Quantum Hall Effect, Topological Phases of Matter and their relation with Conformal Field Theory

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Thesis:

Topological phases of matter

— phases with a gap in the energy spectrum for ~~ex~~ bulk excitations over the ground state —

if non-trivial, have one or more of the following:

1) quasiparticles with non-trivial statistics (possibly mutual) (not bosons or fermions)

2) robust ~~a~~ gapless edge excitations

3) quantized transport properties

4) ground degeneracy > 1 on compact spaces other than sphere.

Overview:

1. Quantum Hall effect - integer and fractional
   Landau levels, Laughlin wavefunctions, fractional statistics

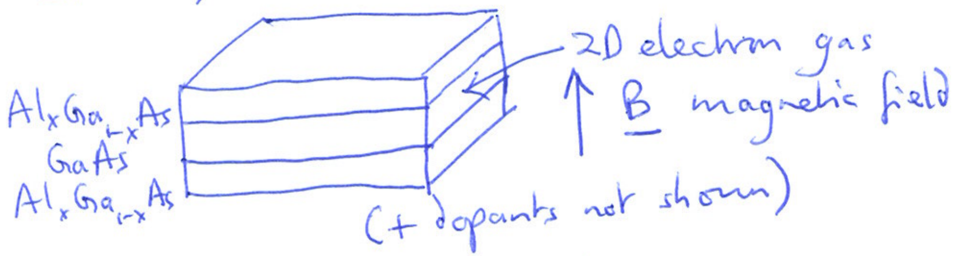
2. Composite ptcle theory: bosons and fermions
   Paired composite fermions
   p+ip BCS paired states, nonabelian statistics
   (Moore-Read state)

3. Trial wavefunctions as conformal blocks
   Nonabelian statistics
   Read-Rezayi series

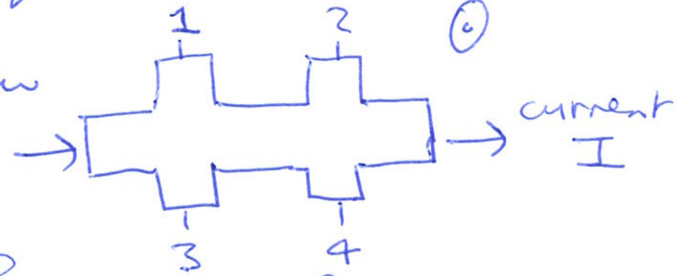
4. Edge theory
   Topological phases of matter

Intro QH effect

Typically is semiconductor heterostructures/quantum wells

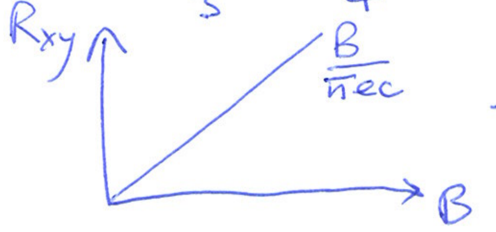


Plane view



Resistance: $R_{xx} = \dfrac{V_{12}}{I}$, $R_{xy} = \dfrac{V_{13}}{I}$

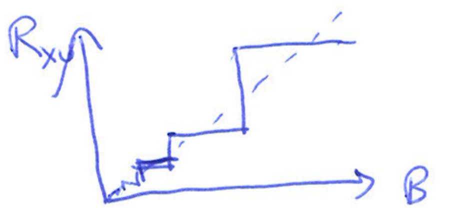
Density $\bar{n} = \dfrac{N}{A} \sim 10^{11}\,\text{cm}^{-2}$ — held fixed



classical Hall effect

Classically (for clean system) $R_{xy} = \rho_{xy} = \dfrac{B}{\bar{n}ec}$ $\left(= \dfrac{B}{\bar{n}} \dfrac{e}{hc} \cdot \dfrac{h}{e^2}\right)$

$\frac{h}{e^2} \simeq 26\,k\Omega$



Quantized Hall Effect

and $R_{xx} = 0$ on "plateaus"

Quantized Hall effect (low T, ~ few K)

$$\boxed{\sigma_{xy} = \frac{1}{\rho_{xy}} = \frac{\nu e^2}{h}}$$

$\rho_{xx} = 0$

$$\boxed{\nu = \frac{\bar{n}}{B} \frac{hc}{e}}$$

dimensionless density or filling factor

— Expt observation: σ_{xy} quantized with

$$\boxed{\nu = \text{integer or rational no.}}$$

IQHE: von Klitzing et al, 1980

FQHE: Stormer, Tsui, Gossard 1982

Landau levels

Single charged ptcle in a magnetic field $H_1 = \frac{1}{2m_e}\left(-i\hbar\nabla - \frac{e}{c}\underline{A}\right)^2$

