§ Setting $\varepsilon = 0, \frac{1}{2}$ (Ramond, Neveu-Schwarz) X_* with $* > 0$, $* = 0$, $* < 0$ (Joint work with T. Koga)

$$\mathfrak{g} = \mathrm{Vir}_\varepsilon = \overbrace{\bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C}^{} \oplus \overbrace{\bigoplus_{m \in \varepsilon + \mathbb{Z}} \mathbb{C}G_m}^{\text{odd}} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}C,$$

$$[\mathfrak{g}, C] = 0,$$

$$[L_m, G_n] = (\tfrac{1}{2}m - n)G_{m+n},$$

$$[G_m, G_n]_+ = 2L_{m+n} + \frac{1}{3}(m^2 - \frac{1}{4})\delta_{m+n,0}C.$$

$$\mathfrak{g}_0 = \begin{cases} \mathbb{C}L_0 \oplus \mathbb{C}C & \varepsilon = \frac{1}{2} \\ \quad \text{〃} \quad \oplus \mathbb{C}G_0 & \varepsilon = 0 \end{cases}$$

● $\varepsilon = \frac{1}{2}$ (NS) ; $(c, h) \in \mathbb{C}^2$, $\mathbb{C}_{c,h} = \mathbb{C}\mathbb{1}_{c,h} \circlearrowleft \mathfrak{g}_+ \oplus \mathfrak{g}_0$

$L_0 . \mathbb{1}_{c,h} := h\mathbb{1}_{c,h}$, $C . \mathbb{1}_{c,h} := c\mathbb{1}_{c,h}$, $\mathfrak{g}_+ . \mathbb{1}_{c,h} := 0$

$M_{\frac{1}{2}}(c,h) := \mathrm{Ind}_{\mathfrak{g}_+ \oplus \mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}_{c,h}$

● $\varepsilon = 0$ (R) ; $(c, h) \in \mathbb{C}^2$, $\mathbb{C}_{c,h} = \mathbb{C}\mathbb{1}_{c,h} \circlearrowleft \mathfrak{g}_0^{\bar{0}}$ as above,

$V_{c,h} := \mathrm{Ind}_{\mathfrak{g}_0^{\bar{0}}}^{\mathfrak{g}_0} \mathbb{C}_{c,h} \circlearrowleft \mathfrak{g}_+$ via $\mathfrak{g}_+|_{V_{c,h}} := 0$

$$V_{c,h} \twoheadrightarrow \overline{V_{c,h}} = \begin{cases} \mathbb{C}_{c,h} & h = \frac{1}{24}c, \\ V_{c,h} & h \neq \frac{1}{24}c. \end{cases}$$

$M_0(c,h) := \mathrm{Ind}_{\mathfrak{g}_0 + \mathfrak{g}_+}^{\mathfrak{g}} \overline{V_{c,h}}$, $N(c) := \mathrm{Ind}_{\mathfrak{g}_0 + \mathfrak{g}_+}^{\mathfrak{g}} V_{c, \frac{1}{24}c}$.

<u>Lemma</u> $\varphi \in \mathrm{Hom}_{\mathrm{Vir}_\varepsilon}(M_\varepsilon(c,h'), M_\varepsilon(c,h))$

i) $\varepsilon = \frac{1}{2}$, $\varphi \neq 0 \to \varphi$: inj., ⟿ 'similar to' Virasoro !

ii) $\varepsilon = 0$, $\varphi \neq 0 \to \varphi$ can be non-inj. and inj. ⟸ Let's study this case !

$\circ$ $V \supset \mathfrak{g}$ s.t. $C|_V = {}^{\exists}c\,\mathrm{id}_V$. $\quad h \in \mathbb{C}$; $\quad V_h := \{v \in V \mid L_0 v = h v\}$

$$= V_h^{\bar{0}} \oplus V_h^{\bar{1}}$$

$v_{c,h} \in M_0(c,h)$ ($\quad v_c \in N(c)$) : even h.w. vect.

<u>Example</u> $\quad (c,h) = (0,0)$. $\quad M_0(0,0) \twoheadrightarrow \mathbb{C}^{1|0}$: the irred quot.

