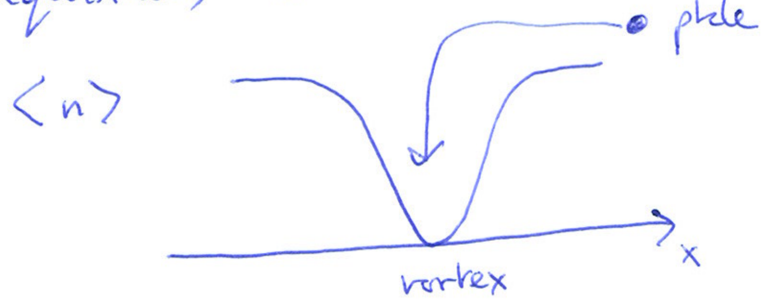
Girvin + MacDonald 1987
N.R. 1989
Zhang et al "
Jain "

Composite particles

Lack of kinetic energy makes it easy for ptcle (e.g. electron) and vortex (quasihole) to bind:



Sim a ptcle and Q vortices.

Adiabatic calculation only used average density of fluid, not details of its correlations. So we find at filling ν:

- ptcle + Q vortices experiences net effective eB of

  $$1 - Q\nu \quad \text{in uniform state}$$

- Q vortices exchanged with Q gives statistical phase $e^{i\pi Q^2 \nu}$.

  So ptcle + Q vortices has stats $e^{i\theta}$, where

$$\frac{\theta}{\pi} \equiv \begin{cases} (1 + Q^2\nu) & (\text{mod } 2) \quad (\text{fermion}) \\ Q^2\nu & (\text{mod } 2) \quad (\text{boson}) \end{cases}$$

Hence at $\nu = \frac{1}{Q}$

1) net eB is zero; states for composite can be lin. combs of plane waves, wavevector $\underline{k}$

2) composites are bosons if Q odd, fermions if Q is even, ~~incase~~ if particles are fermions (electrons); reverse if ptcles are bosons

Expect some effective kinetic energy $\propto \underline{k}^2$ at small $\underline{k}$ (details omitted)

[Sim. for P ptcles, Q vortices, $\nu = \frac{P}{Q}$.]

Now suppose we build ground state by such composites/bound states one by one, with small $\underline{k}$.

Composite bosons

As bosons, can place all in $\underline{k}=0$ — Bose-Einstein condensate.

If $\psi^+(z)$ = creation op for ptcle at z in LLL,
and $U(z)^Q = \prod_i (z_i - z)^Q e^{-\frac{1}{4}|z|^2}$ (in first quantization)
— creates Q qholes at z

then $b^+_{\underline{k}=0} = \int d^2z \; \psi^+(z) U(z)^Q$ is "$\underline{k}=0$" comp. boson creation

$\left(b^+_{\underline{k}=0}\right)^N |0\rangle$ is BEC, and is exactly the Laughlin state
(N.R. 1989)

zero ptcles

— long-range off-diagonal order in $\psi^+(z) U(z)^Q$ — order parameter

— couples to gauge potential ~~with~~ $\underline{a} - \underline{A}$, $\nabla \times (\underline{a} - \underline{A}) = 2\pi Q \langle n \rangle - 1$

Quasiparticles are vortices in order parameter, flux/vorticity quantization ($\Leftarrow$ finite energy) implies they carry charge (number) in multiples of

$$\pm 1/Q$$

(Girvin+MacDonald 1987)

and have fractional statistics — Chern-Simons-Landau-Ginsburg thy
— non-zero energy $\Rightarrow$ Meissner effect for composite, incompressible

Composite fermions

e.g. if ptcles are fermions (electrons), at $\nu = \frac{1}{Q}$, Q even.

