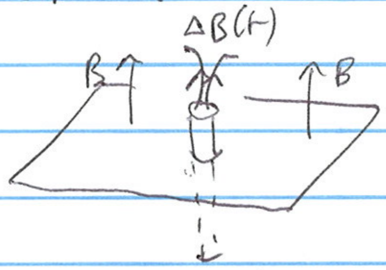
3. Topological phases & CFT

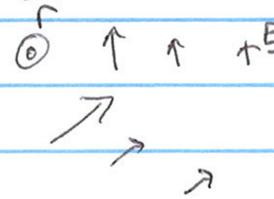
Why are some properties invariant throughout a topological (i.e. gapped) phase, and which properties?

Laughlin fluxtube argument:

Take an incompressible QH fluid at filling ν.
Insert a solenoid, adiabatically increase flux through small center of solenoid



$\frac{\partial B}{\partial t} \neq 0 \Rightarrow$ circulating $\underline{E}$ field, $\sim \frac{1}{r}$
$(= -\dot{\underline{A}})$



$\Rightarrow$ radial current by Hall effect

After increasing flux by $\Phi_0 = \frac{hc}{e}$, have accumulated charge by ν (in el. units)
Integer amount of flux can be gauged away
– essentially have reached another state of original system.

$\Rightarrow$ Incomp fluid at ν has excitation of charge ν

$\nu = P/Q$, P, Q coprime then means by adding ptcles/holes of charge ± 1, we can achieve charges of m/Q, any $m \in \mathbb{Z}$.

Or $\dfrac{\pm 1}{Q}$ by int. no of flux & int. no of ptcles.

Used only long-wavelength quantized σ_{xy} to get this

Variation: we can adiabatically change Ham at same time as flux. Suppose we change H_{int} adiabatically, and no flux. ~~no the~~ Then no Hall current. Charge of an excitation at origin is unchanged. That is, it is indep of H_{int} as long as bulk gap is maintained.

(using H_{int} trans. inv., not nece. rot. inv.)

Effective field theory on $d+1$-dim manifold M

- low energy, long wavelength compared with bulk gap Δ, corres length scale ξ.

Have some fields for deg. of f., also ext fields such as e.m. to measure responses.

Used RG point of view: keep lowest deriv local terms in the fields.

We want asymptotia, set $\xi = \frac{1}{\Delta} = 0$.

Eff Action can contain no scales, ~~Hence~~ $\rightarrow$ scale invariant Cannot contain a metric for spacetime or space, $\rightarrow$ diffeomorphism inv.
because of absence of any gapless excitations
(ie $\Rightarrow$ no long-range ~~spin~~ correlated (connected) correlations of local operators.)

Then only remaining properties must be topological (diffeomorphism) invariants. $\rightarrow$ topological qu. field thy

Example for QH.

$d=2$, manifold M.

Ext. $U(1)$ gauge field A_μ, gauge tran $A_\mu \to A_\mu + \partial_\mu \phi$

$\phi \equiv \phi + 2\pi$ (charge qu in integers)

Dynamical $U(1)$ gauge field a_μ, same $a_\mu \to a_\mu + \partial_\mu \phi$

$\phi \equiv \phi + 2\pi$.

– these are choices of normn.

$$Z[A] = \int \mathcal{D}a \, e^{iS[a,A]}$$

"U(1) Chern-Simons" term

$$S[a,A] = -\frac{k}{4\pi} \int d^3x \, \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

$$+ \frac{1}{2\pi} \int d^3x \, \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda$$

Note e.g. Maxwell term, $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$

$$\frac{1}{e^2} \int d^3x \sqrt{g} \, g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}$$

involves both metric and coupling e^2, and can be dropped
dimensionful $\frac{1}{e^2} \sim$ length

Now

$$\frac{\delta S}{\delta A_\mu} = J^\mu \qquad \text{no. current}$$

$$= \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$$

so $\partial_\mu J^\mu = 0$ without use of eom

Thus $\nabla \times \underline{a} = 2\pi \langle n \rangle$ as last time (rescaled)

Further $$\int_\Sigma d^2 x \, n_\mu J^\mu = \frac{1}{2\pi} \int_\Sigma d^2 x \, n_\mu \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$$

compact surface

is flux of $\nabla \times a$ through Σ, which is quantized $\in \mathbb{Z}$.

— fixes norm^n of $A da$ term.

