D-instanton Amplitudes in String Theory

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String theory

Perturbative string theory in any background gives

1. The spectrum of states

– contains some massless states and infinite tower of massive states

2. A formula for the scattering amplitudes involving these states

A scattering amplitude in closed string theory can be expressed as

$$\sum_{n=0}^{\infty} a_n \, g_s^{2n+\alpha}$$

g_s: string coupling

α: some fixed number for a given scattering process

a_n: coefficients that could depend on the quantum numbers of external states

– can be computed in terms of integrals over the moduli spaces of closed Riemann surfaces

Integrand: Correlation functions of vertex operators on Riemann surfaces

World-sheet with Euler number χ gives contribution $\propto g_s^{-\chi}$

What about non-perturbative corrections?

– need a non-perturbative formulation

– exists only for special backgrounds via various dualities (matrix model, AdS/CFT)

However we do not yet have a complete non-perturbative formulation of string theory in a general background.

Nevertheless, world-sheet formalism $\Rightarrow$ one class of non-perturbative corrections

– additional contributions to an amplitude of the form

$$e^{-C/g_s} \sum_{n=0}^{\infty} b_n \, g_s^{n+\beta}$$

C, β: some constants

b_n's depend on the quantum numbers of external states

– can be computed as integrals over the moduli space of Riemann surfaces with boundaries, with Dirichlet boundary condition along non-compact target space directions

– D-instanton corrections

D-instantons are like ordinary D-branes but have finite action due to Dirichlet b.c

– localized in non-compact directions

In general there may be different D-instantons, differing by boundary condition along the compact directions

– gives different C, b_n

Final result: Weighted sum of these contributions

Systematics of

D-instanton corrections

1. In the presence of D-instantons the spectrum has both closed strings and open strings with ends on the instanton.

However the open strings describe the modes of the instanton and only exist for limited time

– they are not asymptotic states

The external states in a scattering amplitude will always be closed strings

(or open strings on ordinary time filling D-branes if present)

2. Dirichlet boundary condition breaks space-time translation invariance

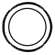
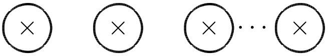
$\Rightarrow$ individual world-sheets with boundaries do not conserve energy / momentum

– disconnected world-sheets contribute even for generic values of external energy / momentum

For getting leading contribution to the D-instanton amplitude, we

– maximize the number of disks since each disk gives $1/g_s$

– can use as many annuli as we want since annuli $\sim (g_s)^0$

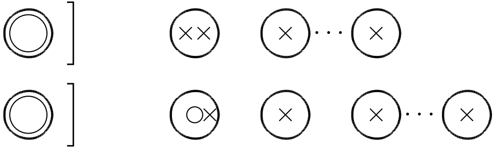
$\exp[-C/g_s]\ \ \exp\Big[$  $\Big]$ 

$\times$: closed string vertex operator C/g_s: D-instanton action

At the next order there are more possibilities

$$\exp[-C/g_s] \ \exp\Big[\quad \Big]$$

$$\exp[-C/g_s] \ \exp\Big[\quad \Big]$$



etc.

This way we can write down the expression for D-instanton induced amplitude to any order in the string coupling g_s

However, the moduli space integrals diverge from regions of the moduli space where the Riemann surface degenerates

Example: 2D bosonic string theory

World-sheet theory has

1. A scalar X describing time direction

2. A Liouville field χ_L with central charge 25

– describes space direction with a potential

3. b,c ghost system with central charge -26

Ghost number assignment: c: 1, b: -1, matter: 0

Closed string spectrum has a single massless scalar field living on a half line along χ_L (tachyon)

This theory has different D-instantons

The one with lowest C has $C = 1$

Zamolodchikov, Zamolodchikov

$$\text{Annulus partition function} = \int_0^\infty \frac{dt}{2t}\left(e^t - 1\right)$$

$$\text{Disk two point function} \propto \text{finite} + \frac{1}{2}\int_0^1 dy\, y^{-2}(1 + 2\omega_1\omega_2 y)$$

Annulus 1-point function

$$\propto \quad \text{finite} + \int_0^1 dv \int_0^{1/4} dx \left\{ \frac{v^{-2} - v^{-1}}{\sin^2(2\pi x)} + 2\,\omega^2\, v^{-1} \right\}$$

For D-instantons in type IIB string theory in ten dimensions

$$\text{Annulus partition function} = \int_0^\infty \frac{dt}{2t}\,(8-8)$$

8 from NS sector, -8 from R sector

Naively the answer vanishes

However this gives results for instanton correction that are inconsistent with the prediction of S-duality

The goal of these lectures will be to understand the physical origin of these divergences and extract unambiguous, finite numbers out of them.