Two possibilities:

1) comp Fermions form Fermi sea, $|\underline{k}| < k_F$ occupied:
   - absence of a BEC implies there is no Meissner effect (for $\underline{a} - \underline{A}$)
   state is compressible + gapless (Halperin, Lee, Read 1993)

2) If comp. fermions form Cooper pairs, pairs are bosons, can condense à la BCS (Moore + NR 1991)
   - state is incompressible once again

As we treated spin as absent or polarized, pairing must be odd angular momentum, e.g. p-wave.

Possible excitations will be:

a) vortices with charge $\pm \frac{1}{2Q}$, due to pairing;

b) fermions from breaking pairs — BCS quasiparticles

BCS theory of $p \pm ip$ paired states

(NR + Green 2000)
— completed at IHP in 1999!

$\{ c_{\underline{k}}, c_{\underline{k}'}^{+} \} = \delta_{\underline{k},\underline{k}'}$

Spinless fermions in 2D, zero magnetic field.
Solve effective mean-field Hamiltonian

$$K_{eff} = \sum_{\underline{k}} \left[\xi_{\underline{k}} c_{\underline{k}}^{+} c_{\underline{k}} + \tfrac{1}{2} \left(\bar{\Delta}_{\underline{k}} c_{-\underline{k}} c_{\underline{k}} + \Delta_{\underline{k}} c_{\underline{k}}^{+} c_{-\underline{k}}^{+} \right) \right]$$

$\xi_{\underline{k}} = \frac{\hbar^2 k^2}{2m^*} - \mu$, $\Delta_{\underline{k}}$ = gap function. Assume here $\Delta_{\underline{k}}$ is ang. mom $l=-1$,

$$\Delta_{\underline{k}} \simeq \hat{\Delta} (k_x - i k_y) \quad \text{at small } k \quad (\text{"}p-ip\text{" form})$$

$(\hat{\Delta} = \text{const})$

Ground state has form $\quad |\Omega\rangle = \prod_{\underline{k}}' \left(u_{\underline{k}} + v_{\underline{k}} c_{\underline{k}}^{+} c_{-\underline{k}}^{+} \right) |0\rangle$

each $(\underline{k}, -\underline{k})$ once

$|u_{\underline{k}}|^2 + |v_{\underline{k}}|^2 = 1$, all $\underline{k}$.

Bogoliubov transf^{n}: $\alpha_{\underline{k}} = u_{\underline{k}} c_{\underline{k}} - v_{\underline{k}} c_{-\underline{k}}^{+}$, $\alpha_{\underline{k}}^{+} = \bar{u}_{\underline{k}} c_{\underline{k}}^{+} - \bar{v}_{\underline{k}} c_{-\underline{k}}$,
and $\alpha_{\underline{k}} |\Omega\rangle = 0$ all $\underline{k}$; Ham becomes

$$K_{eff} = \sum_{\underline{k}} E_{\underline{k}} \alpha_{\underline{k}}^{+} \alpha_{\underline{k}} + \text{const} \qquad (E_{\underline{k}} \geq 0)$$

Conditions give

form of Bogoliubov–de Gennes eq

$$\boxed{\begin{aligned} E_{\underline{k}} u_{\underline{k}} &= \xi_{\underline{k}} u_{\underline{k}} - \bar{\Delta}_{\underline{k}} v_{\underline{k}} \\ E_{\underline{k}} v_{\underline{k}} &= -\xi_{\underline{k}} v_{\underline{k}} - \Delta_{\underline{k}} u_{\underline{k}} \end{aligned}}$$

$$\Rightarrow E_{\underline{k}} = \sqrt{\xi_{\underline{k}}^2 + |\Delta_{\underline{k}}|^2}, \quad |u_{\underline{k}}|^2 = \tfrac{1}{2}\left(1 + \frac{\xi_{\underline{k}}}{E_{\underline{k}}}\right).$$

$$v_{\underline{k}} / u_{\underline{k}} = - \frac{(E_{\underline{k}} - \xi_{\underline{k}})}{\Delta_{\underline{k}}},$$