In $M_0(0,0)$: $\quad M_0(0,0)_1 = \mathbb{C}L_{-1}v_{0,0} \oplus \mathbb{C}G_{-1}v_{0,0}$

$$= M_0(0,0)_1^{\mathfrak{g}_+}$$

In $N_0(0,0)$: $\quad \mathrm{Image}(\Pi M_0(0,0)_1 \hookrightarrow N(0)_1)$

$$= \mathbb{C}G_{-1}G_0v_0 \oplus \mathbb{C}L_{-1}G_0v_0$$

$$\subset N(0)_1^{\mathfrak{g}_+}$$

$N(0) \twoheadrightarrow M_0(0,0)$; $\boxed{L_{-1}v_0,\ G_{-1}v_0} \mapsto L_{-1}v_{0,0},\ G_{-1}v_{0,0}$

$\Uparrow$

But, $\quad G_1L_{-1}v_{0,0} = [G_1, L_{-1}]v_{0,0}$

$$= \tfrac{3}{2}G_0v_{0,0},$$ subsingular !

and $\quad G_{-1}v_{0,0} = [L_1, G_{-1}]v_{0,0}$

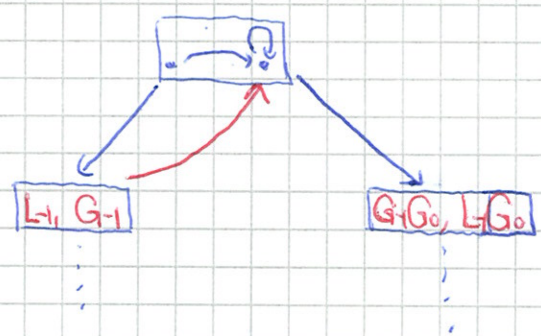
$$= \tfrac{3}{2}G_0v_{0,0}.$$

$L_1L_{-1}v_{0,0} = \cdots = 0$

$G_1G_{-1}v_{0,0} = \cdots = 0$

Hence, we have

$N(0)$



§ Sing. vect. & non-inj. maps

- Suppose $\exists\, S.v_{c,h} \in M_0(c,h)^{\mathfrak{g}_+}_{h'} \setminus \{0\}$

<u>Lemma</u> i) If $h' - \frac{1}{24}c \neq 0 \Rightarrow G_0 S v_{c,h} \in M_0(c,h)^{\mathfrak{g}_+}_{h'} \setminus \{0\}$,

ii) If $h - \frac{1}{24}c \neq 0 \Rightarrow S G_0 v_{c,h} \in M_0(c,h)^{\mathfrak{g}_+}_{h'} \setminus \{0\}$. $\square$

$\rightsquigarrow$ sufficient to look at even sing. vect., or, $(M_0(c,h)^{\bar{0}}_{h'})^{Vir^+}$

<u>Thm.</u> i) $\dim (M_0(c,h)^{\bar{0}}_{h'})^{Vir^+} = 2$

iff $M_0(c,h') \xrightarrow{\exists} M_0(c,h) \xrightarrow{\exists} M_0(c,\frac{1}{24}c)$: non-trivial $\mathfrak{g}$-mod map.

ii) Otherwise, $\dim (M_0(c,h)^{\bar{0}}_{h'})^{\mathfrak{g}_+} \leq 1$.

When $\dim (\quad)^{Vir^+} = 1$, $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(M_0(c,h'), M_0(c,h)) \setminus \{0\}$ is injective! $\square$

In case ii), the structure of Verma modules is like in Vir! $\rightsquigarrow$ not funny!

- When i) can happen?

$p, q \in 2\mathbb{Z}_{>0}$ s.t. $(\frac{p-q}{2}, q) = 1$.

$$c^{\pm}_{p,q} := \frac{3}{2}\left(5 \mp 2\left(\frac{p}{q} + \frac{q}{p}\right)\right) = \frac{3}{2}\left(1 \mp \frac{2(p \mp q)^2}{pq}\right)$$

$$h^{\pm}_{i} := \pm \frac{1}{8} pq i^2 + \frac{1}{24} c^{\pm}_{p,q} \quad (i \in \mathbb{Z}_{\geq 0})$$

<u>Rem.</u> $c^{+}_{p,q} + c^{-}_{p,q} = \frac{3}{2} \cdot 10$

$h^{+}_{i} + h^{-}_{i} = \frac{5}{8}$ $\square$

$\rightsquigarrow$ Consider + -case !