– need string field theory

We shall not try to propose string field theory as a full non-perturbative formulation of string theory

Instead we shall use it to systematically compute the effect of fluctuations around the saddle points represented by D-instantons

– integral along the steepest descent contour of the saddle point

For now, it is enough to know that

1. String field theory is a regular quantum field theory with infinite number of fields, one for each mode of the string

2. It is designed so that the perturbative amplitudes reproduce the world-sheet result (formally).

Each Feynman diagram reproduces integration over part of the moduli space

Sum of all Feynman diagrams reproduces integration over the full moduli space.

Open string propagator $\propto L_0^{-1}$

Closed string propagator $\propto (L_0 + \bar{L}_0)^{-1}$

To make connection with world-sheet formalism we need to use Schwainger parameter representation of the propagator:

$$L_0^{-1} = \int_0^\infty dt \exp\left[-L_0 t\right], \quad (L_0 + \bar{L}_0)^{-1} = \int_0^\infty dt \exp\left[-(L_0 + \bar{L}_0)t\right]$$

After summing over all internal states we recover the world-sheet expression, with the t parameters becoming the moduli of Riemann surfaces

These relations are identities for positive L_0 and $(L_0 + \bar{L}_0)$ eigenvalues

However they fail for $L_0 \leq 0$ and $(L_0 + \bar{L}_0) \leq 0$ since the t integrals diverge at t=0

– origin of all divergences in the world-sheet theory

L_0 and $L_0 + \bar{L}_0$ eigenvalues $\propto k^2 + m^2$

m: mass of the string mode

We can compute the amplitude for positive $k^2 + m^2$ and then explore other regions by analytic continuation in k

e.g. Veneziano or Shapiro-Virasoro amplitude

However, open strings on D-instantons have no continuous momenta due to Dirichlet b.c.

– we need to tackle the divergences differently

$$\text{SFT} \quad \Rightarrow \quad (L_0)^{-1} = \int_0^\infty dt\, e^{-L_0 t} \quad \Leftarrow \quad \text{world-sheet}$$

1. This is an identity for $L_0 > 0$

2. For $L_0 < 0$ the rhs diverges from $t \to \infty$ end but the lhs is finite and we can use lhs as the correct expression

3. For $L_0 = 0$ both sides diverge

However, on the lhs we sit on the pole of a propagator and insights from QFT can be used to make sense of this.

This is the essence of why string field theory is useful for dealing with divergences in the integrals over the moduli spaces of Riemann surfaces

Since closed string propagators do not cause any problem, we can integrate out the closed strings from the internal states and work with an effective string field theory, where

– external states are only closed strings

– internal states are open strings

$\Rightarrow$ Part of string field theory relevant for us is the one that describes the dynamics of open strings with Dirichlet b.c. on all non-compact directions

– a 0-dimensional field theory since open strings do not carry continuous momentum

Path integral $\Rightarrow$ ordinary integrals

As a test of this procedure, we shall verify that when the answer is known from a dual description e.g. matrix model or S-duality, the procedure we shall describe reproduces them correctly.

Given that the string perturbation expansion is expected to be an asymptotic series, does it make sense to compute non-perturbative contribution?

Answer 1:

In many cases the perturbative contribution to specific quantities either vanishes or terminates after a finite order

a) Terms protected by supersymmetry, e.g. R^4 terms in type IIB in D=10, moduli space metric in N=2 supersymmetric theories in D=4, superpotential in N=1 supersymmetric theory in D=4 etc

b) Unitarity violation in c=1 bosonic string theory

c) Barrier penetration in $\hat{c} = 1$ type 0B string theory

Answer 2:

Instantons describe non-trivial saddle points of string theory

Instanton contribution to amplitudes represent the result of the path integral along the steepest descent contour (Lefschetz thimble) of this saddle point

– can be studied independently of the perturbative contribution

– can be used to compare the contribution from individual instantons between a dual pair of theories both of which are weakly coupled, e.g. in c $\leq$ 1 string theories, AdS/CFT correspondence etc

In these lectures we shall focus on single D-instanton amplitudes for simplicity.

n D-instanton contribution $\propto e^{-n\,C/g_s}$

– more suppressed than single D-instanton contribution.