Ground state

$$|\Omega\rangle = \prod_{\underline{k}} |u_{\underline{k}}|^{1/2} \exp\left(\tfrac{1}{2}\sum_{\underline{k}} g_{\underline{k}}\, c^+_{\underline{k}} c^+_{-\underline{k}}\right)|0\rangle, \qquad g_{\underline{k}} = \frac{v_{\underline{k}}}{u_{\underline{k}}}$$

(because $c_{\underline{k}}^{+\,2} = 0$)

In position space, N ptcle part of ground state has wavefunction

$$\langle \underline{r}_1 \cdots \underline{r}_N | \Omega\rangle = \Psi(\underline{r}_1 \ldots \underline{r}_N) = \frac{1}{2^{N/2}(N/2)!}\sum_P \operatorname{sgn} P \prod_{i=1}^{N/2} g(\underline{r}_{P(2i-1)} - \underline{r}_{P(2i)})$$

with $g(\underline{r}) = L^{-2}\sum_{\underline{k}} e^{i\underline{k}\cdot\underline{r}} g_{\underline{k}}$. Right-hand side is the Pfaffian of antisymmetric matrix $M_{ij} = g(\underline{r}_i - \underline{r}_j)$,

$$\operatorname{Pf} M = \frac{1}{2^{N/2}(N/2)!}\sum_P \operatorname{sgn} P \prod_{i=1}^{N/2} M_{P(2i-1),P(2i)} = M_{12}M_{34}M_{56}\cdots \pm \text{distinct perms} = \sqrt{\det M}$$

For $p \pm ip$, $E_{\underline{k}} \sim \sqrt{\mu^2 + \hat{\Delta}^2|\underline{k}|^2}$ at small $\underline{k}$, gap goes to zero at $\mu = 0$.

$\mu > 0$ corresponds to weak-coupling $\mu \simeq \frac{k_F^2}{2m^*}$ – weak-pairing phase

$\mu < 0$ " " strongly attractive int, $\mu \to -\infty$, – strong-pairing phase

Strong-pairing phase: as $\underline{k} \to 0$, $E_{\underline{k}} \to |\mu|$,

$$g_{\underline{k}} = v_{\underline{k}}/u_{\underline{k}} \propto k_x - ik_y, \quad \text{analytic in } k_x, k_y.$$

so $g(\underline{r}) \sim e^{-r/\xi}$ at large r, as in s-wave case

Weak-pairing phase;

$$g_{\underline{k}} = \frac{v_{\underline{k}}}{u_{\underline{k}}} \propto \frac{1}{k_x + ik_y}, \quad \boxed{g(\underline{r}) \propto \frac{1}{x+iy}} \quad \text{large } r$$

— long-range, responsible for non-trivial physics

$\underline{k}$-space topology: $\underline{k} \mapsto (u_{\underline{k}}, v_{\underline{k}})$ (up to overall phase ~~of~~ change of $u_{\underline{k}}$ and $v_{\underline{k}}$)
is a map $\mathbb{R}^2$ (or $\mathbb{T}^2$) $\longrightarrow S^2$, sphere.

With b.c. $\Delta_{\underline{k}} \to 0$ at ∞, $v_{\underline{k}} \to 0$ there. Maps $S^2 \to S^2$ classified by $\pi_2(S^2) = \mathbb{Z}$.

<u>Strong-pairing</u> phase is top. trivial (like s-wave),

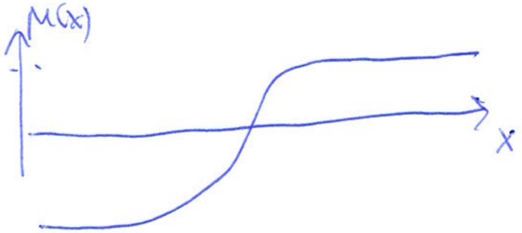
<u>weak-pairing</u> phase has winding no $= 1$.