$(\nabla \times \underline{A} = \underline{B})$ Energy eigenvalues $E_n = (n+\frac{1}{2})\hbar\omega_c$, $n = 0, 1, 2, \ldots$

Larmor/cyclotron freq: $\omega_c = \frac{eB}{m_e c} > 0$

In "symmetric gauge", $\underline{A} = \frac{1}{2}\underline{r} \times \underline{B}$, one set of eigenfunctions for $\underline{n=0}$ (lowest Landau level or LLL) ~~are~~ is

$$u_m(z) = \frac{z^m e^{-\frac{1}{4}|z|^2}}{\sqrt{2\pi 2^m m!}}$$

(we set $l_B^2 = \frac{\hbar c}{eB} = 1$)

$z = x+iy$
$m = 0, 1, 2, \ldots$

$u_m(z)$ peaked at $|z| = \sqrt{2m}$ $\Rightarrow$ no. of states in LLL per unit area $= \frac{1}{2\pi}$ //

$$\boxed{\text{I.e. one state per LL per area covered by one flux quantum } \Phi_0 = \frac{hc}{e}}$$

(Same for higher LLs, of course $u_{m,n>0}$ differs.)

Many non-interacting fermions (electrons), ground state $\Rightarrow$ occupy lowest energy levels. If occupy $n = 0, 1, \ldots, \nu - 1$ (ie ν levels), the density is

$$\bar{n} = \frac{\nu}{2\pi}$$

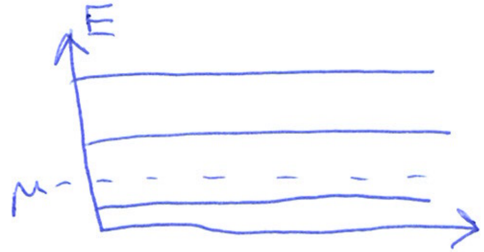
and is uniform. Here $\nu = 2\pi\bar{n} = \frac{\bar{n}}{B}\frac{hc}{e}$

I.e. ν is fractional no. of levels occupied.

For this trans. inv. Ham, can show

$$\sigma_{xy} = \frac{\nu e^2}{h}, \quad \sigma_{xx} = \rho_{xx} = 0$$

Values $\nu = 0, 1, 2, \ldots$ are <u>special</u> because creating/destroying an electron costs energy (ie change in $H - \mu N$) ≥ 0. — <u>energy gap in spectrum</u>



Really as fn of $\bar{n}$, μ jumps at ν = integer from $(\nu - \frac{1}{2})\hbar\omega_c$ to $(\nu + \frac{1}{2})\hbar\omega_c$.
So $\frac{d\mu}{d\bar{n}}$ has δ fn spikes at these values — <u>incompressibility</u>.

$$\kappa = \frac{d\bar{n}}{d\mu}$$

($\kappa = \infty$ for $\nu \neq$ integer.)

To understand quantization of observed σ_{xy}, need trans. symmetry <u>breaking</u> by <u>disorder</u>, and <u>localization</u> of electron/hole excitations about ν = integer states.
(Prof: Bellissard)

Edge states (Halperin, c. 1982)

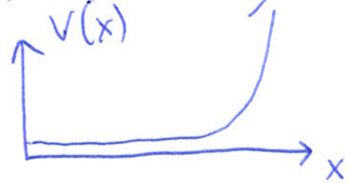
Landau gauge: $A_x = 0$, $A_y = Bx$. Trans. inv in y, $\psi(x) = e^{ik_y y} u_{k_y}(x)$

Schro eq

$$\frac{1}{2m_e}\left[-\hbar^2\frac{d^2}{dx^2} + \left(\hbar k_y - \frac{eB}{c}x\right)^2\right]u_{k_y}(x) = E\, u_{k_y}(x)$$

- SHO centered at $x = \frac{\hbar c}{eB} k_y = k_y$ for $\ell_B^2 = 1$, $\Rightarrow E = (n+\frac{1}{2})\hbar\omega_c$
- LLL is Gaussian in $x - k_y$
- can include periodic b.c. in y, then $k_y = \frac{2\pi m}{L_y}$, $m \in \mathbb{Z}$