E.m. gauge inv $A_\mu \to A_\mu + \partial_\mu \phi$ holds, as $\partial_\mu J^\mu = 0$ (up to surface terms)

E.o.m. for a_μ is

$$-\frac{k}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda + \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = 0$$

ie ~~...~~ if $\nabla \times \underline{A} \neq 0$,

$$\bar{n} = \frac{1}{2\pi k} \nabla \times \underline{A}$$

$$J^i = \frac{1}{2\pi k} E_i$$

ie $\boxed{\nu = \frac{1}{k}, \quad \sigma_{xy} = \frac{\nu}{2\pi}}$ as in QH. [Of course, $\partial_\mu A_\nu - \partial_\nu A_\mu$ is not diffeo. inv. if non-zero.]

Gauge inv under $a_\mu \to a_\mu + \partial_\mu \phi$:

$$S \to S - \frac{k}{4\pi} \int d^3x \; \varepsilon^{\mu\nu\lambda} \partial_\mu \phi \, \partial_\nu a_\lambda$$

int by parts makes it vanish if ϕ single valued.

But if $\pi_1(M) \neq 0$, not so. E.g. $M = \underset{\uparrow \text{ space}}{S^2} \times \underset{\uparrow \text{ time}}{S^1}$.

and ϕ ind of $x^i \in S^2$, winds by 2π ~~m~~ around S^1.

Then

$$\Delta S = -\frac{k\,\cancel{m}}{4\pi} 2\pi \int_{\Sigma} d^2x \; \varepsilon^{ij} \partial_i a_j$$

Σ ← any section

$$= -2\pi \frac{mk}{2}, \qquad m = \text{flux of } a.$$

Invariance of e^{iS} $\Rightarrow$ $\boxed{k \in 2\mathbb{Z}}$, k is even integer.

This agrees with Laughlin states for bosons
$Q = k =$ even.

So only free parameter got fixed to discrete values.

Note: suppose we change microscopic Ham, slightly. We may "renormalize" parameters in general. But only terms that can be generated/changed in eff action are integral of local gauge-inv operators. CS term is not such.
That is why its coeff was quantized, also (and hence cannot be renormed ✓)

Wilson lines (loops):

insert $e^{-iq\oint a_\mu dl^\mu}$ in func. integral.
(no metric required!). These are gauge-inv if $q \in \mathbb{Z}$.
("a-charge").

E.g. for line in time direction, at $\underline{x}=0$, e.o.m. becomes

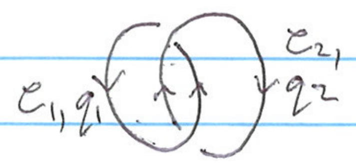
$$-\frac{k}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu\left(a_\lambda - \frac{1}{k}A_\lambda\right) - q\,\delta^\mu_0\,\delta(\underline{x}) = 0$$

For $\nabla \times A$ uniform (or zero) as before, now have excess

$$\Delta J^0 = -\frac{q}{k}\,\delta(\underline{x})$$

— fractional charge.

For given M, exp^{ns} of Wilson line(s) are top. invariants of knots & links. (if $\partial_\mu A_\nu - \partial_\nu A_\mu = 0$).

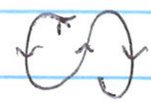


linking no $= 1$

E.g. $\left\langle e^{iq_1\oint_{c_1} a_\mu dl^\mu}\, e^{iq_2\oint_{c_2} a_\mu dl^{\mu'}}\right\rangle$

$$= e^{\frac{2\pi i}{k}q_1 q_2} \quad (\times \text{ self-linking part})$$

— fractional statistics.



$= e^{\frac{i\pi}{k}q_i^2} \times$ (self linking)

Exercise for grad. students: find the number of ground states if space is a 2-torus.

More general – several gauge fields a^α_μ.

$$S = \sum_{\alpha\beta} \frac{k_{\alpha\beta}}{4\pi} \int d^3x \, \varepsilon^{\mu\nu\lambda} a^\alpha_\mu \partial_\nu a^\beta_\lambda + \sum_\alpha \frac{t_\alpha}{2\pi} \int d^3x \, \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a^\alpha_\lambda$$

gauge-inv $\rightarrow$ $k_{\alpha\beta}$ describes <u>even integral lattice Λ</u>

$\nu = P/Q$, $Q = \det k$.

~~QH~~ Corresponds to general "Abelian" QH states.

(fractional charges lie in dual lattice Λ^*)

To do for fermions, require additional "spin" structure.