However the analysis can be (and has been) generalized to multi-instanton amplitudes.

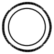
Explicit computations

We have seen that all D-instanton amplitudes have overall normalization factor given by exponential of the annulus amplitude.

We shall first discuss computation of the annulus amplitude.

We shall pick the example of 2D string theory but the same procedure can be applied to all cases.

$$\text{exp[annulus]} = \exp\left[\int_0^\infty \frac{dt}{2t}(e^t - 1)\right]$$

$$\exp\Big[$$  $$\Big] = \exp\left[\int_0^\infty \frac{dt}{2t} Z(t)\right]$$

t $\propto$ ratio of circumference to the width of the cylinder / annulus

$$Z(t) = \mathrm{Tr}\left\{(-1)^F e^{-t L_0} b_0 c_0\right\}$$

Tr is trace over open string states on the D-instanton

F = ghost number - 1

$b_0 c_0$ is needed to remove ghost zero modes

$$Z(t) = \sum_b e^{-t h_b} - \sum_f e^{-t h_f}$$

h_b, h_f: L_0 eigenvalues of bosonic / fermionic open string states that are annihilated by b_0 (Siegel gauge)

If h_b or $h_f \leq 0$, then the integral diverges from large t region.

Strategy for dealing with large t divergence:

1. Use the identities, valid for $h_b, h_f > 0$,

$$\exp\left[\int \frac{dt}{2t}\left(e^{-th_b} - e^{-th_f}\right)\right] = \sqrt{\frac{h_f}{h_b}}$$

$$h_b^{-1/2} = \int \frac{d\psi_b}{\sqrt{2\pi}} e^{-\frac{1}{2}h_b\psi_b^2}, \qquad \psi_b : \text{grassmann even}$$

$$h_f = \int du_f\, dv_f\, e^{-h_f u_f v_f}, \qquad u_f, v_f : \text{grassmann odd}$$

2. Interpret the modes ψ_b, u_f, v_f as open string fields (D=0) and the exponent as open string field theory action in Siegel gauge

3. Modes with $h_b < 0$ are tachyonic modes and integration over them can be carried out along the steepest descent contour producing $1/h_b$

4. Modes with $h_b = 0$ and $h_f = 0$ represent respectively the bosonic and fermionic zero modes and need to be treated carefully.

Origin of zero modes

1. Bosonic zero modes ψ_b^0 can arise from the freedom of translating the instanton along flat directions e.g. Euclidean time

Remedy: Change variables from ψ_b^0 to D-instanton position y.

$\Rightarrow d\psi_b^0 = K_1\, dy$ for some K_1 – to be computed

Integration over y has to be done at the end and produces a factor of $\int dy e^{iEy} = 2\pi\delta(E)$, with E being the total energy of external states

For 2D bosonic string theory

$$\mathbf{Z(t) = \left(e^{t} - 1\right)}$$

$e^t \Rightarrow$ a mode with $h_b = -1 \quad \Rightarrow \quad$ produces $\sqrt{1/h_b} = i$

The bosonic translation zero mode should give +1

Why do we have -1?

2. We have $L_0 = 0$ states coming from ghost sector

$$c_1 c_{-1}|0\rangle, \qquad |0\rangle$$

They are results of wrongly fixing the U(1) 'gauge symmetry' on the instanton

Consider the gauge invariant open string field theory on a Dp-brane

– has a U(1) gauge field in p+1 dimensions.

Action:

$$\int \mathbf{d^{p+1}x} \left[\frac{\mathbf{1}}{\mathbf{4}}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} + \left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\,\partial^{\mu}\mathbf{A}_{\mu} - \phi\right)^{\mathbf{2}}\right]$$

ϕ: mode associated with the state $\mathbf{c_0 e^{ik.X}(0)|0\rangle}$

– not present in the Siegel gauge but is present in the gauge invariant theory

Gauge transformation:

$$\delta\mathbf{A}_{\mu} = \sqrt{\mathbf{2}}\,\partial_{\mu}\theta(\mathbf{x}), \quad \delta\phi = \Box\theta(\mathbf{x})$$

$$S = \int d^{p+1}x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left(\frac{1}{\sqrt{2}} \partial^\mu A_\mu - \phi \right)^2 \right]$$

$$\delta A_\mu = \sqrt{2}\, \partial_\mu \theta(x), \quad \delta\phi = \Box\theta(x)$$

