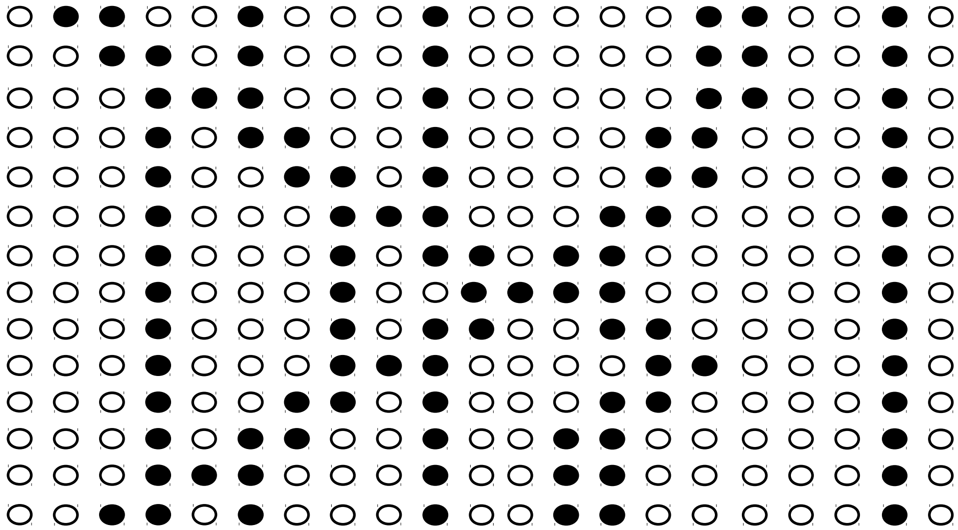
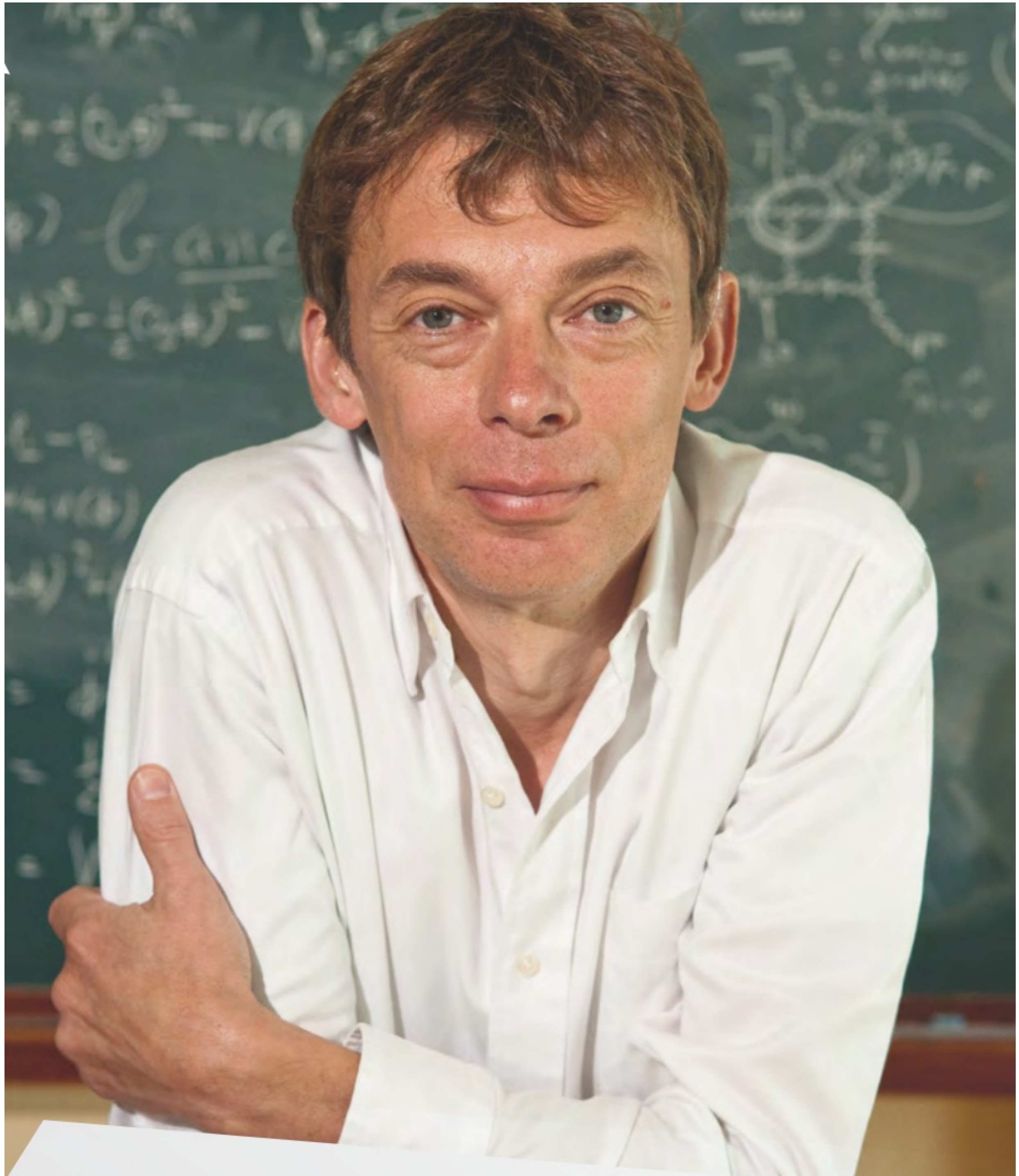
# “Hard rod” spin chains

