In addition, if Wilson lines $\leftrightarrow$ ptcle worldlines, then the <u>same</u> conditions describe consistency for statistics (etc) of (possibly nonabelian) quasiparticles of a gapped phase.

— Fröhlich and others ~1988

— examples seemed to be lacking!

4. Rational conformal field theory

Assume know CFT.

An extended chiral algebra in ~~Rational CFT~~ is a CFT ~~with~~ is some holomorphic Vir primary fields $J_{\alpha}(z)$ that are bosonic and have integer spin, and their ope's generate an algebra that extends Virasoro.

("chiral algebra")

A <u>rational</u> CFT ~~has~~ is a consistent CFT ~~that~~ whose states decompose into a <u>finite</u> set of fields Φ_α that are primary for some chiral algebra, and these repns are simple. ~~semisimplicity~~ In particular, opes of these rep^ns fully decompose as direct sums of these simple objects —

chiral vertex ops

$$\boxed{\Phi_\alpha \times \Phi_\beta = \sum_\gamma N_{\alpha\beta}^{\gamma} \Phi_\gamma}$$

, $N_{\alpha\beta}^{\gamma} \geq 0$ are integers of Clebsch-Gordan series

where this symbolically describes only the multiplicities of the rep^ns. These are the fusion rules.

Let $\alpha = 0$ by $\Phi_0 = I$, identity. Also for each α, $\exists\, \alpha^*$ s.t. $N_{\alpha\alpha^*}^{0} = 1$ (dual)

Correlation functions of corres left-right symm fields $\Phi_\alpha(z,\bar{z})$ (taking "diagonal" modular inv thy for simplicity) have form (e.g. in plane)

exp^n <u>not</u> divided by Z

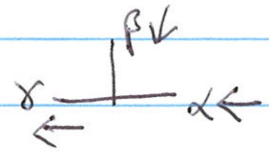
$$\langle \Phi_{\alpha_1}(z_1\bar{z}_1)\,\Phi_{\alpha_2}(z_2\bar{z}_2)\cdots\Phi_{\alpha_N}(z_N\bar{z}_N)\rangle$$

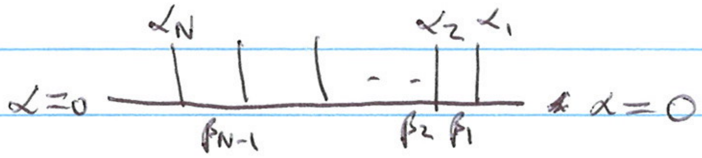
$$= \sum_a \left| \mathcal{F}_{\alpha_1\cdots\alpha_N,a}(z_1\cdots z_N)\right|^2$$

$\mathcal{F}_a$ are holomorphic locally except for possible branch pts at $z_i = z_j$ — multisheeted (actually a labels the distinct sheets at fixed $\{z_i\}$).

No of blocks (→ or range for a) (or dim of space of blocks, ie for $\mathcal{F}_a$) can be computed as

$$|0\rangle \leftrightarrow I, \text{ identity field}$$

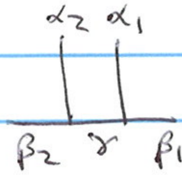
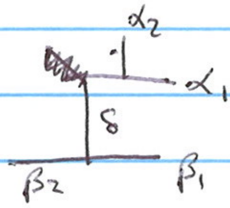
Any fusion  have $N_{\alpha\beta}^{\gamma}$ ways to do it

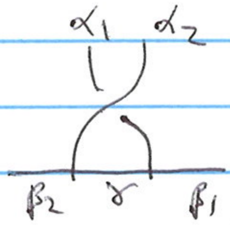
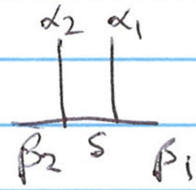


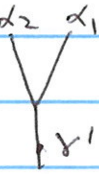
$$\# = \sum_{\beta_1\cdots\beta_{N-1}} N_{0\alpha_1}^{\beta_1}\, N_{\beta_1\alpha_2}^{\beta_2}\cdots N_{\beta_{N-1}\alpha_N}^{0}$$

(note the choices of β_i are made independently for left and right movers.)

Various matrices for changing basis, e.g.

