just the spin mean field), and that the equation is obtained for general spin-dependent interactions. Nevertheless, our result is also less general, since the collisions are assumed to occur only in the s-wave channel and that the collision integral is treated at the relaxation time approximation level (if necessary, these two restrictions could be lifted without any special difficulty).

2.2 Mean field

Another way to derive the modified drift term is to directly use a mean field approximation within field theory. The following second quantized Hamiltonian density is used to describe a trapped ultra-cold gas of bosons or fermions:

$$
\begin{aligned}
\mathcal{H} &= \mathcal{H}_1 + \mathcal{H}_{int} \\
&= \sum_{\alpha=1,2} \left(\frac{\hbar^2}{2m} \nabla \psi_\alpha^\dagger(\mathbf{r}) \cdot \nabla \psi_\alpha(\mathbf{r}) + V_\alpha^{ext}(\mathbf{r}) \psi_\alpha^\dagger(\mathbf{r}) \psi_\alpha(\mathbf{r}) \right) \\
&\quad + \frac{1}{2} \sum_{\alpha,\beta} g_{\alpha\beta} \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}) \psi_\beta(\mathbf{r}) \psi_\alpha(\mathbf{r})
\end{aligned}
\tag{35}
$$

The annihilation and creation field operators of a particle in the internal state α are $\psi_\alpha(\mathbf{r})$ and $\psi_\alpha^\dagger(\mathbf{r})$. They have bosonic or fermionic equal-time commutation or anticommutation relations. The interactions only depend on the three coupling constants g_{11}, g_{12} and g_{22}; the real potential V-matrix elements have been replaced by the appropriate T-matrix elements at low-energy, according to an usual procedure in the study of cold gases (Fermi pseudo-potential method [30], or $V \to T$ renormalization procedure in Ref. [28]).

We wish to obtain the time evolution of the single-particle density matrix $\rho_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) = \langle \psi_\alpha^\dagger(\mathbf{r}', t) \psi_\beta(\mathbf{r}, t) \rangle$, where $\langle \ldots \rangle$ denotes the expectation value with respect to the time-dependent N-body density operator of the system. To this end we write down the Heisenberg equation of motion for $\psi_\alpha^\dagger(\mathbf{r}', t) \psi_\beta(\mathbf{r}, t)$ using the Hamiltonian (35). The non-interacting part of the Hamiltonian density $\mathcal{H}_1$ gives the single-particle Liouville-von Neumann equation:

$$
\begin{aligned}
i\hbar \partial_t \rho_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) = &-\frac{\hbar^2}{2m} \left(\nabla_\mathbf{r}^2 - \nabla_{\mathbf{r}'}^2 \right) \rho_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) \\
&+ \left(V_\alpha^{ext}(\mathbf{r}) - V_\beta^{ext}(\mathbf{r}') \right) \rho_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)
\end{aligned}
\tag{36}
$$

The interacting part $\mathcal{H}_{int}$ couples the single-particle density matrix with the expectation value of the product of four field operators (two-particle density matrix). If we are only interested in time scales large compared to the duration of a collision, we can decompose the averages of such products into averages of products of two operators [2]. In the context of the equation of motion for the single-particle density matrix, this approximation is often called the random phase approximation (RPA) [31]. The interacting part of the equation of motion then becomes:

$$
\sum_{\delta=1,2} V_{\alpha\delta}^{mf}(\mathbf{r}, t) \rho_{\delta\beta}(\mathbf{r}, \mathbf{r}', t) - \sum_{\delta=1,2} \rho_{\alpha\delta}(\mathbf{r}, \mathbf{r}', t) V_{\delta\beta}^{mf}(\mathbf{r}', t)
\tag{37}
$$

where the mean field potential 2×2 matrix is given by:

$$
V_{\alpha\beta}^{mf}(\mathbf{r}) = \delta_{\alpha,\beta} \sum_{\gamma=1,2} g_{\alpha\gamma} \langle \psi_\gamma^\dagger(\mathbf{r}) \psi_\gamma(\mathbf{r}) \rangle + \epsilon g_{\alpha\beta} \langle \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta(\mathbf{r}) \rangle
\tag{38}
$$

which contains a direct term and an exchange mean field term, proportional to ϵ. Within this approximation, each particle evolves under the influence of an effective single-particle

---

[2]This result can be obtained for instance from the Wick theorem. It is valid because the gas is not Bose condensed; otherwise, the mean field could not be obtained in that way, due to the absence of the exchange term.

potential $\hat{U}(\mathbf{r}, t)$, which is the sum of the external potential $\widehat{V}^{ext}(\mathbf{r})$ and the time-dependent mean field $\hat{V}^{mf}(\mathbf{r}, t)$.

In order to compare this result with that of the preceding section, we note that $n_\alpha(\mathbf{r}, t) = \langle \psi_\alpha^\dagger(\mathbf{r}, t)\psi_\alpha(\mathbf{r}, t) \rangle$ is the average density of atoms in internal state $\alpha = 1, 2$ and the off-diagonal elements $n_{12}(\mathbf{r}, t) = n_{21}^*(\mathbf{r}, t) = \langle \psi_1^\dagger(\mathbf{r}, t)\psi_2(\mathbf{r}, t) \rangle$ are the coherences between the internal states, so that the effective potential $\hat{U}$ reproduces exactly the one obtained in the preceding section, equation (26). We therefore arrive at the following operatorial kinetic equation:

$$\frac{d\hat{\rho}_1}{dt} + \frac{1}{i\hbar} \left[\hat{\rho}_1, \hat{H}_1 + \hat{V}^{mf} \right]_- = 0 \tag{39}$$

which is the quantum equivalent of the Vlasov equation (see Ref. [18] for example); in this equation, $\hat{H}_1$ is the single particle hamiltonian defined in (2) and $\hat{V}^{mf}$ is defined by (38). Obviously, if we Wigner transform this operatorial equation, we will obtain a kinetic equation that is equivalent to (25), but without the collision integral. The effect of lateral collisions can then be treated phenomenologically by adding by hand a term corresponding to a relaxation time approximation, as usual [31]; then equations (34) are recovered and the mean field treatment of interactions becomes exactly equivalent to these simplified equations.

2.3 Discussion

We have used three different methods to derive a kinetic equation for a gas of atoms with two internal levels: the S matrix (or LL) Ansatz, the Yvon-Snider Ansatz (YS) and the mean field approximation; we now summarize a comparison between the results.

To zero order in the gradients, LL and YS give exactly the same equation, containing a Boltzmann-type collision integral and a spin mean field. As for the mean field approximation, it provides only the spin mean field, obtained in the limit of low-energy collisions because of the use of the pseudopotential; this limit is in full agreement with the results of the two other methods. We note in passing that this spin mean field occurs for a non-condensed Bose gas, but not in a Bose-Einstein condensate (BEC) at very low temperature, because the exchange term is absent when a single quantum state is involved. In this case, there is however a different mechanism for creating spin waves, the so-called "quantum torque" of purely kinetic origin [32]; it arises naturally because, in quantum mechanics, any gradient of the phase corresponds to a kinetic energy for each component which, in turn, affects the evolution of the relative phase and therefore of the transverse spin orientation.

To first order in the gradients and for forward scattering, LL, YS and the mean field approximation all give the same scalar mean field term (zero sound), again in the limit of low-energy collisions for the third method. We have checked that the complete expression of the LL and YS first order gradient terms coincide exactly for forward scattering, but not for lateral scattering; whether or not the two methods are completely equivalent to this order is still an open question.

In a non-degenerate dilute gas, the spin mean field generally dominates over the scalar mean field, because it appears to lower order in the gradient expansion; it is therefore not surprising that zero sound type collective modes should not propagate as easily as spin waves in dilute non-degenerate gases.

To what extent collisional methods and mean field approximations are equivalent is not a trivial question. Generally speaking, mean field theory is assumed to be valid for systems with either long range interactions (Vlasov's equation for a plasma [18]) or in strongly interacting

degenerate systems described in terms of quasi-particles (Landau's equation for a Fermi liquid [11, 3, 28]); for a dilute gas, neither of these conditions is met. Our conclusion is nevertheless that the equivalence is perfect to first order in the scattering length a but does not hold to higher orders in a. We finally remark that, when describing a non-degenerate dilute atomic gas in terms of quasi-particles (see the work of Bashkin [16], for example), a quasi-particle is just a particle whose kinetic energy $p^2/2m$ is shifted by a local mean field term, so that the kinetic drift term of the quasi-particle transport equation is left unchanged.

3 Spin oscillations and internal conversion

Our purpose now is to use the kinetic equation obtained in the preceding section to discuss spin waves in ultra-cold trapped atomic gases. We first briefly review a recent experiment realized at JILA [1] and the theoretical work stimulated by it [33, 34, 35, 36, 37]. We then discuss the non-hydrodynamic character of the observed spin waves, which allows a comparison between the different theoretical treatments. Finally, we describe the decay of the spin waves and discuss the relevance of decoherence.

3.1 The JILA experiment

In a recent experiment performed at JILA , Lewandowsky *et al* have studied bosonic ^{87}Rb atoms with two hyperfine states of interest (denoted by 1 and 2), confined in an axially symmetric magnetic trap elongated along the Ox direction. Their experiment is described in [1]; the temperature T of the gas in this experiment is about twice the critical temperature for Bose-Einstein condensation, so that the gas is not strongly degenerate and can be treated reasonably well as a Boltzmann gas; however, since the de Broglie thermal wavelength is much larger than the average scattering length, collisions occur in the full quantum regime. Initially, the gas is at equilibrium with only state 1 populated. A $\pi/2$ radio frequency pulse is then applied, which suddenly puts all the atoms into the same coherent superposition of states 1 and 2. The subsequent evolution of the system along the axial direction is then monitored by measuring optically the local densities n_1 and n_2 of atoms in each internal state. Experimentally, one observes that the system "segregates": after about 100 ms, atoms in state 1 are mostly found away from the center of the trap, while atoms in state 2 move towards the center, the total density $n = n_1 + n_2$ remaining practically unchanged. After about 200 ms, the local densities of each species return to equilibrium.