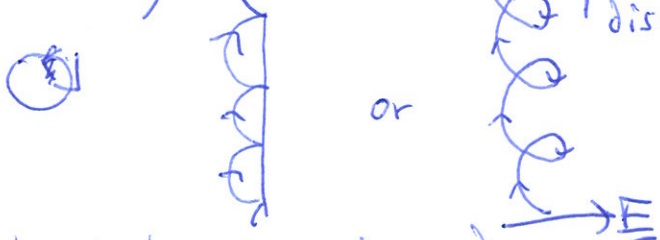
Include pot $V(x)$ for edge:



Energy eigenvalue

$v_n(k_y) = \frac{\partial E_n(k_y)}{\partial k_y} > 0$: these excitations propagate in one direction along edge – they are <u>chiral</u>. Survive in presence of disorder.

Analogs of classical "skipping orbits"



Description of quantization of σ_{xy} using chemical pots in edge modes

Interactions in many-electron system

N ptcles in LLL: general wfn (symmetric gauge)

$$\boxed{\Psi(z_1 \ldots z_N) = f(z_1 \ldots z_N)\, e^{-\frac{1}{4}\sum_j |z_j|^2}}$$

analytic in z_i, all i
(antisymmetric under exchanges for bosons (fermions))

Interaction Ham: $H_{int} = \frac{1}{2}\sum_{i\neq j} V(\underline{r}_i - \underline{r}_j)$, $V(\underline{r}) = \begin{cases} \dfrac{e^2}{|\underline{r}|} & \text{(Coulomb)} \\ \dfrac{4\pi\hbar^2 a}{m_a}\,\delta(\underline{r}) & \text{(atomic pseudopotential)} \end{cases}$

a = scattering length

Should be able to understand fractional QH effect in limit

e.g. $\underline{\underline{e^2\, \bar{n}^{1/2} \ll \hbar\omega_c}}$ — i.e. degenerate pert. th. for pert $= H_{int}$.

So for $\nu < 1$, fermions (or all ν, bosons), all ptcles are in LLL to good approx.

For $\nu > 1$ fermions, lower $[\nu]$ LLs filled, fractional filling $\nu - [\nu]$ of next one can be mapped to LLL. (Ignoring electron spin here.)

Laughlin states (R. Laughlin 1983)

Due to enormous degeneracy $\binom{N_{orb}}{N}$ of LLL fermion states, we need to guess some good trial ("variational") wavefunction, giving a low $\langle H_{int} \rangle$.

$$\boxed{\Psi_{Laughlin}(\{z_i\}) = \prod_{i<j} (z_i - z_j)^Q \, e^{-\frac{1}{4}\sum_j |z_j|^2}}$$

Q is an integer: Q is odd for fermions (Ψ antisymmetric)
Q is even for bosons (Ψ is symmetric)

In terms of orbitals $u_m(z) \propto z^m e^{-\frac{1}{4}|z|^2}$, highest occupied one has

$$\boxed{m_{max} \equiv N_\phi = Q(N-1)}, \quad \text{radius } R = \sqrt{2 m_{max}}$$

If density is uniform, it must be

$$\boxed{\nu = 2\pi \bar{n} = \lim_{N\to\infty} \frac{N}{N_\phi} = \frac{1}{Q}}$$

Filling factors for Laughlin states

Note $\Psi_{Laughlin} \to 0$ as any $z_i \to$ any z_j, should give low $\langle H_{int} \rangle$.

Prob. density for particles:

$$|\Psi_{\text{Laughlin}}|^2 = \exp Q\left[\sum_{i<j} \ln|z_i - z_j|^2 - \frac{1}{2Q}\sum_i |z_i|^2\right]$$

is Boltzmann weight for 2D Coulomb plasma of ptcles of charge +1, with uniform background charge density of density $-\frac{1}{Q}$.
Screening holds in this plasma if $Q \lesssim 70$.
Then density is uniform inside drop, with short range correlations.
So wfn is a homogeneous fluid state.

Special case $Q = 1$:

$$\prod_{i<j}(z_i - z_j) = \det\begin{pmatrix} 1 & z_1 & z_1^2 & \cdots \\ 1 & z_2 & z_2^2 & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

(Vandermonde determinant identity)

so Ψ_{Laughlin} is the wfn of filled LLL!