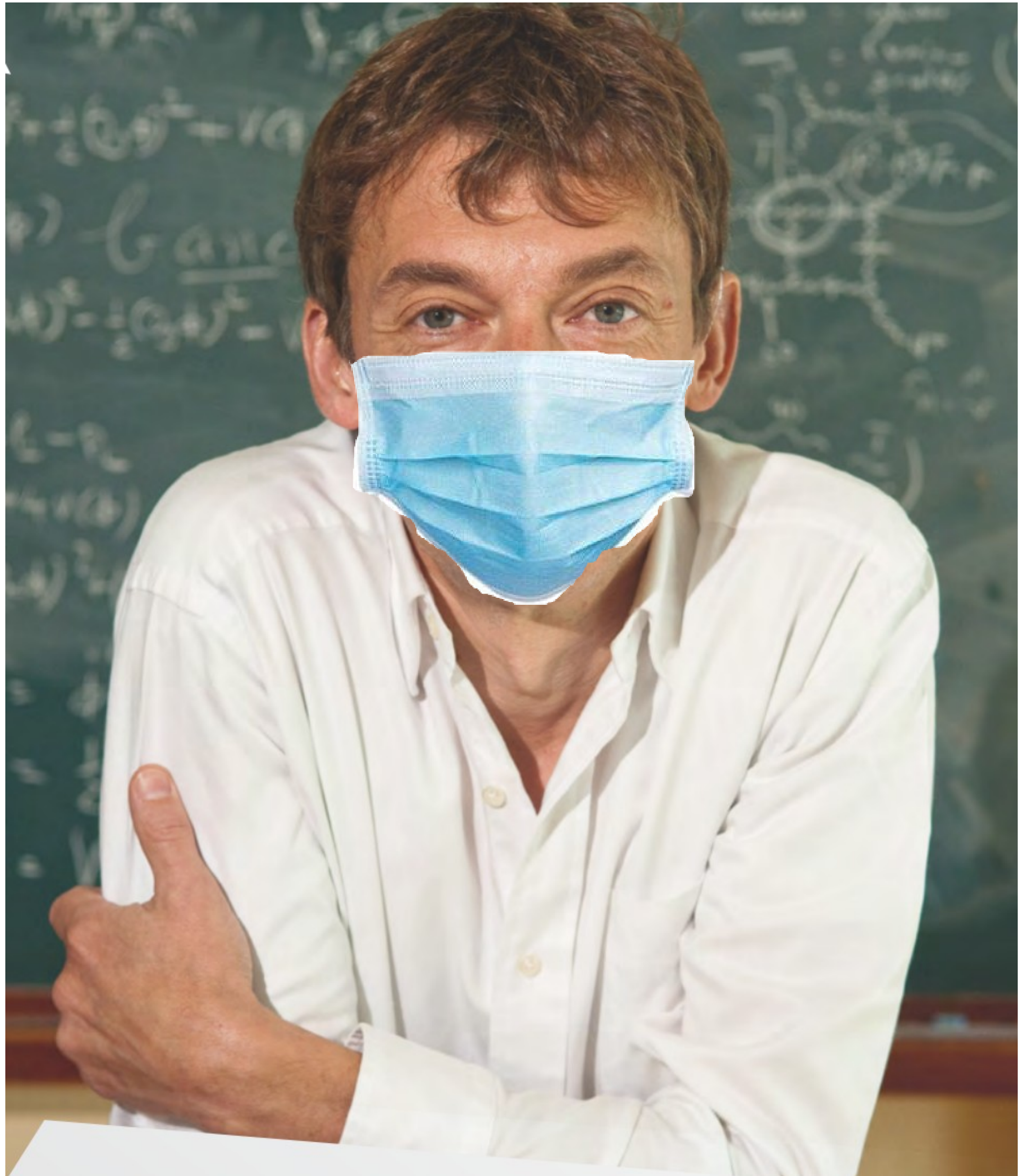
**Eric Vernier** (CNRS – LPSM Paris)