In Ref. [34], this phenomenon was explained in terms of an internal conversion resulting from the identical spin rotation effect (ISRE), which is at the origin of the spin mean field; similar considerations were almost simultaneously provided by two other groups [33, 35]. Qualitatively, this spin oscillation can be understood as follows:

(i) the field gradient creates an inhomogeneous spin precession, so that the gas develops a gradient of transverse spin polarization

(ii) the thermal motion of the atoms then creates correlations between transverse spin and velocity; therefore, a particle moving with a given velocity at point x gets a spin polarization which is not parallel to the average local polarization

(iii) the ISRE then makes its spin polarization leave the transverse plane and get a longitudinal component, in a direction which depends on the sign of the velocity

(iv) this is equivalent to a velocity dependent internal conversion which, at some later time, results in an efficient spatial separation of atoms in the two internal states.

To study this phenomenon quantitatively, we derive from (34) an effective one-dimensional kinetic equation in terms of the local density in phase space $f(x, p, t)$ and spin density $\mathbf{M}(x, p, t)$, by assuming that radial equilibrium is quickly established (the radial trap frequency $\omega_{rad}/2\pi$ is much larger than the axial trap frequency $\omega_{ax}/2\pi$). Integrating (34) over radial coordinates y, z and momenta p_y, p_z provides the following equations:

$$\partial_t f + \frac{p}{m} \partial_x f - \partial_x U_0 \partial_p f - \partial_x \mathbf{U} \cdot \partial_p \mathbf{M} \simeq -(f - f^{eq})/\tau \tag{40}$$

$$\partial_t \mathbf{M} + \frac{p}{m} \partial_x \mathbf{M} - \partial_x U_0 \partial_p \mathbf{M} - \partial_x \mathbf{U} \partial_p f - \frac{2\mathbf{U}}{\hbar} \times \mathbf{M}$$
$$\simeq -(\mathbf{M} - \mathbf{M}^{eq})/\tau \tag{41}$$

(every quantity depending initially on three-dimensional coordinates $\mathbf{r}$ and momenta $\mathbf{p}$ has been integrated over radial coordinates and momenta, so that it now depends only on the coordinate x and the momentum $p \equiv p_x$). In the process of radial averaging, the coupling constants $g_{\alpha\beta}$ are renormalized by a factor $1/2$, as discussed by Levitov [33]. The three-dimensional effective trapping potential (29) therefore becomes:

$$U_0(x, t) = \frac{V_2^{ext} + V_1^{ext}}{2} + \left[\frac{1+\epsilon}{2}(g_{22} + g_{11}) + g_{12}\right]\frac{n}{4}$$
$$+ \frac{1+\epsilon}{2}(g_{22} - g_{11})\frac{m_\parallel}{4} \tag{42}$$

and the three-dimensional effective magnetic field (30) is changed into:

$$\mathbf{U}(x, t) = \frac{\hbar\Omega(x, t)}{2}\mathbf{e}_\parallel + \epsilon\frac{g_{12}\mathbf{m}(x, t)}{4} \tag{43}$$

with:

$$\hbar\Omega(x, t) = V_2^{ext} - V_1^{ext} + \frac{1+\epsilon}{2}\left[(g_{22} - g_{11})\frac{n}{2} + (g_{22} + g_{11} - 2g_{12})\frac{m_\parallel}{2}\right]$$

where $\mathbf{e}_\parallel$ is the unit vector in the longitudinal direction in spin space.

A few simplifying assumptions are appropriate for the experimental conditions of Ref. [1]. The confining energy $(V_2^{ext} + V_1^{ext})/2$ is of order $k_B T \simeq 13$ kHz$\times h$ which is much larger than the mean field interaction energy $gn(0) \simeq 140$ Hz$\times h$. This allows us to keep only the confining energy of the harmonic trap in the effective trapping potential U_0:

$$U_0(x) \simeq \frac{V_1^{ext}(x) + V_2^{ext}(x)}{2} = \frac{1}{2}m\omega_{ax}^2 x^2 \tag{44}$$

The differential trapping energy $V_2^{ext} - V_1^{ext} \sim 10$ Hz$\times h$ is even weaker than the mean field interaction energy. This ensures that the force terms $\partial_x \mathbf{U} \cdot \partial_p(f$ or $\mathbf{M})$ appearing in equations (40,41) are negligible. On the contrary, when the local effective magnetic field $\mathbf{U}$ does not appear under a spatial gradient, as in the term $2\mathbf{U} \times \mathbf{M}/\hbar$, it can not be neglected. This effective magnetic field $\mathbf{U}$ is made of two terms (see equation (43)): one is an effective *external* magnetic field $\hbar\mathbf{\Omega}/2$; the other is an *exchange* magnetic field or spin mean field $g_{12}\mathbf{m}/4$ which results from the ISRE. The effective external magnetic field is the sum of the contributions of a differential Zeeman and of a differential mean field:

$$\hbar\mathbf{\Omega}(x) = \hbar\ \Omega(x)\mathbf{e}_\parallel \simeq [V_2^{ext}(x) - V_1^{ext}(x) + (g_{22} - g_{11})n(x)/2]\mathbf{e}_\parallel \tag{45}$$

where, following ref. [1], we have assumed that $2g_{12} \simeq g_{11} + g_{22}$ for simplicity. The average value over the sample of the effective external magnetic field can be removed by going to a uniformly rotating frame (Larmor frame). For the numerical simulations, we need to know the x dependence of this effective magnetic field. As the profile $n(x)$ will be shown to be a Gaussian (as for a non-interacting non-degenerate gas) which does not vary in time, we model the experimentally measured effective external magnetic field by:

$$\mathbf{\Omega}(x) = -\delta\Omega \exp -(m\omega_{ax}^2 x^2 / 2k_B T)\mathbf{e}_{\parallel} \tag{46}$$

where the parameter $\delta\Omega$ is the variation of $\Omega(x)$ between the center and the edge of the cloud.

Taking into account all approximations mentioned above, we finally arrive at equations:

$$\partial_t f + \frac{p}{m}\,\partial_x f - m\omega_{ax}^2 x\,\partial_p f \simeq -(f - f^{eq})/\tau \tag{47}$$

$$\partial_t \mathbf{M} + \frac{p}{m}\,\partial_x \mathbf{M} - m\omega_{ax}^2 x\,\partial_p \mathbf{M} - (\mathbf{\Omega} + \frac{g_{12}\mathbf{m}}{2\hbar}) \times \mathbf{M}$$
$$\simeq -(\mathbf{M} - \mathbf{M}^{eq})/\tau \tag{48}$$

The initial equilibrium Maxwell-Boltzmann distribution $f(x,p)$ solves the kinetic equation (47), so that the dynamics after the $\pi/2$ pulse can be expressed in terms of $\mathbf{M}$ only. In addition, the density $n(x)$ is obtained by integrating the equilibrium Maxwell-Boltzmann distribution over momentum p, which shows that it is time independent; from now on, we will only consider equation (48). The spin distribution immediately after the $\pi/2$ pulse is assumed to be:

$$\mathbf{M}(x,p,t=0) = f(x,p)\mathbf{e}_{\perp,1} \tag{49}$$

Equation (48) can be solved numerically with values of the parameters taken from Ref. [1]: the trap frequencies are $\omega_{ax}/2\pi = 7$ Hz and $\omega_{rad}/2\pi = 230$ Hz; the average time between collisions is $\tau \sim 10$ ms; the temperature is $T \simeq 0.6$ μK; the density at the center of the trap is $n(0) = 1.8 \times 10^{13}$ cm^{-3}; $\delta\Omega/2\pi$ is typically ~ 12 Hz, and the scattering lengths are $a_{11} = 100.9a_0$, $a_{22} = 95.6a_0$ and $a_{12} = 98.2a_0$ where a_0 is the Bohr radius. The solution at the center of trap is plotted in Figure 1; it shows very good agreement with the experimental observations, without any adjustable parameter.

3.2 Large, non-hydrodynamic, spin waves

Ordinary spin waves in non-degenerate gases usually occur either in the hydrodynamic regime or/and in a regime of small amplitudes [8, 9, 10]. One can then assume a small departure of the spin distribution (from either local or global equilibrium); this is similar to the first order Chapman-Enskog gradient expansion, which leads to the Navier-Stokes equations (see for instance [30]). The result is the well known Leggett equations [4], transposed from degenerate Fermi liquids to dilute non-degenerate gases [6, 16]:

$$\partial_t \mathbf{m} + \partial_x \mathbf{j} = \mathbf{\Omega} \times \mathbf{m}$$
$$\partial_t \mathbf{j} - (\mathbf{\Omega} + \frac{g_{12}\mathbf{m}}{2\hbar}) \times \mathbf{j} + \frac{k_B T}{m}\partial_x \mathbf{m} + \omega_{ax}^2 x\mathbf{m} \quad \simeq -\frac{\mathbf{j}}{\tau} \tag{50}$$

where the spin current along Ox is defined by:

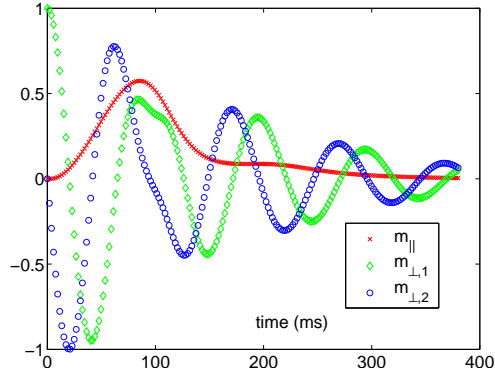$$\mathbf{j}(x,t) \equiv \int dp\,\frac{p}{m}\mathbf{M}(x,p,t) \tag{51}$$

15

Figure 1: Time evolution of the spin polarization $\mathbf{m}$ at the center of the trap ($x = 0$) when $\delta\Omega/2\pi = 12$ Hz.