 $= \sum_\gamma F_{\gamma\delta}$  (cf 6j symbol)

Braiding:  $= \sum_\delta B_{\gamma\delta}$ 

but in form (transform using F's)  $= B'$ 

it becomes diagonal. ~~B~~ ~~not~~ evals not well defined if $\alpha_1 \neq \alpha_2$, but those of $(B')^2$ are, so are $\alpha_1 = \alpha_2$.

F's obey "pentagon condition", F & B obey "hexagon condition" and B's alone obey another "hexagon", equiv to "Yang-Baxter" braid gp rel".

Note from (chiral) CFT ope's we know that if

$$\Phi_\alpha(z)\,\Phi_\beta(0) \sim \frac{c_{\alpha\beta}^{\ \gamma}}{z^{h_\alpha+h_\beta-h_\gamma}}\,\Phi_\gamma(0) + \ldots$$

so eigenvalues of $\left((B')^2\right)^{\gamma}_{\alpha\beta\,\alpha\beta}$ are $e^{2\pi i(h_\alpha+h_\beta-h_\gamma)}$

Also $\theta_\alpha = e^{2\pi i h_\alpha}$ is "twist" of α.

 $= \theta_\alpha$  (2π rotation)

Example (Moore + N.R. 1991)

Chiral part of a compactified scalar field:

$$\langle \varphi(z)\varphi(0)\rangle = -\ln z$$

$$\langle e^{iq\varphi(z)}\, e^{iq'\varphi(z')}\rangle = (z-z')^{qq'} \quad \text{if } q+q'=0$$

Then $e^{iq\varphi}$ has conf. wt. $q^2/2$.

Compactification with rational radius^2 means $\exists$ primary fields

$$e^{\pm i\sqrt{k}\,\varphi(z)} \qquad (k>0)$$

in chiral algebra. Weight $k/2$ should be integer, so k even.

Chiral alg: 1, $j = \frac{-i}{\sqrt{k}}\partial\varphi$, $T = \frac{1}{2}:\partial\varphi^2:$, $e^{\pm i\sqrt{k}\varphi}$

Representations: primaries local (single valued) wrt chiral alg inc $e^{\pm i\sqrt{k}\varphi}$

so

$$e^{i\sqrt{k}\varphi(z)}\, e^{iq\varphi(0)} \sim z^{\sqrt{k}q}\, e^{i(q+\sqrt{k})\varphi(0)}$$

$\Rightarrow$ require $q = m/\sqrt{k}$, $m \in \mathbb{Z}$

Also q and $q+\sqrt{k}$ in same rep.

So distinct primaries $q = 0, \frac{1}{\sqrt{k}}, \frac{2}{\sqrt{k}}, \ldots \frac{k-1}{\sqrt{k}}$

— k of them. or $\alpha = 0, \frac{1}{k}, \ldots, 1-\frac{1}{k}$ = charge in no units

Fusion rules: addition of q (mod $\sqrt{k}$). $N_{\alpha\beta}^{\gamma} = \delta_{\alpha+\beta-\gamma \,(\mathrm{mod}\, 1)}$

Correlators

given as $\prod_\ell e^{-i\varepsilon\varphi(z_\ell)}$ small charges on a grid

$$\left\langle \prod_i e^{i\sqrt{k}\varphi(z_i)} \cdot e^{-i\nu\int d^2z\, \varphi(z)} \right\rangle$$

uniform neutralizing background

$$= \prod_{i<j} (z_i - z_j)^k e^{-\frac{1}{4}\sum_i |z_i|^2} \times \text{singular phase}$$

So field $e^{i\sqrt{k}\varphi}$ in chiral alg represents ptcle (boson)

Include also

$$\left\langle \prod_j e^{i\sqrt{k}\varphi(z_j)} \prod_\ell e^{\frac{i}{\sqrt{k}}\varphi(w_\ell)} e^{-i\nu'\int d^2z'\varphi(z')} \right\rangle$$

$$= \prod_{\ell<m} (w_\ell - w_m)^{1/k} \cdot \prod_{j\ell} (z_j - w_\ell) \cdot \prod_{i<j} (z_i - z_j)^k$$

$$\times e^{-\frac{1}{4}\sum_i |z_i|^2 - \frac{1}{4k}\sum_\ell |w_\ell|^2}$$

$$= \Psi(z_1 \ldots z_N; w_1 \ldots w_n)$$

This form of $q$hole states is from Halperin 1983

(z-dep is same as before).