Pozsgay, Gombor, Hutsalyuk, Jiang, Pristyák, EV arXiv:2105.02252
+ work in progress



Hubert’s $60^{th}$, 23/09/2021

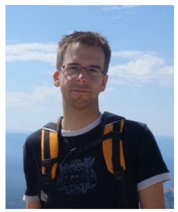


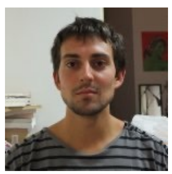


27/04/2015

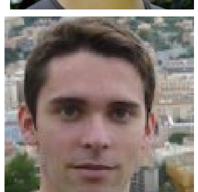
2007 

2009 

2010 

2011 

2015 

2016 

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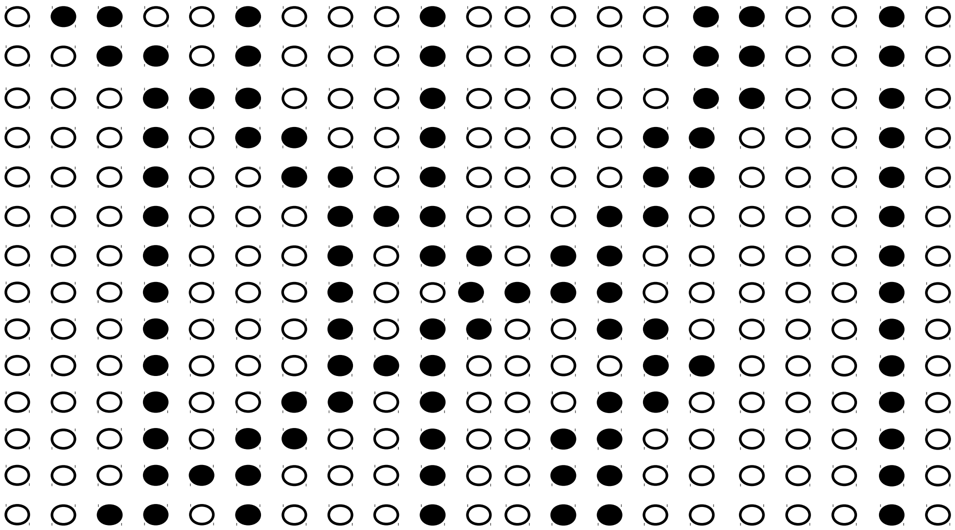
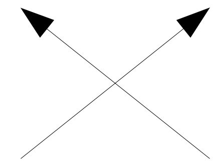


## Simplest quantum integrable models : non-interacting

$$H = \sum_j \left[\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+\right]$$