The momentum dependence has disappeared from these equations, which contain only position variables. These equations can describe spin waves, not only in the hydrodynamic regime, but also in the collisionless regime as, for example, in liquid ^{3}He (Silin spin waves [38]) or in H↓ gas [39]. For a discussion of the hydrodynamic-like description of spin waves in the collisionless regime of non-degenerate dilute gases, see Refs. [16, 40].

The spin waves observed at JILA actually occurred in the intermediate regime between hydrodynamic and collisionless, where $\omega_{ax}\tau \sim 1$, so that there is no a priori reason to believe in the validity of the Leggett equations; we therefore need to resort to a numerical solution of the kinetic equation in terms of both position and momentum variables. Figure 2 shows the results of this calculation, and the spin density distribution (longitudinal and transverse) in p-space for different times. For comparison, the local equilibrium spin distribution is also plotted (dashed line); we see that the spin distribution in p-space can indeed get very distorted, which illustrates a strongly non-hydrodynamic situation. For very short times, $t \lesssim 40$ ms, the transverse and longitudinal spin distributions are still close to local equilibrium ($M_{||}^{eq} = 0$), so that the Leggett equations are valid. For $t \gtrsim 40$ ms, this is no longer true, so that one would lose a significant part of the physics by not retaining the full momentum dependence of the distribution. In other words, studying the two first moments of the distribution ($\mathbf{m}$ and $\mathbf{j}$) is not equivalent to studying the full distribution $\mathbf{M}(p)$. What actually takes place is a phenomenon analogous to a density "shock wave" rather than a "sound wave". Eventually, for $t \gtrsim 160$ ms (end of the "segregation"), since equilibrium is almost reached, the Leggett equations become valid again.

The initial part of the internal conversion (or state separation) can be treated analytically in several approximations, allowing for comparison between different approaches such as those of Refs. [34, 33, 35]. In Ref. [34], we solved the kinetic equation analytically, for times $t \ll \tau$, by expanding in Taylor series in time and obtained:

$$\frac{m_{||}(x,t)}{n(x)} = \frac{g_{12}n(x)}{2\hbar}\left[\frac{k_B T}{m}\Omega''(x) - 2\omega_{ax}^2 x\Omega'(x)\right]\frac{t^4}{4!} \tag{52}$$

where Ω' and Ω'' are the first and second spatial derivative of $\Omega(x)$. This formula predicts an initial quartic in t behavior, which correctly reproduces the numerical results; each power of t corresponds to one of the four physical processes (i) to (iv) described in § 3.1.
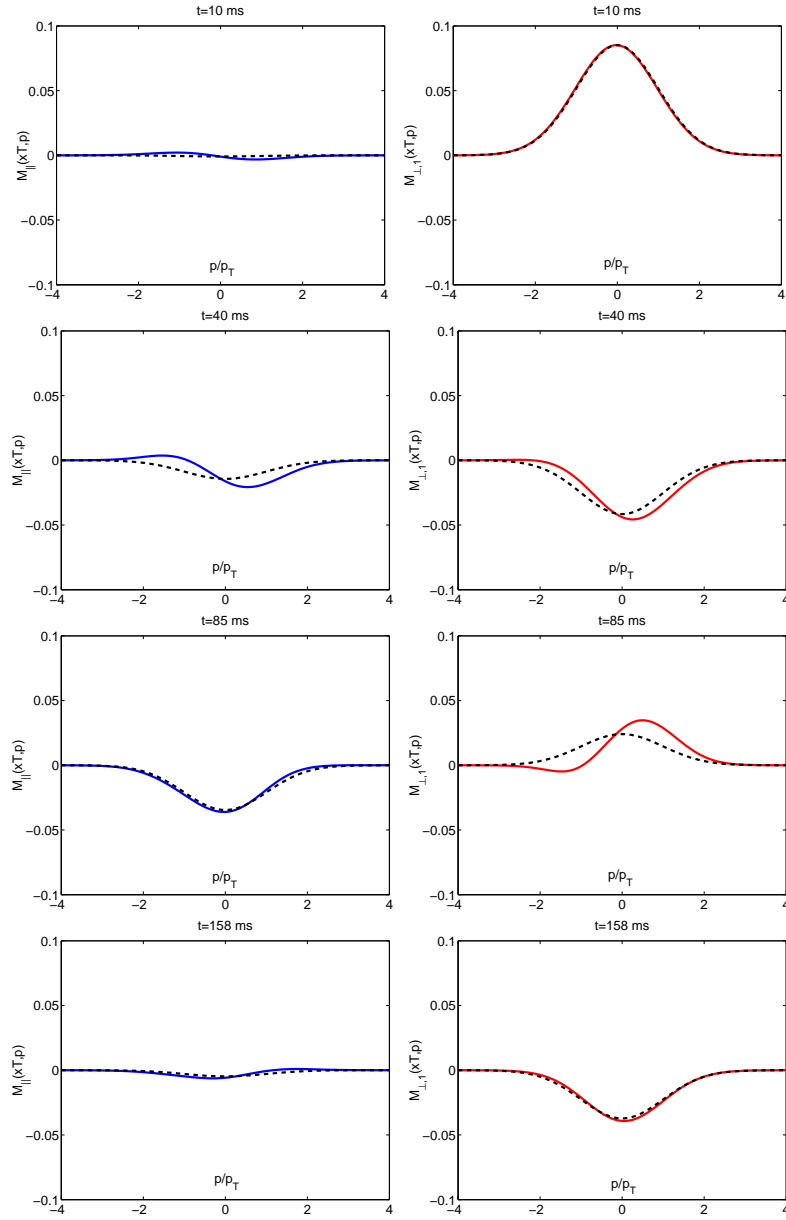
Figure 2: Longitudinal spin distribution $M_{||}(x = x_T, p, t)$ and transverse spin distribution $M_{\perp,1}(x = x_T, p, t)$ at position $x = x_T$ for $t = 10; 40; 85; 160$ ms, where $x_T = \sqrt{k_B T / m\omega_{ax}^2}$ and $p_T = \sqrt{mk_B T}$. The local equilibrium spin distribution is plotted with a dashed line.
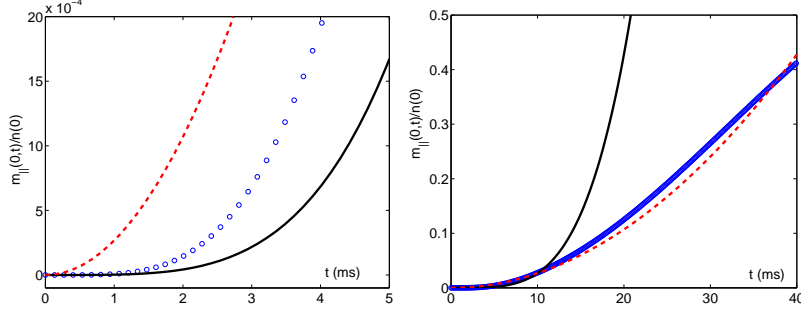
Figure 3: Beginning of the internal conversion at the center of the trap. The numerical simulation is plotted with circles, the t^4 result with a full line and the t^2 result with a dashed line.

Another approach to calculate the short time behavior is that used by Williams *et al* [35]. They use the Leggett equations which are valid for $t \ll \tau$ because the spin distribution is at equilibrium at $t = 0$. Solving the Leggett equations in the small time limit, where $-\mathbf{j}/\tau$ is negligible, they recover (52).

As a third approach is to consider, following Oktel and Levitov [33]), the Leggett equations in the hydrodynamic regime. The approximation consists in assuming that the spin current remains close to its stationary value, which allows to neglect the term $\partial_t \mathbf{j}$ in equation (50). Using the resulting equations, we obtain a different behavior for small times:

$$\frac{m_{||}(x,t)}{n(x)} \sim \frac{g_{12}n(x)}{2\hbar}[\frac{k_B T}{m}\Omega''(x) - 2\omega_{ax}^2 x\Omega'(x)]\frac{(\tau t)^2}{4!} \tag{53}$$

(the effect of the hydrodynamic approximation is to replace t^2 by $t\tau$). The t^4 and t^2 results are plotted in Fig. 3; we see that times smaller than a time between collisions τ can not be dealt with in this approach. Nevertheless, after each atom has made one collision on average, the spin current is close to its stationary value and that the hydrodynamic approximation is reasonably valid. Therefore, when $t > \tau \simeq 10$ ms, we get:

$$\frac{m_{||}(x,t)}{n(x)} \sim \frac{g_{12}n(x)}{2\hbar}[\frac{k_B T}{m}\Omega''(x) - 2\omega_{ax}^2 x\Omega'(x)]\frac{\tau^2(t-t_0)^2}{4!} \tag{54}$$

(the parameter t_0 includes the accumulated effect of the retardation of the spin current $\mathbf{j}$ for small time). The $(t - t_0)^2$ behavior is plotted in Fig. 4 (the reason why, after 40 ms, the numerical solution of the kinetic equation is not well approximated by the $(t - t_0)^2$ law is that the spin distribution gets very distorted, so that the Leggett equations are no longer valid).