Siegel gauge $\phi = 0$ leads to gauge fixed action including ghosts:

$$\int d^{p+1}x \left[-\frac{1}{2} A^\mu \Box A_\mu - u \Box v \right], \qquad u, v: \text{ ghosts}$$

On D-instanton, $p = -1$, there is no A_μ and all fields are x independent

$\Rightarrow u\Box v = 0$

$\Rightarrow$ **leads to ghost zero modes**

– arise since we are attempting to gauge fix a rigid symmetry with parameter θ under which $\delta\phi = 0$

Remedy: Undo the gauge fixing by using a gauge invariant form of the path integral

1. Integrate over ϕ and drop the integration over the ghosts

$$\Rightarrow \int d\phi\, e^{-\phi^2} = \sqrt{\pi}$$

2. Divide by the volume of the gauge group

$$\Rightarrow \int d\theta$$

– can be found by carefully comparing the string field theory gauge transformation laws with $\psi \to e^{i\alpha}\psi$ where α has period 2π.

ψ: any state of the open string with one end on the instanton

If $\theta = K_2\,\alpha$ then $\int d\theta = K_2\, 2\pi$

$$\exp\left[\int_0^\infty \frac{dt}{2t} Z(t)\right] = \exp\left[\int_0^\infty \frac{dt}{2t} (e^t - 2 + 1)\right]$$

reduces to

$$i \frac{\sqrt{\pi}}{2\pi K_2} \frac{1}{\sqrt{2\pi}} K_1 \, 2\pi\delta(E)$$

To find K_1, K_2 we need more details of open string field theory.

Open (bosonic) string field theory

$H^{(n)}$: Vector space of open string states of ghost number n

Before gauge fixing, an open string field $|\psi\rangle$ is an arbitrary element of $H^{(1)}$

Let $|\xi_r^{(n)}\rangle$ be a set of basis states in $H^{(n)}$

Then $|\psi\rangle = \sum_r \psi_r |\xi_r^{(1)}\rangle$

ψ_r are the dynamical variables over which we do (path) integration.

Action:

$$\mathbf{S} = \frac{1}{2}\langle\psi|\mathbf{Q_B}|\psi\rangle + \text{interaction terms}$$

$$\mathbf{Q_B} = \oint_0 dz\, [c(z)\, T_m(z) + b(z)c(z)\partial c(z)]$$

$T_m(z)$: matter stress tensor

$Q_B^2 = 0$

The action S is invariant under gauge transformation:

$$\delta|\psi\rangle = Q_B|\lambda\rangle + \cdots$$

$|\lambda\rangle$: arbitrary state in $H^{(0)}$

If we expand $|\lambda\rangle$ as $\sum_r \lambda_r|\xi_r^{(0)}\rangle$, then λ_r are the 'gauge transformation' parameters

Siegel gauge: $b_0|\psi\rangle = 0$.

$$S = \frac{1}{2}\langle\psi|c_0 L_0|\psi\rangle + \cdots$$

The gauge fixing leads to Faddeev-Popov ghosts

Result: The full action including the ghosts has the form

$$S = \frac{1}{2}\langle\widetilde{\psi}|c_0 L_0|\widetilde{\psi}\rangle + \cdots$$

with $|\widetilde{\psi}\rangle \in \sum_n H^{(n)}$ subject to $b_0|\widetilde{\psi}\rangle = 0$

Components of $|\widetilde{\psi}\rangle$ with ghost number other than 1 are the Faddeev-Popov ghosts

Propagator $\propto (L_0)^{-1}$

$|\zeta_r^{(n)}\rangle$: **A basis of states of ghost number n, satisfying** $b_0|\zeta_r^{(n)}\rangle = 0$

$$\{|\xi_r^{(n)}\rangle\} = \{|\zeta_r^{(n)}\rangle\} \cup \{c_0|\zeta_r^{(n-1)}\rangle\}$$

In order that the gauge fixed action $\frac{1}{2}\langle\widetilde{\psi}|c_0 L_0|\widetilde{\psi}\rangle$ **has the form**

$$-\frac{1}{2} h_b \psi_b^2 + h_f u_f v_f$$

we need to normalize the basis states as

$$\langle\zeta_r^{(1)}|c_0|\zeta_s^{(1)}\rangle = \delta_{rs}, \qquad \langle\zeta_r^{(2)}|c_0|\zeta_s^{(0)}\rangle = \delta_{rs}$$

etc.