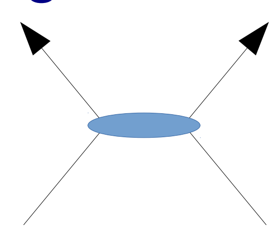
$$H = \sum_j \left[c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right]$$



$$\mathcal{S} = -1$$

## Complicated quantum integrable models : interacting

$$H = \sum_j \left[\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \frac{\Delta}{2} \sigma_j^z \sigma_{j+1}^z\right]$$



$$\mathcal{S}(p_1, p_2)$$

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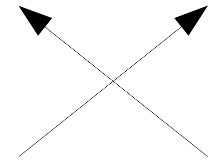
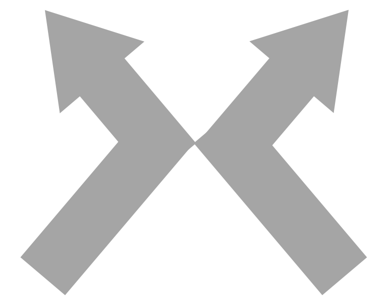


$$\mathcal{S} = -1$$

## “Next to simplest” integrable models ?

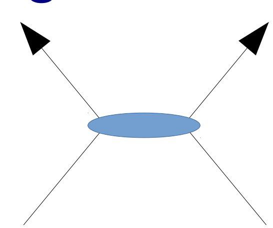
“Hard rod deformations”, obtained from free models by giving a finite width to particles



$$\mathcal{S}(p_1, p_2) = -e^{i(p_1 - p_2)}$$

## Complicated quantum integrable models : interacting

$$H = \sum_j \left[\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \frac{\Delta}{2} \sigma_j^z \sigma_{j+1}^z\right]$$



$$\mathcal{S}(p_1, p_2)$$

# Motivation 1 : out-of-equilibrium dynamics

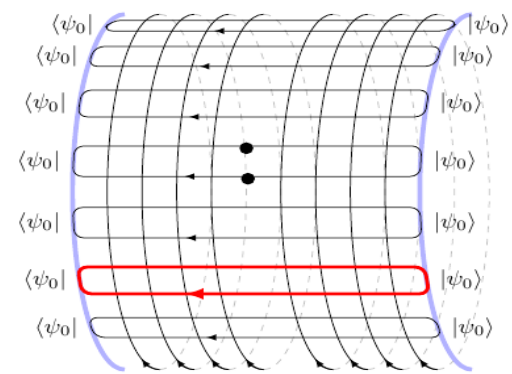
quantum quench : $$\langle \Psi_0 | e^{iHt} \mathcal{O} e^{-iHt} | \Psi_0 \rangle = ?$$

Expansion over basis of eigenstates
("Quench Action") Caux Essler PRL 2013

$$\sum_{i,j} \mathrm{e}^{i(E_i - E_j)t} \langle \Psi_0 | i \rangle \langle i | \mathcal{O} | j \rangle \langle j | \Psi_0 \rangle$$

Finite time dynamics tractable analytically only for non-interacting systems

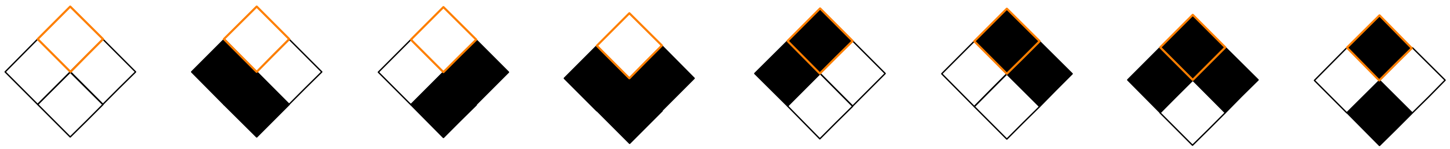
Computation in transverse channel
("Boundary Quantum Transfer Matrix")
Pozsgay 2013, Pozsgay Piroli EV 2017