3.3 Decay of the phenomenon

Up to this point, we have studied mostly the initial part of the phenomenon, when the amplitude of the spin wave grows and the two species tend to separate from each other; we now discuss its final part, when the spin oscillations decay and, eventually, the system returns to equilibrium. A first remark is that it is relatively easy to obtain an order of magnitude of the time at which the transition between the two regimes occurs, i.e. the time at which the maximum of the separation takes places. For short times, we have seen that
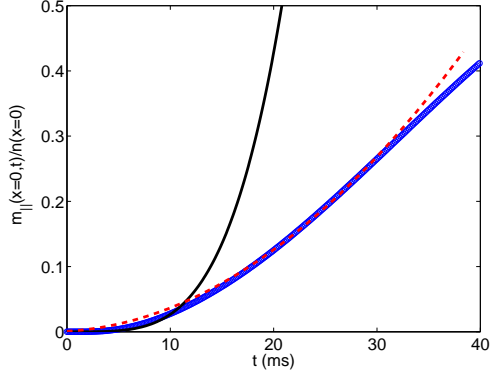
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Figure 4: The numerical simulation is plotted with circles, the t^4 result with a full line and the $(t - t_0)^2$ result with a dashed line ($t_0 \sim 2$ ms here). Note the good agreement between the $(t - t_0)^2$ law and the kinetic equation in the range $\tau \simeq 10$ ms$< t \lesssim 40$ ms.

this separation is an indirect result of a differential precession of the transverse component of the spin at different points along the Ox axis; the thermal motion of the atoms then creates correlations between the transverse orientation of the spins and the sign of the x component of the velocities. But, for long times, the differential rotation of the spins will be so large that each sign of this velocity will become associated with a widely open fan of transverse orientation, with almost zero average; clearly, the selective internal conversion effect will then also average to almost zero. Let us consider the simple situation of a homogeneous gas in a box of size L with a linear gradient of external effective magnetic field Ω'. We focus, for instance, on the situation at the center of the box. At time t, atoms initially at $x = v \times t$ will cross this point with a transverse orientation rotated by an angle of the order of:

$$\Omega' \times x \times t = \Omega' \ v \ t^2 \tag{55}$$

Therefore, at times greater than:

$$t_m \sim \frac{1}{\sqrt{\Omega' \ v}} \tag{56}$$

the transverse directions of the spins average out so that the apparent segregation effect no longer takes place. In the case of the trapped gas, $\Omega' \sim \delta\Omega/L$ and $v \sim \omega_{ax}L$ where L is the size of the cloud, and we obtain:

$$t_m \sim \frac{2\pi}{\sqrt{\delta\Omega \ \omega_{ax}}} \tag{57}$$

Typical values give $t_m \sim 100$ ms in accordance with the results of the simulation. After this maximum, the system tends to return to equilibrium during a time period which we call the decay of the phenomenon.

Fig.1 shows that the decay of the longitudinal and transverse spin polarizations are significantly different. The longitudinal spin polarization $m_\parallel(x,t)$ returns to equilibrium in two steps: it first quickly decreases from its maximum value to almost zero in a time scale of the order of 50 ms; then, it then goes to zero much more slowly, on a time scale of the order of 200 ms. In order to understand this, we note in Fig.1 that, at the maximum of separation, the wave packets (or the clouds) associated with each internal state overlap only little; the

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first step is then easily interpreted as the free motion of the clouds under the restoring force of the trap. It takes a quarter of a period $T_{ax}/4 = \pi/2\omega_{ax} = 36$ ms to go from the maximum of oscillation to the center of the trap. Actually, as soon as the two clouds start to overlap, they interact through a repulsive mean field ($g > 0$), so that it takes a little more time than a quarter of a period to reach the situation where the two clouds overlap significantly [3]. At this point, the subsequent evolution of the system can be understood as the mutual diffusion of two gases [4]; this happens ~ 150 ms after the initial pulse and can again be described by the Leggett equations (we checked on the p distribution that the system is close to local equilibrium). The relaxation time scale is then given by the spin diffusion time $t_{diff} \sim L^2/D$ (where $L \sim 2\sqrt{k_B T/m\omega_{ax}^2}$ is the size of the cloud and $D = k_B T\tau/m$ is the spin diffusion constant). Using the same parameters as in the simulation, we obtain $t_{diff} \sim 200$ ms, in good agreement with the numerical result.

We now discuss the transverse spin polarization $\mathbf{m}_\perp(x,t)$. As seen on Fig.1, it oscillates many times before going to zero. After ~ 150 ms, when the Leggett equations are valid again, we know [6] that spin waves occur in the hydrodynamic regime. The spectrum derived from these equations predicts damped transverse spin waves [6]. The damping time is of the order of the spin diffusion time $t_{diff} \sim 200$ ms, which gives an order of magnitude for the time it takes the transverse spin polarization to go from its value at the maximum of separation to zero. This is in agreement with the relaxation time obtained by numerically solving the kinetic equation (Fig.1). We finally note that the frequency of the transverse spin waves is essentially given by the external effective magnetic field precession frequency $\delta\Omega/2\pi = 12$ Hz.

3.4 "Ghost" wave packet and decoherence

We now come back to the time at which the separation is maximum. In order to simplify the discussion, we consider two clouds of atoms, one corresponding to state 1 and located around $x = d/2$, the other corresponding to state 2 located around $x = -d/2$, where d is the distance between the center of the two clouds; we note δx the width of each of these wave packets, assuming that $\delta x < d$. In this case, elementary quantum mechanics predicts that the local transverse spin polarization vanishes everywhere (this is because the local spin density corresponds to an operator which is local in ordinary space). Nevertheless, the superposition of the two wave packets remains coherent so that, rigorously speaking, the system is not equivalent to a classical mixture of two gases, in opposition to what we have assumed in the previous section.

One may wonder how this coherence translates in terms of the Wigner distribution. The answer is well known: the presence of the coherence is contained in a so-called "ghost" wave packet, which exists around the middle point between the two real wave packets (or clouds). For spinless particles, it "carries interferences with it" [43], and reconstructs them as soon as the real packets overlap again. For particles with spin, when the two separated wave packets have opposite spin orientations, the ghost wave packet centered at $x = 0$ appears only in the

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[3]That the two clouds interact mainly trough the mean field and not trough (lateral) collisions is confirmed by the numerical solution of the kinetic equation. Indeed, we checked that removing the collision integral does not change the decay time scale. The role of the collision integral is merely to damp revivals of the longitudinal spin waves, which are observed when the collision integral is discarded.

[4]Actually, an instability of the transverse spin polarization is possible when there is a strong longitudinal spin polarization gradient (which is the case when the two clouds overlap again) and a spin mean field. This is known as Castaing's instability [41]. In Ref. [42], we checked that it does not play a role in the experiment done at JILA [1].

non-diagonal spin matrix elements (transverse spin components of the Wigner distribution). In both cases, the wave packet rapidly oscillates as a function of momentum p, taking positive and negative values, with an oscillation period $2\pi\hbar/d$; the further apart the two clouds, the faster the oscillation. On the other hand, for a statistical mixture of the internal states, the ghost wave packet does not exist. Such oscillating wave packets are well-known in the context of macroscopic superpositions of states [43].

The next question is to what extent the ghost wave packet is preserved by the time evolution of the kinetic equation. Since the Wigner formalism in itself implies no approximation (it is strictly equivalent to the use of operators), the question arises only because our kinetic equation does not provide the exact quantum evolution: it was actually obtained from the operatorial (quantum) equation for the single-particle density operator by truncating the gradient expansion of the Wigner transform. In other words, we made a semi-classical approximation. But, for the ghost wave packet, the p oscillation period is $2\pi\hbar/d$ while the width of the wave packet is δx, so that the parameter of the gradient expansion is $d/\delta x$, which may be larger than 1: therefore, our kinetic equation does not necessarily remain correct for this case; the semi-classical gradient expansion may smooth the oscillations of the ghost wave packet, and artificially introduce decoherence. Another related source of artificial decoherence[5] might come from the discretization of phase-space (necessary for the numerical solution of the kinetic equation), which may introduce a lattice spacing in momentum space larger than δp.

Up to this point, no real physical decoherence mechanism was included in our discussion: we have assumed that the system is perfectly isolated. But this is not the case in practice. In order to obtain an estimate of the physical decoherence time, we proceed by analogy with a well-understood situation in quantum optics: a macroscopic superposition of two coherent states of the electro-magnetic field in a cavity. In that case, it is known that the coherence is lost as soon as one photon on average escapes the cavity and goes in the environment [44]. We therefore assume here that for our trapped gas, a source of decoherence comes from the fact that atoms can leave the trap, because of 3-body collisions for example. The time it takes one atom to leave the trap is $1/\alpha_{3b}n(0)^2 N$ where $\alpha_{3b} \sim 4 \times 10^{-29}$ cm^6/s is the 3-body recombination rate constant [45], $N \sim 10^6$ is the total number of atoms and $n(0) \simeq 2 \times 10^{13}$ cm^{-3} is the density at the center of the trap. This gives ~ 0.1 ms as a rough upper bound to the physical decoherence time, which is still much shorter than the characteristic time scale of the spin state segregation. Our conclusion is therefore that, when the wave packets recombine, coherence is not likely to play a role in these experiments; this justifies our discussion in the preceding section.

4   Conclusion

We have derived a kinetic equation for a Boltzmann gas with two internal levels using different methods. A first conclusion which emerges from this work is that, for dilute gases at low temperature, several significantly different theoretical approaches lead essentially to the same result. Here we have discussed the Yvon-Snider method, with its peculiar way to close

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[5]Concerning the second source of artificial decoherence, we think that it is not a problem in our simulations. We checked that, when removing the collision integral so that the kinetic equation becomes reversible, the transverse spin polarization disappears when the two wave packets do not overlap but reappears when they mix again.

the BBGKY hierarchy, the S matrix method which is very close to the spirit of the original Boltzmann equation, as well as the popular mean field method. In the latter case, the equivalence actually holds only in the low temperature limit, when forward scattering dominates over lateral scattering by a factor of the order of λ_T/a. Whether or not it is possible to find experimental situations where the results of these theories are significantly different, and can be tested experimentally, is still an open question (for instance, whether or not ISRE in lateral collisions may become dominant).