If $|\psi\rangle = \sum_r \chi_r|\zeta_r^{(1)}\rangle + \sum_r \phi_r c_0|\zeta_r^{(0)}\rangle$ **and** $|\lambda\rangle = \sum_r \lambda_r|\zeta_r^{(0)}\rangle$ **then**

$$\delta|\psi\rangle = Q_B|\lambda\rangle \qquad \Rightarrow \qquad \delta\phi_r = h_r\lambda_r$$

h_r: **L_0 eigenvalue of** $|\zeta_r\rangle$

Open string field on D-instanton before gauge fixing:

$$|\psi\rangle = \chi\, c_1|0\rangle + \psi_b^0\, c_1\alpha_{-1}|0\rangle + i\,\phi\, c_0|0\rangle + \cdots$$

Gauge transformation parameters:

$$|\lambda\rangle = i\,\theta|0\rangle + \cdots$$

Siegel gauge field:

$$|\widetilde{\psi}\rangle = \chi\, c_1|0\rangle + \psi_b^0\, c_1\alpha_{-1}|0\rangle + u|0\rangle + v\, c_1 c_{-1}|0\rangle + \cdots$$

χ**: tachyon corresponding to the h $= -1$ state**

α_{-1}**: oscillator of X satisfying** $[\alpha_1, \alpha_{-1}] = 1$

Factors of i ensure that ϕ and θ are real

If $|0\rangle$ had carried L_0 eigenvalue h, e.g. by carrying momentum k, then gauge transformation law would give $\delta\phi = h\,\lambda$ and the Siegel gauge $\phi = 0$ would give a Faddeev-Popov determinant h

– would be reproduced by the ghost action huv

For h=0 this procedure breaks down.

Go back to the original gauge invariant formulation:

$$\int \frac{d\chi}{\sqrt{2\pi}} \int \frac{d\psi_b^0}{\sqrt{2\pi}} \int d\phi\, e^{-S} / \int d\theta$$

$$S = -\frac{1}{2}\chi^2 + \phi^2$$

Note: Comparison with the world-sheet result fixes the normalization of the path integral measure over the open string fields.

We could have replaced $d\phi$ by $d\phi/\sqrt{2\pi}$ but then $d\theta$ will also be replaced by $d\theta/\sqrt{2\pi}$ so that the Faddeev-Popov determinant remains L_0.

Relation between ψ_b^0 and y:

1. The dependence of an amplitude on the D-instanton position y must be of the form

$$e^{i\omega y}$$

where ω is the total energy carried by all the closed string states

y insertion in an amplitude should product a $i\omega$ factor

Compare this with the result of the ψ_b^0 insertion

State multiplying ψ_b^0 in string field expansion

$$c_1\alpha_{-1}|0\rangle = c(0)\, i\sqrt{2}\partial X(0)|0\rangle$$

X is normalized so that

$$\partial X(z)\partial X(w) = -\frac{1}{2(z-w)^2} + \text{non-singular}$$

$\Rightarrow$ vertex operator for ψ_b^0:

$$\text{unintegrated}: \ c(z)\, i\sqrt{2}\partial X(z), \qquad \text{integrated}: \ i\sqrt{2}\partial X(z)$$

The disk amplitude with one insertion of ψ_b^0 and n closed string vertex operators $V_1, \cdots V_n$ of energy $\omega_1, \cdots \omega_n$ is

$$A \propto g_o \left\langle \int dz\, i\sqrt{2}\partial X(z) \prod_{k=1}^{n} V_k(z_k) \right\rangle$$

$g_o = (1/2\pi^2 T)^{1/2}$: open string coupling constant

T: D-instanton action C/g_s

Using OPE

$$\partial X(z) V_k(z_k) = -\frac{i\omega_k}{2(z - z_k)} V_k(z_k) + \text{non-singular}$$

we get

$$A = i\pi\sqrt{2}\, g_o\, \omega \left\langle \prod_{k=1}^{n} V_k(z_k) \right\rangle, \qquad \omega \equiv \sum_k \omega_k$$

$\Rightarrow \psi_b^0$ **insertion in an amplitude produces a factor of** $i\pi\sqrt{2}\, g_o\, \omega$

Since y insertion produces a factor of $i\omega$**, we have**

$$\psi_b = K_1\, y, \qquad K_1 = \frac{1}{\pi\sqrt{2}\, g_o}$$

Relation between θ and α:

α: rigid gauge transformation parameter

An open string stretched between the original D-instanton and a second spectator D-instanton picks up a phase $e^{i\alpha}$.