This is an exactly-soluble point for the plasma with continuous $\Gamma = 2Q$ also. (Jancovici)

Fractionally-charged quasiparticles (Laughlin 1983)

Laughlin quasihole wfn: $\Psi_{1\,\text{qhole}} = \prod_i (z_i - w) \cdot \Psi_{\text{Laughlin}}$ $(w \in \mathbb{C})$.

Zero probability to find a ptcle at w. How "big" is this "hole"?

- Counting argument: (e.g. for $w = 0$) All ptcles pushed outwards, $m_{max} \to m_{max} + 1$, so net ~~charge~~ number ~~deficiency~~ excess at origin is $-\frac{1}{Q}$.
- Plasma mapping argument: qhole is "impurity" of charge $\frac{1}{Q}$ in plasma, fixed at w. Plasma screens, so net $-\frac{1}{Q}$ (exactly) of particle is localized within a screening length of w.

$\to$ $\boxed{\text{Fractionally charged quasihole}}$

Make any number by $\prod_{i,k} (z_i - w_k)\, \Psi_{\text{Laughlin}}$, $w_k \in \mathbb{C}$.

Quasielectrons of charge $+\frac{1}{Q}$: use $\prod_i \left(2 \frac{\partial}{\partial z_i} - \overline{w}\right)$ (adjoint)

Due to localized number excess/deficiency, expect a well-separated qhole + qel to have non-zero energy $\sim \Delta$ charge from ground state (as $N \to \infty$)

$\Rightarrow$ $\boxed{\text{Laughlin state is an incompressible fluid}}$, like filled LLs.

— trial wfns represent a phase of matter.

Statistics of quasiparticles — bosons, fermions ... ?

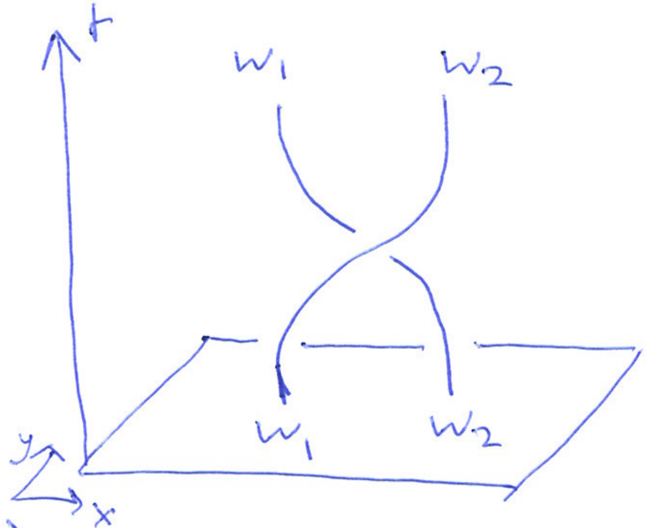
In 2D, easy to change symmetry of wavefn by multiplying it by any power of

$$\frac{\prod_{k<l}(w_k - w_l)}{\prod_{k<l}|w_k - w_l|}$$

(a "singular gauge transfn")

We need a gauge-invariant definition. One way is adiabatic transport.

Drag quasiparticles slowly



and calculate (Berry) phase change in state.

This makes sense when there is energy gap in spectrum above the n-quasiparticle state.

One quasihole: Think of a Ham which has a pot term (dep. on w) s.t. qhole at w is an eigenstate, with gap above. Subtract off energy of qhole state.

Arovas, Schrieffer, + Wilczek 1984

Vary $w = w(s)$ adiabatically, phase change $\gamma(s)$

Berry: $\dfrac{d\gamma}{ds} = i \left\langle \Psi(s) \middle| \dfrac{d\Psi}{ds}(s) \right\rangle$ if $|\Psi(s)\rangle$ normd for all s,

If at $s = S$, $|\Psi(S)\rangle = |\Psi(0)\rangle$, then $\gamma(S) - \gamma(0)$ is desired phase

$$\Psi_L^{+w} = \prod_i (z_i - w) \Psi_{L = \text{Laughlin}}, \quad \frac{d\Psi_L^{+w}}{ds} = -\frac{dw}{ds} \sum_i \frac{1}{z_i - w} \Psi_L^{+w}$$