A second conclusion is that a correct treatment of the correlations between internal variables and velocities at each point of space may be important; this is illustrated by the figures contained in figure 2. In other words, hydrodynamic equations are not always appropriate. What is remarkable is that the effects of local field inhomogeneity and velocities combine nonlinearly to create this almost complete separation of the two internal states; the effect is not a segregation of states but rather an internal conversion that depends on the direction of the velocity. It is probably even more remarkable that this effect was discovered experimentally, without any theoretical prediction as a guide to the appropriate experimental conditions, and appeared basically as a 100% effect from the beginning. Clearly, ISRE is not a small quantum correction to the effect of collisions, but may dominate the entire dynamics of the quantum gas!

It would be interesting to extend the present work in a few directions, for instance exploring the influence of a temperature gradient on the spin current in hydrodynamical situations, as predicted in [6]. For bosons, it would also be important to understand better how spin waves evolve progressively from the non-condensed regime, which we have studied in this article, to the condensed low temperature regime; the theory recently developed in [32] seems to provide an appropriate tool for this purpose.

APPENDICES: INTERACTION TERM OF THE S-MATRIX ANSATZ

Appendix I: spin-independent interactions

In this appendix, we give the detailed calculation of the Wigner transform of the collision term of the S matrix Ansatz, equation (18). We will closely follow the notations of a similar calculation done in the Appendix of Ref. [14], where the Wigner transform of the Yvon-Snider Ansatz, equation (6) without internal levels and without exchange, is obtained to first order in the gradient expansion. We define the on-shell T matrix $\mathcal{T}$ by:

$$\hat{S} = \hat{1} - i2\pi\hat{\mathcal{T}} \tag{58}$$

and rewrite the core of the Ansatz in equation (18) as:

$$\hat{S}\hat{\rho}_1(1)\hat{\rho}_1(2)\hat{S}^\dagger - \hat{\rho}_1(1)\hat{\rho}_1(2) = -i2\pi\hat{\mathcal{T}}\hat{\rho}_1(1)\hat{\rho}_1(2) + \text{c.c.}$$
$$+ (2\pi)^2\hat{\mathcal{T}}\hat{\rho}_1(1)\hat{\rho}_1(2)\hat{\mathcal{T}}^\dagger \tag{59}$$

where c.c. is the complex conjugate of the preceding term. The first two terms of the preceding equation are linear in the T matrix, the last is quadratic. The Wigner transform $\hat{F}_T$ of the linear in T matrix operator between curly brackets in (18) is:

$$\hat{F}_T(\mathbf{R}, \mathbf{r}, \mathbf{P}, \mathbf{p}) = -i(2\pi)^{-5}\hbar^{-6} \int d^3K \int d^3\kappa \ e^{i\mathbf{K}\cdot\mathbf{R}}e^{i\kappa\cdot\mathbf{r}}$$

$$\times \langle \mathbf{K}_+, \mathbf{k}_+| \frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}} \hat{\mathcal{T}}\hat{\rho}_1(1)\hat{\rho}_1(2)\frac{\hat{1} + \epsilon\hat{P}_{ex.}}{\sqrt{2}}|\mathbf{K}_-, \mathbf{k}_-\rangle + \text{c.c.} \tag{60}$$

where:

$$\mathbf{K}_\pm = \frac{\mathbf{P}}{\hbar} \pm \frac{\mathbf{K}}{2}; \ \mathbf{k}_\pm = \frac{\mathbf{p}}{\hbar} \pm \frac{\mathbf{k}}{2} \tag{61}$$

With two closure relations for relative wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$, the r.h.s. of (60) becomes equal to:

$$\frac{1}{2i(2\pi)^5\hbar^6} \int d^3K \int d^3\kappa \int d^3k_1 \int d^3k_2 \ e^{i\mathbf{K}\cdot\mathbf{R}}e^{i\kappa\cdot\mathbf{r}}$$

$$\times \left(T(\mathbf{k}_+, \mathbf{k}_1) + \epsilon T(-\mathbf{k}_+, \mathbf{k}_1)\hat{P}_S \right) \delta(E_{k_+} - E_{k_1})$$

$$\times \langle \mathbf{K}_+, \mathbf{k}_1 |\hat{\rho}_1(1)\hat{\rho}_1(2)|\mathbf{K}_-, \mathbf{k}_2\rangle \left(\delta(\mathbf{k}_2 - \mathbf{k}_-) + \epsilon\delta(\mathbf{k}_2 + \mathbf{k}_-)\hat{P}_S \right) + \text{c.c.} \tag{62}$$

where $\hat{P}_S$ is the exchange operator in spin space. Now using the inverse Wigner transform formula, we can introduce into (62) the Wigner transform $\hat{\rho}(\mathbf{r}, \mathbf{p})$ of $\hat{\rho}_1$ and replace the matrix element of the product of $\hat{\rho}$'s by:

$$\hbar^6 \int d^3R' \int d^3r' e^{-i\mathbf{K}\cdot\mathbf{R}'}e^{i(\mathbf{k}_2 - \mathbf{k}_1)\cdot\mathbf{r}'}$$

$$\times \hat{\rho}\left(\mathbf{R}' + \frac{\mathbf{r}'}{2}, \frac{\mathbf{P} + \hbar\mathbf{k}_1 + \hbar\mathbf{k}_2}{2}\right) \hat{\rho}\left(\mathbf{R}' - \frac{\mathbf{r}'}{2}, \frac{\mathbf{P} - \hbar\mathbf{k}_1 - \hbar\mathbf{k}_2}{2}\right) \tag{63}$$

When inserting the result into (62), a delta function appears:

$$\int d^3K e^{i\mathbf{K}\cdot(\mathbf{R}-\mathbf{R}')} = (2\pi)^3\delta(\mathbf{R} - \mathbf{R}') \tag{64}$$

Tracing over the orbital and the spin space of particle 2, we finally obtain the following expression for the Wigner transform $\hat{I}_T(\mathbf{r}_1, \mathbf{p}_1)$ for the terms linear in T in the Ansatz (18):

$$\hat{I}_T(\mathbf{r}_1, \mathbf{p}_1) = \frac{1}{2i(2\pi)^2 \ \Delta t} \int d^3q \int d^3r \int d^3\kappa \int d^3k_1' \int d^3k_2' \int d^3r'$$

$$\times e^{i\kappa\cdot\mathbf{r}}e^{i(\mathbf{k}_2' - \mathbf{k}_1')\cdot\mathbf{r}'}\delta(E_{k_+} - E_{k_1'})$$

$$\times \left[\delta(\mathbf{k}_2' - \mathbf{k}_-)T(\mathbf{k}_+, \mathbf{k}_1')\hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} - \mathbf{r}'}{2}, \mathbf{p}_1''\right) f\left(\mathbf{r}_1 - \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{p}_2''\right) \right.$$

$$+ \epsilon\delta(\mathbf{k}_2' - \mathbf{k}_-)T(-\mathbf{k}_+, \mathbf{k}_1')\hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{p}_2''\right) \hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} - \mathbf{r}'}{2}, \mathbf{p}_1''\right)$$

$$+ \epsilon\delta(\mathbf{k}_2' + \mathbf{k}_-)T(\mathbf{k}_+, \mathbf{k}_1')\hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} - \mathbf{r}'}{2}, \mathbf{p}_1''\right) \hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{p}_2''\right)$$

$$+ \delta(\mathbf{k}_2' + \mathbf{k}_-)T(-\mathbf{k}_+, \mathbf{k}_1')f\left(\mathbf{r}_1 - \frac{\mathbf{r} - \mathbf{r}'}{2}, \mathbf{p}_1''\right) \hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{p}_2''\right) \right]$$

$$+ \text{c.c.} \tag{65}$$

with notation (11) and with:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \; ; \qquad \mathbf{k}_\pm = \frac{\mathbf{p}}{\hbar} \pm \frac{\boldsymbol{\kappa}}{2}$$
$$\mathbf{p}_1'' = \mathbf{p}_1 - \frac{\mathbf{q}}{2} + \hbar\frac{\mathbf{k}_1' + \mathbf{k}_2'}{2} \; ; \quad \mathbf{p}_2'' = \mathbf{p}_1 - \frac{\mathbf{q}}{2} - \hbar\frac{\mathbf{k}_1' + \mathbf{k}_2'}{2} \tag{66}$$

The distribution f is the spin trace of $\hat{\rho}$, see equation (10).