This gives infinitesimal transformation law:

$$\delta\xi = i\,\alpha\,\xi$$

ξ: Any state of the open string stretched between the pair of D-instantons

We can compare this with the known gauge transformation law of ξ in open string field theory:

$$\delta\xi = g_o\,K\,\theta\,\xi$$

K: three point function of normalized vertex operators of θ, ξ, ξ^c

In the expansion of the string field, θ multiplies i $|0\rangle$

$\Rightarrow$ θ vertex operator is i $\times$ identity

$\Rightarrow$ three point function of θ, ξ, ξ^c reduces to i $\times$ two point function of ξ, ξ^c

i as long as ξ, ξ^c are normalized

Compare $\delta\xi = i\, g_o\, \theta\xi$ with $\delta\xi = i\alpha\xi$

This gives

$$\theta = K_2\, \alpha, \qquad K_2 = \frac{1}{g_o}$$

Net result:

$$i\, \frac{\sqrt{\pi}}{2\pi K_2}\, \frac{1}{\sqrt{2\pi}} K_1\; 2\pi\delta(E) = \frac{i}{4\pi^2}\; 2\pi\delta(E)$$

agrees with a dual matrix model result

If there are $L_0 = 0$ states in the Ramond sector zero modes, they represent fermion zero modes

– goldstino modes associated with supersymmetries broken by the D-instanton

⇒ we need to perform integral over these grassmann odd modes at the end

– same as differentiation

⇒ insert vertex operators of these modes into the world-sheet diagram

Example: D-instantons in IIB in D=10 has 16 such fermion zero mode

– 16 open string vertex operators will have to be distributed in all possible ways over the boundaries of the world-sheet diagrams

When the result is known from a dual description, this procedure produces the correct result in all cases that have been studied.

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2. c<1 bosonic string theory — Eniceicu, Mahajan, Murdia, A.S.

3. Type IIB in D=10 — A.S.

4. Type IIA / IIB on CY_3 — Alexandrov, A.S., Stefanski

5. $\hat{c} = 1$ type 0B string theory — Chakravarty, A.S.

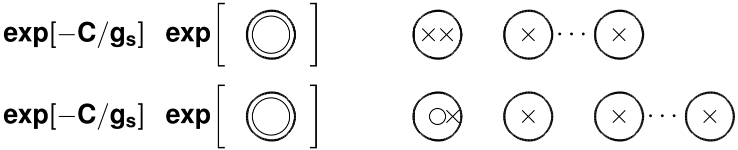
6. IIA/IIB on CY_3 orientifolds — Alexandrov, Firat, Kim, A.S., Stefanski

7. Sine-Liouville deformation of c=1 bosonic string theory

Alexandrov, Mahajan, A.S., work in progress

Higher order contributions

At the next order we need to compute



etc.

Define:

$g_s\, f(\omega_1, \omega_2)$: Ratio of disk two point function to product of two disk one point functions

$g_s\, g(\omega)$: Ratio of annulus one point function to disk one point function

$g_s\, C$: Partition function for disk with two holes and torus with one hole.

Order g_s contribution to the n-point amplitude:

$$g_s \times \text{leading order contribution} \times \left[\sum_{j<k} f(\omega_j,\omega_k) + \sum_j g(\omega_j) + C\right]$$

f, g and C have divergences.

$$f = f_{finite} + f_{div}, \qquad g = g_{finite} + g_{div}, \qquad C = C_{finite} + C_{div}$$

$$f_{div}(\omega_1,\omega_2) = \frac{1}{2}\int_0^1 dy\, y^{-2}(1+2\omega_1\omega_2 y) \equiv A_f + B_f\,\omega_1\omega_2$$

$$g_{div}(\omega) = \int_0^1 dv \int_0^{1/4} dx \left\{\frac{v^{-2}-v^{-1}}{\sin^2(2\pi x)} + 2\,\omega^2\,v^{-1}\right\} \equiv A_g + B_g\omega^2$$

n-point function at order g_s:

$$= g_s \times \text{leading order contribution}$$

$$\times \left[\frac{n(n-1)}{2}A_f + n\,A_g + C + \left\{B_g - \frac{B_f}{2}\right\}\sum_j \omega_j^2 + \text{finite}\right]$$

We again need to make use of string field theory

Strategy:

1. Express the amplitudes as sum over SFT Feynman diagrams

The external states in a Feynman diagram are closed strings

The internal propagators are of open strings since we integrate out the closed strings.