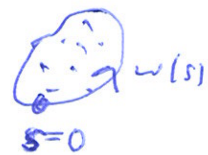
$$= -\frac{dw}{ds} \int d^2z' \, \frac{n(z')}{z' - w} \Psi_L^{+w}$$

where $n(z) = \sum_i \delta(z_i - z)$ is ptcle density

$$\text{So } \frac{d\gamma}{ds} = -i \int d^2z' \, \tfrac{1}{2}\left(\frac{1}{z' - w} \frac{dw}{ds} - \frac{1}{\bar{z}' - \bar{w}} \frac{d\bar{w}}{ds} \right) \langle n(z') \rangle^{+w}$$

$\nwarrow$ from normalization

$$\Rightarrow \gamma(S) - \gamma(0) = -i \oint dw \int d^2z' \, \tfrac{1}{2}\left(\frac{1}{z' - w} - \frac{1}{\bar{z} - \bar{w}} \right) \langle n(z) \rangle^{+w}$$



$$= -2\pi \int_{\text{interior}} d^2z' \, \langle n(z') \rangle$$

$$= -\oint d\underline{w} \cdot \underline{a}(w)/Q \quad \text{where } \nabla \times \underline{a} = 2\pi Q \langle n \rangle$$

$\Rightarrow$ $\boxed{\text{Like vortices in a (super-) fluid, quasiholes experience particle density as "Aharonov–Bohm" flux or magnetic field}}$

$\underline{a}$ = 2D vec pot representing this flux.

So

- In uniform fluid, $\langle n \rangle = \bar{n} = \frac{\nu}{2\pi}$, quasihole experiences $\frac{1}{Q}$ times the magnetic field on particles
 – consistent with fractional charge
- If path encloses another quasihole, we get in addition

$$\Delta\gamma = \frac{2\pi}{Q}$$

or similarly if we exchange two quasiholes, phase

$$e^{i\theta} = e^{i\pi/Q}$$

– <u>fractional statistics</u> if $Q > 1$ — Leinaas + Myrheim, Wilczek ~1982

(they are <u>anyons</u>) (fermions if $Q = 1$ ✓)

"Special Hamiltonian" (Haldane 1983)

an interaction Hamiltonian for which Laughlin state is eigenstate

Note for ptcles i, j, $(z_i + z_j)^M (z_i - z_j)^m e^{-\frac{1}{4}|z_i|^2 - \frac{1}{4}|z_j|^2}$ $(m, M = 0, 1, 2 ..)$

is eigenstate of relative angular momentum, $m_{ij} = m$.

Let $P_{ij}(m)$ be projector onto this subspace within LLL

Then

$$H_{int} = \sum_{i<j} \sum_{m=0}^{Q-1} V_m P_{ij}(m)$$

annihilates Laughlin state with exponent Q, and also the multiquasihole states, because $m_{ij} \geqslant Q$ in all terms in wfn, for all i, j.

This is short-range.

So for $V_m > 0$, $m \leqslant Q$, H_{int} is positive, all other eigenvalues are higher than zero. But we don't know how to calculate them, nor can we show this H_{int} has a gap above ground state in thermo limit — though this assumption is reasonable.

Edge excitations

For N large, $\prod (z_i - w)$ for $w = O(1)$ changes angular momentum by a lot. But note

$$\prod_i (z_i - w) = \sum_{m=0}^{N} (-w)^{N-m} e_m ,$$

$e_N = z_1 z_2 \ldots z_N$

$e_0 = 1, \; e_1 = \sum_i z_i, \; e_2 = \sum_{i<j} z_i z_j, \ldots$ are elementary symmetric polynomials

Note also s_m,

$$s_0 = N, \quad s_1 = \sum_i z_i, \quad s_2 = \sum_i z_i^2, \ldots \quad s_N = \sum_i z_i^N$$

(sums of powers)

Either set of $N+1$ symmetric polynomials generates (algebraically) the algebra of all symmetric polynomials in N variables

Sums of powers resemble Fourier modes of density on circle $z = Re^{i\theta}$, ie $\rho(\theta) = \sum_j e^{im\theta_j}$

The states

$$\prod_{m=1}^{N} s_m^{n_m} \cdot \Psi_{\text{Laughlin}} , \qquad n_m = 0, 1, 2, \ldots, \text{ for } m = 1, \ldots, N$$

span all zero energy states of special H_{int}. (analogous to those of $\nu = 1$ case)

For low degree $= \sum_m m n_m$, they can be viewed as density excitations of edge,