Analysis on sing. vect.

$$S.v_{c,h} \in M_0(c,h)^{\mathfrak{g}_+}_{h+N} \qquad S = XG_0 + Y \;,\quad X, Y \in U(\mathfrak{g}_-)$$

$$X = C_X G_{-1} L_{-1}^{N-1} + \sim\sim\sim \in U(\mathfrak{g}_-)\mathfrak{g}_{\leq -2} = \mathfrak{g}_{\leq -2} U(\mathfrak{g}_-)$$

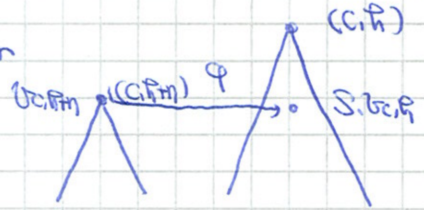
$$Y = C_Y L_{-1}^{N} + \sim\sim\sim \in$$

<u>FACT</u> (C_X, C_Y) completely determines S. $\square$

<u>SMALL EXERCISE</u> $S = XG_0 + Y$ $(X, Y \in U(\mathfrak{g}_-))$ s.t. $S.v_{c,h} \in M_0(c,h)^{Vir^+}_{h+n} \setminus \{0\}$: even

$\tilde{S} = PG_0 + Q$ $(P, Q \in$ " $)$ s.t. $\tilde{S}.v_{c,h+n} \in M_0(c,h+n)^{Vir^+}_{h+n+m} \setminus \{0\}$: even.

Consider



and compute $\varphi(\tilde{S}.v_{c,h+n})$:

$$\varphi(\tilde{S}.v_{c,h+n}) = (PG_0 + Q)(XG_0 + Y).v_{c,h}$$

$$= \Big((P[G_0, X] + PY + QX) G_0 - \underbrace{PXG_0^2 + P[G_0, Y] + QY}_{} \Big) v_{c,h}$$

leading term : $(2C_P C_X + C_P C_Y + C_Q C_X) G_{-1} L_{-1}^{m+n-1} + \sim\sim\sim$ no term $+ \underbrace{C_Q C_Y L_{-1}^{m+n}}_{}$

$$\therefore \varphi(\tilde{S}.v_{c,h+n}) = \tilde{S}S.v_{c,h} = \Big(\big(\boxed{2C_P C_X + C_P C_Y + C_Q C_X}\big) G_{-1} L_{-1}^{m+n-1} G_0 + \boxed{C_Q C_Y} L_{-1}^{m+n} \Big) v_{c,h} + \sim\sim\sim \quad \square$$

$$P = C_P G_{-1} L_{-1}^{m-1} + \cdots \;, \qquad X = C_X G_{-1} L_{-1}^{n-1} + \cdots$$

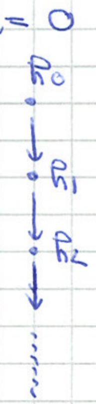
$$Q = C_Q L_{-1}^{m} + \cdots \;, \qquad Y = C_Y L_{-1}^{n} + \cdots$$

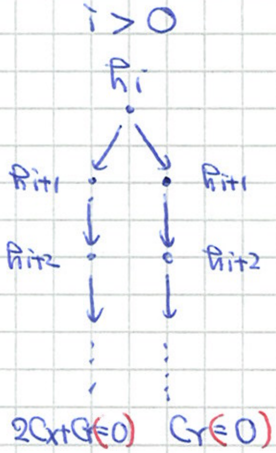
Some consequences of this "SMALL Exercise" are as follows :

§ Structure of Verma modules & non injective maps.