Open string propagator of a Siegel gauge state

$$L_0^{-1} = \int_0^\infty dt\, e^{-t L_0} = \int_0^1 dq\, q^{L_0-1}, \qquad q \equiv e^{-t}$$

q's become the moduli in the world-sheet description.

Divergences come from states with L_0 eigenvalue $h \leq 0$ from q=0 end

2. In SFT, $1/L_0 = 1/h$ is finite for $h < 0$

– removes the power law divergences in the q integral

Also remove the zero mode contribution to the propagators since they are to be integrated at the end or removed altogether.

$$\int_0^1 q^{-1+h} \Rightarrow h^{-1}, \qquad \int_0^1 q^{-1} \Rightarrow 0$$

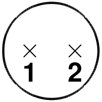
3. Add the propagator of the field ϕ that was not present in the world-sheet formulation but should be present.

$$S = -\phi^2 \qquad \Rightarrow \qquad \text{Propagator} = \frac{1}{2}$$

4. Account for corrections to the jacobian factors for change of variable from ψ_b^0 to y and θ to α

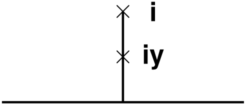
We shall first describe the analysis of $f_{div}(\omega_1, \omega_2)$.

– related to the divergent part of disk / UHP two point function:



$$f_{div}(\omega_1, \omega_2) = \frac{1}{2} \int_0^1 dy\, y^{-2}(1 + 2\omega_1\omega_2 y)$$

On the UHP, closed string vertex operators are located at i and iy



$$f_{div}(\omega_1, \omega_2) = \frac{1}{2} \int_0^1 dy\, y^{-2}(1 + 2\omega_1\omega_2 y)$$

Feynman diagrams:

(a)

(b)

Thick lines: Closed strings

Thin lines: open strings

The open-closed interaction vertices are UHP two point functions

To compute these amplitudes we need the two point open-closed interaction term for off-shell external states.

Need to choose a 'local coordinate' w_i at the location of each vertex operator.

If the UHP coordinate z is related to w as $z = f(w)$ then we insert the vertex operator $f \circ V(w)$ – conformal transform of V by f

e.g. for dimension h primeries, $f \circ V(w) = f'(w)^h V(f(w))$

Since only the open strings are off-shell, we need a choice of local coordinates at the open string puncture.

C-O interaction vertex

Put C at i, O at 0

Choose local coordinate at O to be

$$\mathbf{w} = \lambda\,\mathbf{z} \quad \Rightarrow \quad \mathbf{f(w) = w}/\lambda \quad \Rightarrow \quad \mathbf{f \circ V(w)} = \lambda^{-\mathbf{h_V}}\mathbf{V(z)}$$

λ: an arbitrary constant, taken to be large for convenience

$\Rightarrow$ the two point function of a closed string state C and open string state O is

$$\langle \mathbf{V_C(i) V_O(0)} \rangle_{\mathbf{UHP}}\, \lambda^{-\mathbf{h_O}}$$

We need to find the relation between y and the Schwinger parameter $q = e^{-t}$ for diagram (a).

Diagram (a) corresponds to two UHP's sewed via

$$ww' = -q \quad \Rightarrow \quad \lambda^2 zz' = -q, \quad q \equiv e^{-t}, \quad t : \text{Schwinger parameter}$$

On the sewed surface the punctures are located at

$$z = i, \qquad z' = i \quad \Rightarrow \quad z = i\, q/\lambda^2 \equiv i\, y$$

This gives $y = q/\lambda^2$.

$$\mathbf{y = q/\lambda^2}$$

$$\mathbf{0 \leq q \leq 1 \qquad \Rightarrow \quad 0 \leq y \leq 1/\lambda^2}$$

The region $1/\lambda^2 < y \leq 1$ comes from diagram (b) and gives finite result.