Edge, domain wall, vortices

At an edge, potential $V(x)$ can be subsumed into $-\mu + V(x) = -\mu(x)$.

If bulk is weak-pairing, $\mu > 0$, then as $V(x) \to \infty$ outside edge, $\mu(x) \to -\infty$ there.

Generically equiv to domain wall between weak and strong



In position space, BdG eq at small $\underline{k}$ becomes

$$\boxed{\begin{aligned} Eu &= -\mu(\underline{x})\,u + i\hat{\Delta}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)v \\ Ev &= \mu(\underline{x})\,u + i\hat{\Delta}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)u \end{aligned}}$$

— a form of the Dirac eq, but with reality condition $u(\underline{r},t) = \bar{v}(\underline{r},t)$ makes it "Majorana eq^n" in 2+1

Put y-dep $\to e^{ik_y y}$. At $k_y = 0$, there is a bound state

$$-iv(x) = \boxed{u(x) \propto e^{-i\pi/4} \exp -\frac{1}{\hat{\Delta}} \int^x \mu(x)\,dx}$$

(Jackiw-Rebbi 1976)

For $k_y \neq 0$, we have $E = -\hat{\Delta} k_y$, a chiral Majorana fermion mode.

Vortices

$\hat{\Delta}(\underline{x}) \to 0$ at vortex core, so model minimal vortex in weak-pairing phase as circular domain wall.
In addition, we should include $\frac{hc}{2e}$ of magnetic flux. (as δ-fn in center).

BdG :

$$\hat{\Delta} i e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right) v = \mu u$$

$$\hat{\Delta} i e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right) u = -\mu v$$

for $E=0$,
gauge choice: $A=0$
and $u(r,\theta) = -u(r,\theta+2\pi)$
$v(r,\theta) = -v(r,\theta+2\pi)$

and has a sol^n with $v=\bar{u}$. This means the term in the field operators is a <u>Majorana zero-mode operator</u> $\gamma=\gamma^{+}$.

No such exact zero mode in strong-pairing (or in s-wave)

For n vortices, well-separated, we obtain one such operator γ_i for each $i=1\ldots n$.
$\{\gamma_i,\gamma_j\}=2\delta_{ij}$

The zero energies may be split by amounts $\sim e^{-r_{ij}/\xi}$, $\xi \sim \frac{\hat{\Delta}}{\mu}$

By forming complex ops $f_1=\gamma_1+i\gamma_2$, $f_1^{+}=\gamma_1-i\gamma_2$, $f_2=\gamma_3+i\gamma_4$, ...
with $\{f_i, f_j^{+}\}=\delta_{ij}$ $(i=1,\ldots n/2)$ we find that $2^{n/2}$ states are needed (for n even; $2^{(n+1)/2}$ for n odd).

Can argue that adiabatically exchanging the vortices gives a <u>matrix</u> operation on these degenerate states,

$$\propto \quad 1 + \gamma_i \gamma_j \qquad\qquad i \;\; j \quad \text{or} \;\ldots$$

for exchanging i and j (enclosing no others)

— they possess <u>non-Abelian statistics</u>

Back to QH states: a trial state with p-ip form for composite fermions is "Pfaffian" state

$$\boxed{\Psi_{MR} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \cdot \prod_{i<j} (z_i - z_j)^Q \, e^{-\frac{1}{4}\sum_i |z_i|^2}}$$

(Moore + NR 1991)

Note it agrees with $g(r) \propto \frac{1}{z}$ in p-ip. Here Q even for fermions, odd for bosons, and $\nu = \frac{1}{Q}$.

Physics is hybrid of p-ip and Laughlin states: quasiparticles of charge $\pm \frac{1}{2Q}$ have non-Abelian stats.

MR state is exact for a short-range <u>three-body</u> int. special Hamiltonian (Greiter, Wen, Wilczek 1991)