Fix $p, q \in 2\mathbb{Z}_{>0}$ s.t. $(\frac{p-q}{2}, q) = 1$, $c = c^{+}_{p,q}$

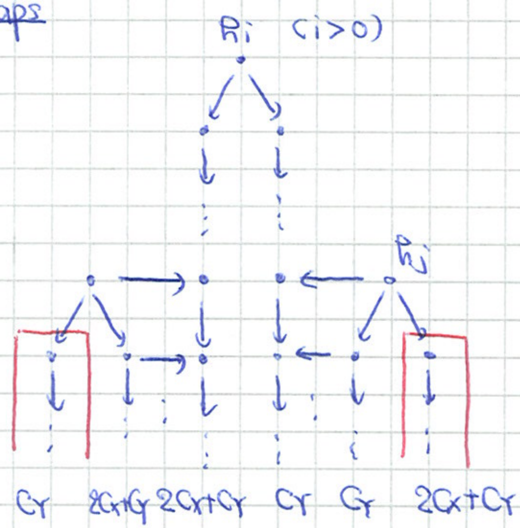
Structures of $M_0(c, h_i^{+})$

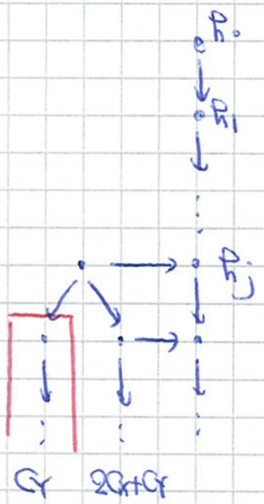


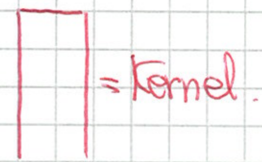


□

Non inj. maps







□

§ Structure of $N(c)$ $\quad c = c^+_{p,q}$ $(p,q \in 2\mathbb{Z}_{>0}$ s.t. $(\frac{p-q}{2}, q) = 1$.

$$\prod M_0(c,\tfrac{1}{24}c) \hookrightarrow N(c) \twoheadrightarrow M_0(c,\tfrac{1}{24}c) \qquad (\sigma : M_0(c,\tfrac{1}{24}c) \dashrightarrow N(c))$$

σ : the section def. as follows:

$X.v_{c,\frac{1}{24}c}$ $(X \in U(\mathfrak{g}_-)) \in M_0(c,\frac{1}{24}c)$,

$\sigma(X.v_{c,\frac{1}{24}c}) := X.v_c \in N(c)$.

$i \in \mathbb{Z}_{>0}$; $\quad M_0(c,\frac{1}{24}c)(i) := \mathrm{Im}\left(M_0(c,h_i^+) \hookrightarrow M_0(c,\frac{1}{24}c)\right)$.

<u>Key Lemma</u> i) $(XG_0 + Y).v_c \in N(c)^{\mathfrak{g}_+}$ with $X, Y \in U(\mathfrak{g}_-)\mathfrak{g}_-$

$\Rightarrow \quad Y = 0$!!

ii) $u \in M_0(c,\frac{1}{24}c)^{\mathfrak{g}_+}_{h_i} \cap G_0 M_0(c,\frac{1}{24}c)(i)$: non-zero $\quad (i>0)$

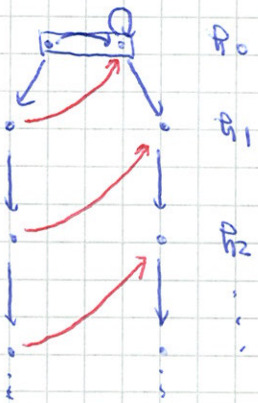
$\Rightarrow \quad \mathfrak{g}_+ . \sigma(u) \subseteq \prod M_0(c,\frac{1}{24}c)(i-1)$. $\square$

<u>Cor.</u> $V \subsetneq N(c)$: proper submod.

$\Rightarrow$ i) $V \simeq \prod M_0(c,\frac{1}{24}c)(i)$ $\quad ({}^{\exists} i > 0)$, or

ii) $\prod M_0(c,\frac{1}{24}c)(i) \hookrightarrow V \twoheadrightarrow M_0(c,\frac{1}{24}c)(j)$ $\quad ({}^{\exists} i < j)$: exacte. $\square$

<u>Rem.</u> <u>Structure of $N(c)$</u>



K. Iohara & Y. Koga

Adv. Math. 178, 1-65, (2003),

Lett. Math. Phys. 78, 89-96, (2006).

$\square$