(Moore + Belov)

<u>Non abelian CS thy</u> (Witten 1989)

$$S = \frac{k}{\cancel{8}\pi} \int d^3x \, \varepsilon^{\mu\nu\lambda} \mathrm{Tr} \Big[\cancel{2} a_\mu (\partial_\nu a_\lambda - \partial_\lambda a_\nu) + \frac{2}{3} a_\mu [a_\nu, a_\lambda] \Big]$$

For simply-connected gauge group G, now get $k \in \mathbb{Z}$ because of $\pi_3(G) = \mathbb{Z}$ for compact simple G.

(for some norm^n of Tr)

Now $W_R(\mathcal{C}) = \mathrm{Tr}_R \, P e^{i\int_{\mathcal{C}} a_\mu dl^\mu}$

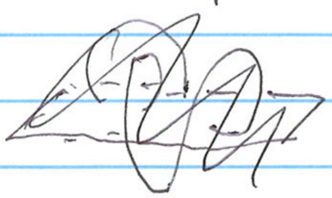
Tr_R: rep^n R of G; P: path ordering

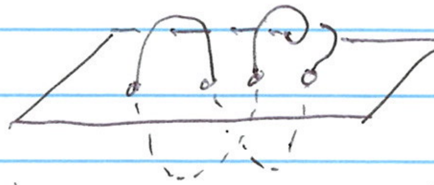
Sol^n is not so direct but uses 2D rational conformal field thy (RCFT), e.g. WZW thy for $G = SU(N)$.
— rather tricky. (see Elitzur et al, Nucl Phys B 1989)

But if we only want to compute exp^n values of links, computation is simple.
~~Becau~~ If R is the defining rep^n of $SU(N)$, the

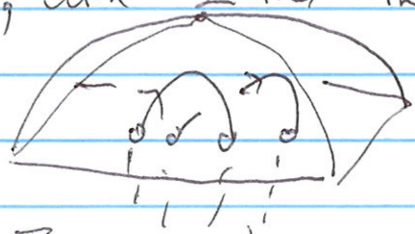
$$R \otimes R = (\cancel{\text{adjoint}}\ \text{antisymm}) \oplus (\text{symmetric})\ \cancel{\text{tensor}})\ \text{tensors},$$

and total $SU(N)$ rep^n on a time slice is trivial

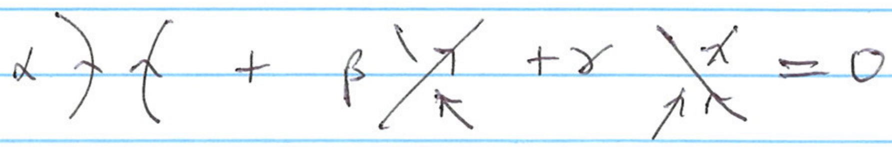




so only care about inv tensors. Then isolating a 3-ball, with 2 lines in, 2 out



There are 3 ways to connect but only 2 distinct.
We get a rel^n (among three states of CSW +Wilson line)



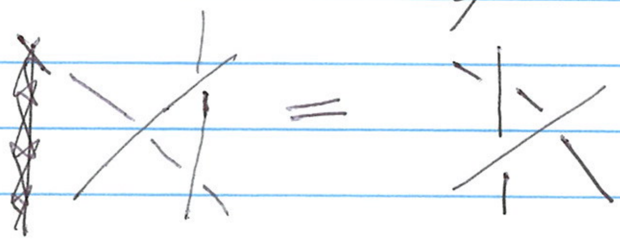
(skein rel^n) Kauffman

~~&~~ this suffices to compute recursively the Jones-HOMFLY knot poly

[It is well-defined ind of presentation as 2D picture provided α, β, γ are well chosen. Exactly these values arise from CSW.]

These are related to braiding matrices from $SU(N)_k$ CFT.

In all cases we need "consistency condition" like



(any direction of each arrow)

and others, which put polynomial conditions on the parameters.

One can construct the entire structure giving exp^n of links & 3-manifold invariants out of detailed specification of such ingredients, without any Feynman path integral or QFT.

— Reshitikhin + Turaev, ~1990, (coming originally from quantum groups, but then more abstractly)

— structure is called a modular tensor category.

(Funar, next week)

Very similar structures were isolated in 2D RCFT (operator formalism) by Moore + Seiberg ~1989 / G. Segal 1989

— "modular invariance" of correlators.

I.e. well-definition of all corr. fns when built out of certain units, "chiral vertex operators"