The same kind of calculation can be done for the term quadratic in the T matrix in (18). It gives the Wigner transform $\hat{I}_{T^2}(\mathbf{r}_1, \mathbf{p}_1)$ as:

$$\begin{aligned}
\hat{I}_{T^2}(\mathbf{r}_1, \mathbf{p}_1) = &\frac{1}{4\pi\Delta t} \int d^3q \int d^3r \int d^3\kappa \int d^3k_1' \int d^3k_2' \int d^3r' \\
&\times e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \, e^{i(\mathbf{k}_2' - \mathbf{k}_1')\cdot\mathbf{r}'} \delta(E_{k_+} - E_{k_1'})\delta(E_{k_-} - E_{k_2'}) \\
&\times \left[T(\mathbf{k}_+, \mathbf{k}_1')T(\mathbf{k}_-, \mathbf{k}_2')^* \hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} - \mathbf{r}'}{2}, \mathbf{p}_1''\right) f\left(\mathbf{r}_1 - \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{p}_2''\right) \right. \\
&+ \epsilon T(-\mathbf{k}_+, \mathbf{k}_1')T(\mathbf{k}_-, \mathbf{k}_2')^* \hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} - \mathbf{r}'}{2}, \mathbf{p}_2''\right) \hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{p}_1''\right) \\
&+ \epsilon T(\mathbf{k}_+, \mathbf{k}_1')T(-\mathbf{k}_-, \mathbf{k}_2')^* \hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} - \mathbf{r}'}{2}, \mathbf{p}_1''\right) \hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{p}_2''\right) \\
&\left. + T(-\mathbf{k}_+, \mathbf{k}_1')T(-\mathbf{k}_-, \mathbf{k}_2')^* f\left(\mathbf{r}_1 - \frac{\mathbf{r} - \mathbf{r}'}{2}, \mathbf{p}_1''\right) \hat{\rho}\left(\mathbf{r}_1 - \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{p}_2''\right) \right] \tag{67}
\end{aligned}$$

As in Ref. [26, 14], we now assume that $\hat{\rho}(\mathbf{r}, \mathbf{p})$ varies slowly in space over microscopic distances and expand the product of $\hat{\rho}$ and f in (65) and (67) according to

$$\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1'')f(\mathbf{r}_1, \mathbf{p}_2'') - \frac{\mathbf{r}}{2} \cdot \nabla_{\mathbf{r}_1}\left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1'')f(\mathbf{r}_1, \mathbf{p}_2'')\right]$$
$$+ \frac{\mathbf{r}'}{2} \cdot \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1'')\nabla_{\mathbf{r}_1}f(\mathbf{r}_1, \mathbf{p}_2'') - \left(\nabla_{\mathbf{r}_1}\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1'')\right)f(\mathbf{r}_1, \mathbf{p}_2'') + ...\right] \tag{68}$$

The first term in (68) corresponds to the local term, the two which follow to first order in gradients non-local terms. A very similar equation exists for the product of $\hat{\rho}$'s in (65) and (67).

(1) Local term We now calculate the zeroth order in gradients. The following three integrals occur:

$$\int d^3r \, e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} = (2\pi)^3\delta(\boldsymbol{\kappa}); \int d^3r' \, e^{i(\mathbf{k}_2' - \mathbf{k}_1')\cdot\mathbf{r}'} = (2\pi)^3\delta(\mathbf{k}_2' - \mathbf{k}_1')$$
$$\int d^3r' \, e^{-i(\mathbf{k}_2' + \mathbf{k}_1')\cdot\mathbf{r}'} = (2\pi)^3\delta(\mathbf{k}_2' + \mathbf{k}_1') \tag{69}$$

The linear in T matrix terms become:

$$
\begin{aligned}
\hat{I}_T(\mathbf{r}_1, \mathbf{p}_1) = \frac{(2\pi)^4}{2i\,\Delta t} \int d^3q \int d^3k_1'\ &\delta(E_k - E_{k_1'}) \\
\times \Big[&\delta(\mathbf{k}_1' - \mathbf{k})T(\mathbf{k}, \mathbf{k}_1')\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} + \hbar\mathbf{k}_1'\right) f\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} - \hbar\mathbf{k}_1'\right) \\
+ &\epsilon\delta(\mathbf{k}_1' - \mathbf{k})T(-\mathbf{k}, \mathbf{k}_1')\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} - \hbar\mathbf{k}_1'\right)\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} + \hbar\mathbf{k}_1'\right) \\
+ &\epsilon\delta(\mathbf{k}_1' + \mathbf{k})T(\mathbf{k}, \mathbf{k}_1')\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} + \hbar\mathbf{k}_1'\right)\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} - \hbar\mathbf{k}_1'\right) \\
+ &\delta(\mathbf{k}_1' + \mathbf{k})T(-\mathbf{k}, \mathbf{k}_1')f\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} + \hbar\mathbf{k}_1'\right)\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} - \hbar\mathbf{k}_1'\right) \Big] \\
+ \text{c.c.} &
\end{aligned}
\tag{70}
$$

In the preceding equation, we note the appearance of the square of delta functions of the energy $\delta(E_k)\delta(\mathbf{k}) \propto [\delta(E_k)]^2$. They are handled by the following well-known simplification from scattering theory:

$$
[\delta(E_k)]^2 = \frac{\Delta t}{2\pi\hbar}\delta(E_k)
\tag{71}
$$

where Δt is a time larger than the duration of a collision (see Ref. [46], for example), which simplifies with the one introduced in the S matrix Ansatz (18). The same manipulations can be done with the quadratic in T matrix term, they lead to:

$$
\begin{aligned}
\hat{I}_{T^2}(\mathbf{r}_1, \mathbf{p}_1) = \frac{(2\pi)^4}{2\hbar} \int d^3q \int d^3k'\ &\delta(E_k - E_{k'}) \\
\times \Big[&T(\mathbf{k}, \mathbf{k}')T(\mathbf{k}, \mathbf{k}')^*\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} + \hbar\mathbf{k}'\right) f\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} - \hbar\mathbf{k}'\right) \\
+ &\epsilon T(-\mathbf{k}, \mathbf{k}')T(\mathbf{k}, \mathbf{k}')^*\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} - \hbar\mathbf{k}'\right)\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} + \hbar\mathbf{k}'\right) \\
+ &\epsilon T(\mathbf{k}, \mathbf{k}')T(-\mathbf{k}, \mathbf{k}')^*\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} + \hbar\mathbf{k}'\right)\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} - \hbar\mathbf{k}'\right) \\
+ &T(-\mathbf{k}, \mathbf{k}')T(-\mathbf{k}, \mathbf{k}')^*f\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} + \hbar\mathbf{k}'\right)\hat{\rho}\left(\mathbf{r}_1, \mathbf{p}_1 - \hbar\mathbf{k} - \hbar\mathbf{k}'\right) \Big]
\end{aligned}
\tag{72}
$$

We will now use different properties of the T matrix. In addition to the differential $\sigma_k(\hat{k}, \hat{k}')$ and total $\sigma_T(k)$ cross sections defined in (12), we introduce, following Ref. [13], the following cross sections:

$$
T(-\mathbf{k}, \mathbf{k}) = \frac{\hbar^2 k}{i8\pi^3 m}\left(\sigma_{fwd.}^{ex.}(k) - i\tau_{fwd.}^{ex.}(k)\right)
$$

$$
T(-\mathbf{k}', \mathbf{k})T(\mathbf{k}', \mathbf{k})^* = \frac{\hbar^4}{4\pi^4 m^2}\left(\sigma_k^{ex.}(\hat{k}, \hat{k}') - i\tau_k^{ex.}(\hat{k}, \hat{k}')\right)
\tag{73}
$$

Unitarity of the S matrix implies the optical theorem:

$$
\frac{T(\mathbf{k}, \mathbf{k})}{2i} + c.c. = \operatorname{Im} T(\mathbf{k}, \mathbf{k}) = -\frac{\hbar^2 k}{(2\pi)^3 m}\sigma_T(k)
\tag{74}
$$

and the rotational invariance of the interaction Hamiltonian can be used to show that:

$$
T(-\mathbf{k}, -\mathbf{k}') = T(\mathbf{k}, \mathbf{k}')
\tag{75}
$$

Performing the integral over the length of wavevector $\mathbf{k}'$ in (70) and (72) and using the previous properties of the T matrix, we obtain the Wigner transform to zero order of the S

matrix Ansatz $\hat{I}_W = \hat{I}_T + \hat{I}_{T^2}$:

$$
\hat{I}_W(\mathbf{r}_1, \mathbf{p}_1) = -\int d^3q \, \frac{q}{m} \Bigg\{ \Big[\sigma_T(k)\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1)f(\mathbf{r}_1, \mathbf{p}_2)
$$

$$
+ \frac{\epsilon}{2} \left(\sigma_{fwd.}^{ex.}(k) \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1), \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2)\right]_+ + i\tau_{fwd.}^{ex.}(k) \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1), \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2)\right]_- \right) \Big]
$$

$$
- \int d^2\hat{k}' \Big[\sigma_k(\hat{k}, \hat{k}')\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1')f(\mathbf{r}_1, \mathbf{p}_2') + \frac{\epsilon}{2} \Big(\sigma_k^{ex.}(\hat{k}, \hat{k}') \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1'), \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2')\right]_+
$$

$$
+ i\tau_k^{ex.}(\hat{k}, \hat{k}') \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1'), \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2')\right]_- \Big) \Big] \Bigg\} \tag{76}
$$

with the notations (11). This local term is the LL "collision integral" [13]. It contains terms linear in T matrix, usually written on the l.h.s. of the kinetic equation, as well as terms quadratic in T matrix.

The low-energy limit of the cross sections is obtained from the low-energy expression of the T matrix (5):

$$
\begin{array}{ll}
\sigma_T(k) \sim 4\pi a^2 & \sigma_k(\hat{k}, \hat{k}') \sim a^2 \\
\sigma_{fwd.}^{ex.}(k) \sim 4\pi a^2 & \sigma_k^{ex.}(\hat{k}, \hat{k}') \sim a^2 \\
\tau_{fwd.}^{ex.}(k) \sim -4\pi a/k & \tau_k^{ex.}(\hat{k}, \hat{k}') \to 0
\end{array}
$$

when $k \to 0$, see Ref. [13]. When introduced into (76), the collision integral of equation (22) is obtained, as well as the spin mean field contained in the commutator in equation (20). Other mean field terms (contained in the anticommutator) appear to first order of the gradient expansion.

(2) First order terms We retain only first order terms that are in addition linear in T matrix.