Analyze f_{div} using this:

$$\frac{1}{2}\int_0^1 dy\, y^{-2}(1 + 2\omega_1\omega_2 y) = \frac{1}{2}\left\{\int_0^{1/\lambda^2} + \int_{1/\lambda^2}^1\right\} dy\, y^{-2}(1 + 2\omega_1\omega_2 y)$$

$$= \frac{1}{2}\int_0^1 dq\{\lambda^2 q^{-2} + 2\omega_1\omega_2 q^{-1}\} + \frac{1}{2}\int_{1/\lambda^2}^1 dy y^{-2}(1 + 2\omega_1\omega_2 y)$$

$$\Rightarrow -\frac{1}{2}\lambda^2 + \frac{1}{2}\int_{1/\lambda^2}^1 dy y^{-2}(1 + 2\omega_1\omega_2 y) = -\frac{1}{2} + 2\,\omega_1\omega_2 \ln\lambda$$

$-\omega_1, \omega_2$ are energies of incoming and outgoing C.

For the choice of local coordinates we have made, the C-ϕ vertex vanishes.

$\Rightarrow$ no need to include ϕ exchange contribution.

Final result:

$$f_{div}(\omega_1, \omega_2) = -\frac{1}{2} + 2\omega_1\omega_2 \ln\lambda \equiv A_f + B_f\omega_1\omega_2$$

$$A_f = -\frac{1}{2}, \qquad B_f = \ln\lambda^2$$

Note: If we had chosen a different local coordinate for the C-O vertex, the result will be different

– compensated by ϕ exchange diagram for A_f.

For B_f some part may also cancel against contribution to $2\,B_g$.

We now turn to the divergent part of the annulus one point function:

– four types of contributions

1. $g_{feynman}$ from the Feynman diagrams with zero mode contribution to the propagators removed

2. g_{ϕ} with one or more ϕ propagators

3. Correction to the relation between ψ_b^0 and y

$$\psi_b^0 = K_1 y \left[1 + g_s \int d\omega C(\omega) F(\omega) \right]$$

F: computable function

Then the path integral gets an additional Jacobian factor while changing variables from ψ_b^0 to y

$$\left[1 + g_s \int d\omega C(\omega) F(\omega) \right] \simeq \exp \left[g_s \int d\omega C(\omega) F(\omega) \right]$$

$\Rightarrow$ new contribution $g_{jac}(\omega)$

4. There is a similar correction to the $\theta - \alpha$ relation

$$\theta = K_2\, \alpha \left[1 + g_s \int d\omega C(\omega) G(\omega) \right]$$

$\Rightarrow$ $g_{gauge}(\omega)$

Results:

$$g_{feynman}(\omega) = -\frac{2}{\pi}\int_{(2\widetilde{\lambda})^{-1}}^{1} d\beta\,(1+\beta^2)^{-1}\,\widetilde{\lambda}^2\,f(\beta)^2 + \frac{\widetilde{\lambda}}{4\pi} + \frac{1}{2}\omega^2 \ln\frac{\alpha^2\widetilde{\lambda}^2}{4}$$

$$g_{\phi}(\omega) = \frac{2}{\pi}\int_{(2\widetilde{\lambda})^{-1}}^{1} d\beta\,(1+\beta^2)^{-1}\,\widetilde{\lambda}^2\,f(\beta)^2 + \frac{\widetilde{\lambda}}{4\pi}$$

f: an arbitrary function that enters the construction of C-O-O interaction vertex

$$g_{jac}(\omega) = -\frac{\widetilde{\lambda}}{\pi} - \omega^2 \ln\frac{\widetilde{\lambda}^2}{\lambda^2}$$

$$g_{gauge}(\omega) = \frac{\widetilde{\lambda}}{2\pi}$$

Total

$$g_{div}(\omega) = \frac{1}{2}\,\omega^2 \ln\frac{\lambda^2}{4} \equiv A_g + B_g\omega^2$$

$$\Rightarrow \qquad A_g = 0, \qquad B_g = \frac{1}{2}\ln\frac{\lambda^2}{4}$$

Recall $B_f = \ln\lambda^2$

$\Rightarrow B_f - 2B_g = \ln 4$ **is independent of** λ

Unitarity

Based on our understanding of D-instanton amplitudes, one can also analyze unitarity of these amplitudes

Result: The only source of unitarity violation is in the imaginary part of the exponential of the annulus partition function

– related to the tachyonic modes on the instanton

Conclusion

World-sheet theory, aided by string field theory, provides a fully systematic procedure for computing D-instanton contribution to an amplitude

Besides being of practical use, this can be used to gain deeper understanding of string theory, e.g,

– testing duality conjectures

– role of resurgence

etc.