Redo adiabatic & statistics calculation:

First note that if $|\Psi(w)\rangle$ is holo in w & normalized, then

$$A = i\langle \Psi(w) | \frac{\partial}{\partial w} \Psi(w) \rangle$$

$$= i\frac{\partial}{\partial w} \cdot 1$$

$$= 0.$$

Ψ is norm'd up to a const for well-sep w's, by a plasma argument

In fact Ψ is not holo, but only because of the $e^{-\frac{1}{4}\sum_h |w_h|^2}$, each w_h — no problem.
This reproduces the Aharonov–Bohm phase due to background density of ptcles that appears even for 1 qhole.
& Drop this throughout.

Then for 1 qh exchange with another, we have $\underline{\underline{A=0}}$. But on exchange we see

$$(w_1-w_2)^{1/k} \to (w_2-w_1)^{1/k}$$

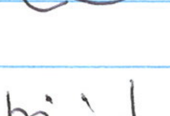
$$= e^{i\pi/k}(w_1-w_2)^{1/k}$$

— monodromy of ~~wfns~~ conformal blocks under analytic continuation.

So net effect of exchange (holonomy) equals monodromy only.

Of course this is the CFT that corresponds to the U(1) CS thy we looked at. Note that a Wilson line of charge ± 1,

$$e^{\pm ik\oint a_\mu dl_\mu}$$

is invisible to other Wilson lines — 

gives phase $= 1$. So view as trivial — charges defined mod 1 as here.

More states/phases

Use more general CFT, ~~like~~ product of φ (charge sector) with something else. Simplest non-trivial unitary CFT is Ising CFT, a massless Majorana fermi field:

$$\psi(z)\psi(0) \sim \frac{1}{z} \quad , \quad h_\psi = 1/2$$

Also have spin field σ, $\qquad h_\sigma = \frac{1}{16}$

$$\psi(z)\sigma(0) \sim \frac{1}{z^{1/2}}\sigma(0)$$

Also identity I.

Fusion rules:

$$I \times \psi = \psi, \quad I \times \sigma = \sigma, \quad I \times I = I$$
$$\psi \times \psi = I, \quad \psi \times \sigma = \sigma$$
$$\sigma \times \sigma = I + \psi \quad - \sigma \text{ is "nonabelian"}$$

Consider as wavefn

$$\Psi(z_1 \ldots z_N) = \Big\langle \prod_j e^{i\sqrt{Q}\varphi(z_j)} e^{-i\nu^{-1}\int d^2z'\varphi(z')} \Big\rangle_\varphi \Big\langle \prod_j \psi(z_j) \Big\rangle_\psi$$
$$= \text{Pf}\left(\frac{1}{z_j - z_k}\right) \cdot \prod_{j<k}(z_j - z_k)^Q e^{-\frac{1}{4}\sum_j |z_j|^2}$$

Moore + NR (1991) state again

Here Q = even $(Q>0)$ for fermions (electrons), odd for bosons and indeed

$$\psi(z) e^{i\sqrt{Q}\varphi(z)} \quad \text{has total weight } \frac{1}{2} + \frac{Q}{2}$$

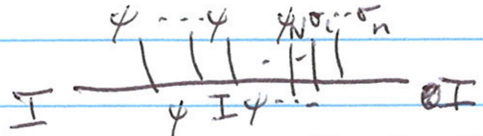
which is integer for Q odd, e.g. $Q=1$ $(\nu=1)$

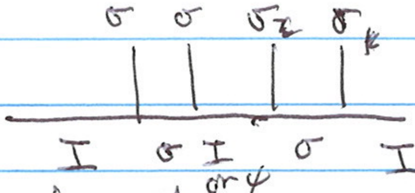
In fact for $Q=1$ $\langle \psi(z)\, e^{\pm i\varphi(z)}$, with $j = -i\partial\varphi$, generate $SU(2)$ current algebra at level $k=2$. The primaries are $s=0$ ($\leftrightarrow I$), $s=\frac{1}{2}$ ($\leftrightarrow \sigma(z)\, e^{\frac{i}{\sqrt{2}}\varphi(z)}$), $s=1$ ($\leftrightarrow e^{\pm i\varphi(z)}, \psi(z)$)

Anyway, write down possible qhole states

$$\Psi(z_1 \ldots z_N; w_1 \ldots w_n) = \Big\langle \prod_j \psi(z_j)\, e^{i\sqrt{Q}\varphi(z_j)} \cdot \prod_k \sigma(w_k)\, e^{\frac{i}{\sqrt{2}}\varphi(w_k)} \times e^{-i\nu^{-1}\int d^2z'\, \varphi(z')}\Big\rangle$$

but now there are $2^{n/2-1}$ conf blocks (fixed N)





ie $2 = 2^{4/2-1}$

— Same as in p+ip Majorana zero mode picture!