(2-a) r gradients The first-order terms introduce the gradient of a delta function:

$$
\int d^3r \, \mathbf{r} \, e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} = -i(2\pi)^3 \nabla_{\boldsymbol{\kappa}} \delta(\boldsymbol{\kappa})
$$

which implies taking the derivative with respect to $\boldsymbol{\kappa}$ of the function under the integral. We obtain:

$$
\frac{(2\pi)^3}{2} \int d^3q \Bigg\{ \operatorname{Re} T(\mathbf{k}, \mathbf{k}) \, \nabla_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1)f(\mathbf{r}_1, \mathbf{p}_2) \right]
$$

$$
- \nabla_{\mathbf{p}} \left[\operatorname{Re} T(\mathbf{k}, \mathbf{k}) \right] \cdot \nabla_{\mathbf{r}} \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1)f(\mathbf{r}_1, \mathbf{p}_2) \right]
$$

$$
+ \frac{\epsilon}{2} T(\mathbf{k}, \mathbf{k}) \, \nabla_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1)\hat{\rho}(\mathbf{r}_1, \mathbf{p}_2) \right]
$$

$$
- \frac{\epsilon}{2} \nabla_{\mathbf{p}} \left[T(\mathbf{k}, \mathbf{k}) \right] \cdot \nabla_{\mathbf{r}} \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1)\hat{\rho}(\mathbf{r}_1, \mathbf{p}_2) \right]
$$

$$
+ \frac{\epsilon}{2} T(\mathbf{k}, \mathbf{k})^* \, \nabla_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_2)\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) \right]
$$

$$
- \frac{\epsilon}{2} \nabla_{\mathbf{p}} \left[T(\mathbf{k}, \mathbf{k})^* \right] \cdot \nabla_{\mathbf{r}} \left[\hat{\rho}(\mathbf{r}_1, \mathbf{p}_2)\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) \right] \Bigg\} \tag{77}
$$

(2-b) r′ gradients   The following integrals occur:

$$\int d^3r' \ e^{i(\mathbf{k}_2' - \mathbf{k}_1') \cdot \mathbf{r}'} \mathbf{r}' = -i(2\pi)^3 \nabla_{\mathbf{k}_1'} \delta(\mathbf{k}_1' - \mathbf{k}_2')$$

$$\int d^3r' \ e^{-i(\mathbf{k}_2' + \mathbf{k}_1') \cdot \mathbf{r}'} \mathbf{r}' = -i(2\pi)^3 \nabla_{\mathbf{k}_1'} \delta(\mathbf{k}_2' + \mathbf{k}_1') \tag{78}$$

so that we obtain:

$$\frac{(2\pi)^3}{2} \int d^3q \ \nabla_{\mathbf{p}} \cdot \Big\{ \mathrm{Re}\, T(\mathbf{k}, \mathbf{k}) \ [\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) \nabla_{\mathbf{r}} f(\mathbf{r}_1, \mathbf{p}_2) - f(\mathbf{r}_1, \mathbf{p}_2) \nabla_{\mathbf{r}} \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1)]$$

$$+ \frac{\epsilon}{2} T(\mathbf{k}, -\mathbf{k}) \ [\hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) \nabla_{\mathbf{r}} \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2) - \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2) \nabla_{\mathbf{r}} \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1)]$$

$$+ \frac{\epsilon}{2} T(\mathbf{k}, -\mathbf{k})^* \ [\nabla_{\mathbf{r}} \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2) \ \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) - \nabla_{\mathbf{r}} \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1) \ \hat{\rho}(\mathbf{r}_1, \mathbf{p}_2)] \Big\} \tag{79}$$

The expression of the T matrix to lowest order in the scattering length (see equation (5))

$$(2\pi)^3 T(\mathbf{k}, \mathbf{k}) \simeq g$$

is independent of $\mathbf{k}$. Using the preceding expression in (77) and (79), we obtain the following first order term:

$$\frac{1}{2} \left[\nabla_{\mathbf{p}} \hat{\rho}(\mathbf{r}_1, \mathbf{p}_1), \cdot \nabla_{\mathbf{r}} \big(g n(\mathbf{r}_1) \hat{1} + \epsilon \, g \hat{n}(\mathbf{r}_1) \big) \right]_+$$

This is the mean field term appearing in the anticommutator in the kinetic equation (20).

Appendix II: spin-dependent interactions (forward scattering)

In this appendix, we show how to obtain the full mean field in the case of spin-dependent interactions, using the S-matrix Ansatz. We will limit ourselves to collisions at low-energy, using the T matrix only at lowest order in the scattering lengths (equation (23)):

$$\langle \alpha \, ; \beta | \ \hat{T}_k \ | \delta \, ; \gamma \rangle = \delta_{\alpha, \gamma} \delta_{\beta, \delta} \frac{g_{\alpha\beta}^{(d)}}{(2\pi)^3} + (1 - \delta_{\alpha, \beta}) \delta_{\alpha, \delta} \delta_{\beta, \gamma} \frac{g_{\alpha\beta}^{(t)}}{(2\pi)^3} \tag{80}$$

where $g_{\alpha\alpha}^{(d)} \equiv g_{\alpha\alpha}$. In this limit the T matrix elements are real.

As we are interested in mean field terms, we only keep terms linear in the T matrix in the r.h.s. of the S matrix Ansatz, equation (18):

$$\frac{2\pi}{i\Delta t} Tr_2 \left\{ \frac{\hat{1} + \epsilon \hat{P}_{ex.}}{\sqrt{2}} \left[\hat{\mathcal{T}} \ \hat{\rho}_1(1) \ \hat{\rho}_1(2) - \hat{\rho}_1(1) \ \hat{\rho}_1(2) \ \hat{\mathcal{T}} \right] \frac{\hat{1} + \epsilon \hat{P}_{ex.}}{\sqrt{2}} \right\} \tag{81}$$

where we introduced the $\mathcal{T}$ matrix (the on-shell T matrix), whose definition is given in Appendix I, equation (58). Using the properties of the trace on particle 2 and the fact that the exchange operator commutes with the $\mathcal{T}$ matrix, we can rewrite the preceding equation in the form of a commutator:

$$\frac{1}{i\hbar} \left[\frac{2\pi\hbar}{\Delta t} Tr_2 \left\{ (\hat{1} + \epsilon \hat{P}_{ex.}) \hat{\mathcal{T}} \ \hat{\rho}_1(2) \right\}, \ \hat{\rho}_1(1) \right]_- \tag{82}$$

It seems therefore natural that the first operator in the commutator should play the role of an effective single-particle Hamiltonian.

We now consider a general matrix element of the preceding commutator. To simplify the notation, we only write the first term of the commutator:

$$\langle 1 : \mathbf{p}_1, \alpha | \frac{2\pi\hbar}{\Delta t} Tr_2 \left\{ (\hat{1} + \epsilon \hat{P}_{ex.}) \hat{\mathcal{T}} \; \hat{\rho}_1(2) \right\} \hat{\rho}_1(1) | 1 : \mathbf{p}'_1, \alpha' \rangle \tag{83}$$

We calculate this matrix element by introducing three closure relations of the form:

$$\hat{1} = \int d^3p \sum_\beta |\mathbf{p}, \beta\rangle\langle\mathbf{p}, \beta| \tag{84}$$

and by using the expression (80) of the T matrix. We obtain:

$$\frac{2\pi\hbar}{(2\pi)^3 \Delta t} \times \int d^3p_2 \int d^3p_3 \; \delta(E_{(\mathbf{p}_1 - \mathbf{p}_2)/2} - E_{(\mathbf{p}_3 - \mathbf{p}_4)/2})$$

$$\times \sum_\beta \left\{ g_{\alpha\beta}^{(d)} \langle\mathbf{p}_3, \alpha|\hat{\rho}_1|\mathbf{p}'_1, \alpha'\rangle\langle\mathbf{p}_4, \beta|\hat{\rho}_1|\mathbf{p}_2, \beta\rangle \right.$$

$$+ (1 - \delta_{\alpha,\beta}) g_{\alpha\beta}^{(t)} \langle\mathbf{p}_3, \beta|\hat{\rho}_1|\mathbf{p}'_1, \alpha'\rangle\langle\mathbf{p}_4, \alpha|\hat{\rho}_1|\mathbf{p}_2, \beta\rangle$$

$$+ \epsilon g_{\alpha\beta}^{(d)} \langle\mathbf{p}_3, \beta|\hat{\rho}_1|\mathbf{p}'_1, \alpha'\rangle\langle\mathbf{p}_4, \alpha|\hat{\rho}_1|\mathbf{p}_2, \beta\rangle$$

$$\left. + \epsilon(1 - \delta_{\alpha,\beta}) g_{\alpha\beta}^{(t)} \langle\mathbf{p}_3, \alpha|\hat{\rho}_1|\mathbf{p}'_1, \alpha'\rangle\langle\mathbf{p}_4, \beta|\hat{\rho}_1|\mathbf{p}_2, \beta\rangle \right\} \tag{85}$$

where $\mathbf{p}_4 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3$. By defining the three following coupling constants:

$$g_{\alpha\alpha} = g_{\alpha\alpha}^{(d)} \qquad ; \qquad g_{\alpha\beta} = g_{\alpha\beta}^{(d)} + \epsilon g_{\alpha,\beta}^{(t)} \quad \text{if } \alpha \neq \beta \tag{86}$$

we can rewrite equation (85) as:

$$\frac{2\pi\hbar}{\Delta t} \times \int d^3p_2 \int d^3p_3 \; \delta(E_{(\mathbf{p}_1 - \mathbf{p}_2)/2} - E_{(\mathbf{p}_3 - \mathbf{p}_4)/2})$$

$$\times \sum_\beta \frac{g_{\alpha\beta}}{(2\pi)^3} \left\{ \langle\mathbf{p}_4, \beta|\hat{\rho}_1|\mathbf{p}_2, \beta\rangle\langle\mathbf{p}_3, \alpha|\hat{\rho}_1|\mathbf{p}'_1, \alpha'\rangle \right.$$

$$\left. + \epsilon\langle\mathbf{p}_4, \alpha|\hat{\rho}_1|\mathbf{p}_2, \beta\rangle\langle\mathbf{p}_3, \beta|\hat{\rho}_1|\mathbf{p}'_1, \alpha'\rangle \right\} \tag{87}$$

We therefore see that, for both statistics (bosonic or fermionic), only three coupling constants (g_{11}, g_{22} and g_{12}) are involved.

Now, taking the Wigner transform of the operator (82) and expanding in gradients (see Appendix I) introduces another delta function of energy conservation, which is present at each order in gradients. This allows to use the "$\delta(E)^2$" simplification (equation (71) in Appendix I) to make the Δt disappear. The net result is that:

$$\frac{2\pi\hbar}{\Delta t} \delta(E_{(\mathbf{p}_1 - \mathbf{p}_2)/2} - E_{(\mathbf{p}_3 - \mathbf{p}_4)/2}) \tag{88}$$

gets replaced by 1. Formally the matrix element of equation (83) becomes:

$$\int d^3 p_2 \int d^3 p_3 \sum_\beta \frac{g_{\alpha\beta}}{(2\pi)^3} \times \left\{ \langle \mathbf{p}_4, \beta | \hat{\rho}_1 | \mathbf{p}_2, \beta \rangle \langle \mathbf{p}_3, \alpha | \hat{\rho}_1 | \mathbf{p}'_1, \alpha' \rangle \right.$$
$$\left. + \epsilon \langle \mathbf{p}_4, \alpha | \hat{\rho}_1 | \mathbf{p}_2, \beta \rangle \langle \mathbf{p}_3, \beta | \hat{\rho}_1 | \mathbf{p}'_1, \alpha' \rangle \right\} \tag{89}$$

This result can be rewritten:

$$\langle 1 : \mathbf{p}_1, \alpha | \hat{V}^{mf}(1) \hat{\rho}_1(1) | 1 : \mathbf{p}'_1, \alpha' \rangle \tag{90}$$

where:

$$V_{\alpha\beta}^{mf}(\mathbf{r}) = \delta_{\alpha\beta} \sum_{\gamma=1,2} g_{\alpha\gamma} \langle \mathbf{r}, \gamma | \hat{\rho}_1 | \mathbf{r}, \gamma \rangle + \epsilon g_{\alpha\beta} \langle \mathbf{r}, \alpha | \hat{\rho}_1 | \mathbf{r}, \beta \rangle \tag{91}$$

is the mean field potential. Therefore, equation (82) can be written formally:

$$\frac{1}{i\hbar} \left[\hat{V}^{mf}(1), \ \hat{\rho}_1(1) \right]_- \tag{92}$$

This equation is valid provided it is used to compute the Wigner transform and do a subsequent gradient expansion.

In the body of the article, we define the effective single-particle Hamiltonian $\hat{U}$ which is the sum of the external potential $\hat{V}^{ext}$ and of the mean field potential $\hat{V}^{mf}$. The operatorial kinetic equation is now:

$$\frac{d\hat{\rho}_1}{dt} + (i\hbar)^{-1} \left[\hat{\rho}_1, \hat{H}_1 + \hat{V}^{mf} \right]_- = 0 \tag{93}$$

The kinetic equation is then derived following the procedure of Appendix I (Wigner transform and gradient expansion up to first order); the result is:

$$\partial_t \hat{\rho}(\mathbf{r}, \mathbf{p}) + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} \hat{\rho}(\mathbf{r}, \mathbf{p}) + \frac{1}{i\hbar} [\hat{\rho}(\mathbf{r}, \mathbf{p}), \widehat{U}(\mathbf{r})]_-$$
$$- \frac{1}{2} [\nabla_{\mathbf{p}} \hat{\rho}(\mathbf{r}, \mathbf{p}), \cdot \nabla_{\mathbf{r}} \widehat{U}(\mathbf{r})]_+ = 0 \tag{94}$$

which is equal to the l.h.s. of (25).

References

[1] H.J. Lewandowski, D.M. Harber, D.L. Whitaker and E.A. Cornell, Phys. Rev. Lett. **88**, 07403 (2002).

[2] J.M. McGuirk, H.J. Lewandowski, D.M. Harber, T. Nikuni, J.E. Williams and E.A. Cornell, Phys. Rev. Lett. **89**, 090402 (2002).

[3] V.P. Silin, Soviet Phys. JETP **6**, 945 (1958).

[4] A.J. Leggett, J. Phys. C **3**, 448 (1970).

[5] E.P. Bashkin, JETP Lett. **33**, 8 (1981); JETP **60**, 1122 (1984) .

[6] C. Lhuillier and F. Laloë, J. Physique **43**, 225 (1982).

[7] K. Miyake, W.J. Mullin and P.C.E. Stamp, J. Physique **46**, 663 (1985).

[8] B.R. Johnson, J.S. Denker, N. Bigelow, L.P. Lévy, J.H. Freed, and D.M. Lee, Phys. Rev. Lett. **52**, 1508 (1984).

[9] P.J. Nacher, G. Tastevin, M. Leduc, S.B. Crampton and F. Laloë, J. Phys. Lett **45**, L-441 (1984). G. Tastevin, P.J. Nacher, M. Leduc, F. Laloë, J. Phys. Lett. **46**, 249 (1985).

[10] W.J. Gully and W.J. Mullin, Phys. Rev. Lett. **52**, 1810 (1984).

[11] E.M. Lifshitz, L.P. Pitaevskii, Landau and Lifshitz Course of Theoretical Physics volume 10, *Physical Kinetics*, Pergamon Press (1981).

[12] T.Y. Wu and T. Ohmura, *Quantum theory of scattering*, chap. A §7, Prentice Hall (1962).

[13] C. Lhuillier and F. Laloë, J. Physique **43**, 197 (1982).

[14] F. Laloë and W.J. Mullin, J. Stat. Phys. **59**, 725 (1990).

[15] A.E. Meyerovich, Phys. Rev. B **39**, 9318 (1989).

[16] E.P. Bashkin, Sov. Phys. Usp. **29**, 238 (1986).

[17] J.A. McLennan, *Introduction to Non-Equilibrium Statistical Mechanics*, Prentice Hall (1989).

[18] D. Zubarev, V. Morozov, G. Röpke, *Statistical Mechanics of Nonequilibrium Processes*, vol. I, Akademie Verlag (1996).

[19] E.P. Wigner, Phys. Rev. **40**, 749 (1932).

[20] M. Hillery, R.F. O'Connell, M.O. Scully and E.P. Wigner, Phys. Rep. **106**, 121 (1984).

[21] H. Groenewold, Physica **12**, 405 (1946).

[22] R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics*, vol. I, formula (30-18), Addison-Wesley (1965).

[23] R.G. Newton, *Scattering theory of waves and particles*, Springer Verlag (1982).

[24] J. Yvon, C. R. Acad. Sci. **246**, 2858 (1958).

[25] R.F. Snider, J. Chem. Phys. **32**, 1051 (1960).

[26] M.W. Thomas and R.F. Snider, J. Stat. Phys. **2**, 61 (1970).

[27] M. Pinard and F. Laloë, J Physique **41**, 769 (1980).

[28] E.M. Lifshitz, L.P. Pitaevskii, Landau and Lifshitz Course of Theoretical Physics volume 9, *Statistical Physics, Part 2*, Pergamon Press (1980).

[29] A.E. Meyerovich, K.A. Musaelian, Phys. Rev. Lett. **72**, 1710 (1994); J.W. Jeon, W.J. Mullin, J. Low Temp. Phys. **88**, 433 (1992); J.W. Jeon, H.R. Glyde, Phys. Rev. B **43**, 5338 (1991).

[30] K. Huang, *Statistical Mechanics*, Wiley, 2$^{\mathrm{nd}}$ ed. (1987).

[31] H. Smith, H.H. Jensen, *Transport Phenomena*, chapter 1, Clarendon Press, Oxford (1989).

[32] T. Nikuni and J. Williams, arXiv:cond-mat/0304095 (2003); submitted to J. Low Temp. Phys.

[33] M.Ö. Oktel and L. Levitov, Phys. Rev. Lett. **88**, 230403 (2002).

[34] J.N. Fuchs, D.M. Gangardt and F. Laloë, Phys. Rev. Lett. **88**, 230404 (2002).

[35] J. Williams, T. Nikuni and C.W. Clark, Phys. Rev. Lett. **88**, 230405 (2002).

[36] T. Nikuni, J. Williams and C.W. Clark, Phys. Rev. A **66**, 043411 (2002).

[37] A.S. Bradley, C.W. Gardiner, J. Phys. B **35**, 4299 (2002).

[38] N. Masuhara, D. Candela, D.O. Edwards, R.F. Hoyt, H.N. Scholtz, D.S. Sherrill, R. Combescot, Phys. Rev. Lett. **53**, 1168 (1984).

[39] N.P. Bigelow, J.H. Freed, D.M. Lee, Phys. Rev. Lett. **63**, 1609 (1989).

[40] A.E. Meyerovich in Prog. Low Temp. Phys. volume 11, ed. D.F. Brewer, pp. 1-73, North-Holland, Amsterdam (1987).

[41] B. Castaing, Physica **126B**, 212 (1984).

[42] J.N. Fuchs, O. Prévoté, D.M. Gangardt, arXiv:cond-mat/0209506 (2002); to be published in Eur. Phys. J. D.

[43] D. Dragoman, Phys. Lett. **A285**, 109 (2001).

[44] S. Haroche, 5$^{\text{ème}}$ cours du Collège de France 2002-2003, http://www.lkb.ens.fr/recherche/qedcav/college/college.html .

[45] E.A. Burt, R.W. Ghrist, C.J. Myatt, M.J. Holland, E.A. Cornell, C.E. Wiemann, Phys. Rev. Lett. **79**, 337 (1997).

[46] P. Roman, *Advanced Quantum Theory*, chapter 4-1, Addison-Wesley (1965).