Note $e^{\frac{i}{\sqrt{2}}\varphi(w)}$ with $\sigma(w)$ is necessary ~~because~~ to get single-valuedness of "ptcle" field $\psi(z)\, e^{i\varphi(z)}$ around w.

$n=2$ we get

$$(w_1-w_2)^{-\frac{1}{8}+\frac{1}{4Q}}\, \mathrm{Pf}\left(\frac{(z_i-w_1)(z_j-w_2)+(z_i-w_2)(z_j-w_1)}{z_i-z_j}\right)$$

$$\times \prod_{i<j}(z_i-z_j)^Q\, e^{-\frac{1}{4}\sum|z_i|^2 - \frac{1}{8Q}(|w_1|^2+|w_2|^2)}$$

Those effectively carry $\frac{1}{2}$ flux qu each, hence charge $\frac{1}{2Q}$ (if state screens in charge sector).

Statistics: our conjecture was that for the trial wfns given by conformal blocks Berry connection vanishes, and holonomy equals monodromy (adiabatic transport) (analytic continuation) as for case of Laughlin states.

Not proved then, but later (NR 2009, Bonderson et al, 2011) for some cases, given that screening holds in some plasma.

<u>Read-Rezayi series</u> w. E. Režayi 1999.

Recall MR with $Q=1$ exact for 3-body δ-fn int

$$H_{int} = \underset{\substack{\uparrow \\ \text{projector}}}{P_{LLL}} \sum_{i<j<k} \delta(z_i - z_j)\,\delta(z_j - z_k)\,P_{LLL}$$

while Laughlin $Q=2$ for $H_{int} = P_{LLL} \sum_{i<j} \delta(z_i - z_j)\,P_{LLL}$.

Try for $k+1$-body δ-fn interaction.
Soln uses $\mathbb{Z}_k$ parafermions: $\ell = 0, \dots, k-1$
($\psi_0 = I$)

$$\psi_\ell(z)\,\psi_{\ell'}(z') \sim \frac{c_{\ell\ell'}}{(z-z')^{\Delta_\ell + \Delta_{\ell'} - \Delta_{\ell+\ell'}}}\,\psi_{\ell+\ell'} \qquad (\ell+\ell' < k)$$

$$\psi_{k-\ell} = \psi_{k-\ell}^{*} \quad \text{to some other}$$

$$\Delta_\ell = \frac{\ell(k-\ell)}{k}, \quad c = \frac{2(k-1)}{k+2}$$

For $k=2$, $\psi_1 = \psi$, Majorana fermion.

$$\Psi(z_1 \ldots z_N) = \langle \psi_1(z_1) \ldots \psi_1(z_N) \rangle \prod_{i<j} (z_i - z_j)^{M + \cancel{2}/k} e^{-\frac{1}{4}\sum |z_i|^2}$$

exp chosen to make it single valued

$M \geq 0$ integer. M even $\leftrightarrow$ boson
M odd $\leftrightarrow$ fermion

$M=0$ solves int above. $\psi_1(z) e^{\pm \frac{i\sqrt{2}}{\sqrt{k}} \varphi(z)}$ has conf weight 1 and generates $SU(2)$ level k current alg.

$$\nu = \frac{k}{Mk+2} = \frac{1}{M + 2/k}$$

Ope's imply for $M=0$, Ψ vanishes if any $k+1$ z_i's are equal
Wavefns.

Write qhole states using σ_1, $\sigma_\ell = \Phi_\ell^\ell$, all zero energy for special Ham.

These exhaust <u>all</u> zero energy states for $H_{int,k}$

Massive parts of CFT N. R. 2009
and issue about <u>use of</u> non-unitary CFT

Edge theory Wen 1990

TQC. Kitaev ~1997