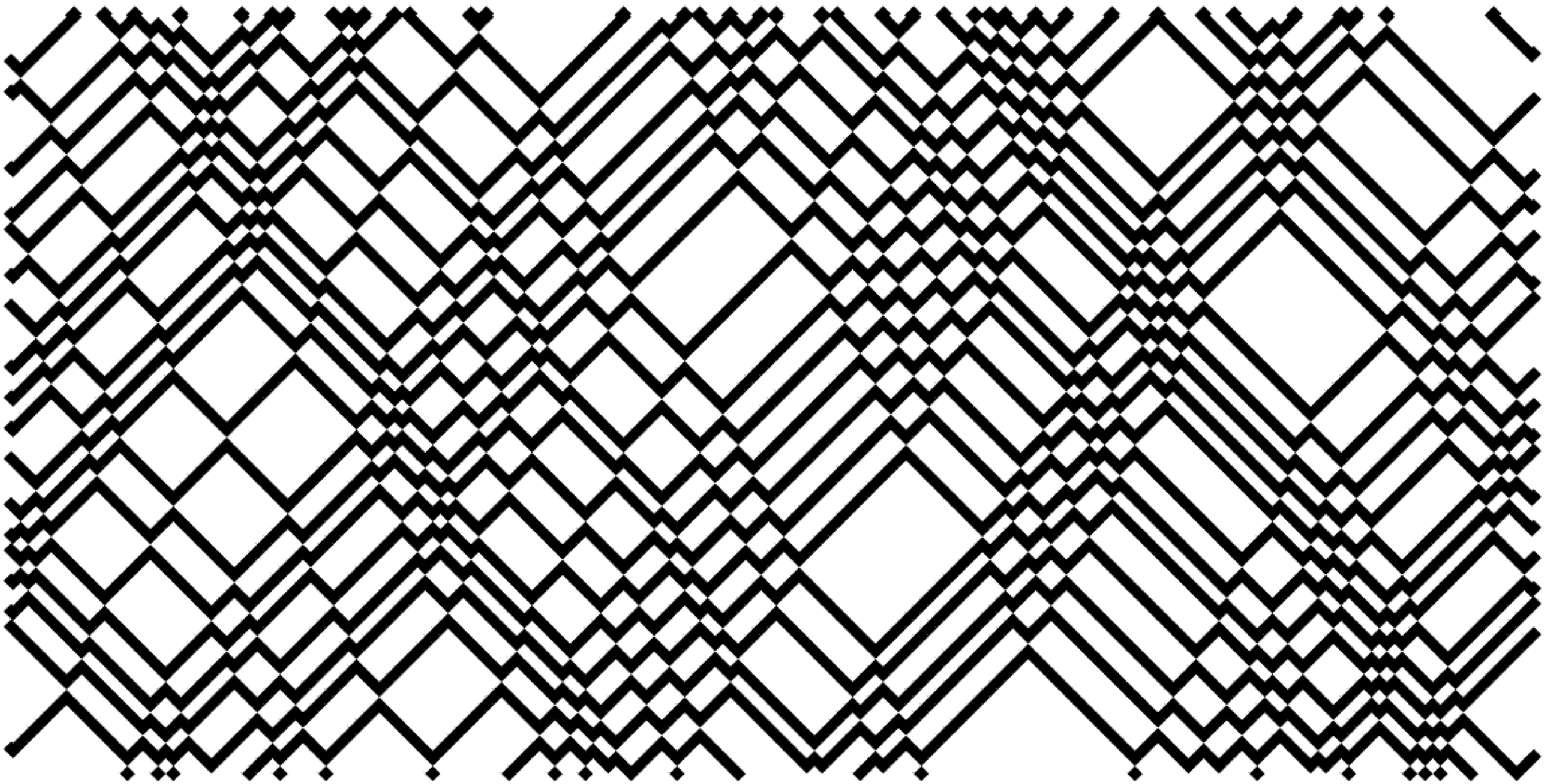


Exact results for the "Loschmidt echo", but dynamics of local observables requires very heavy form factor machinery

Most exact results lately using MPS techniques on discretized time evolutions (unitary circuits)
Klobas Bertini Piroli PRL (2021) ; Piroli Bertini Cirac Prosen, PRB (2020)

Rule 54 cellular automaton : "*the simplest interacting integrable model*"

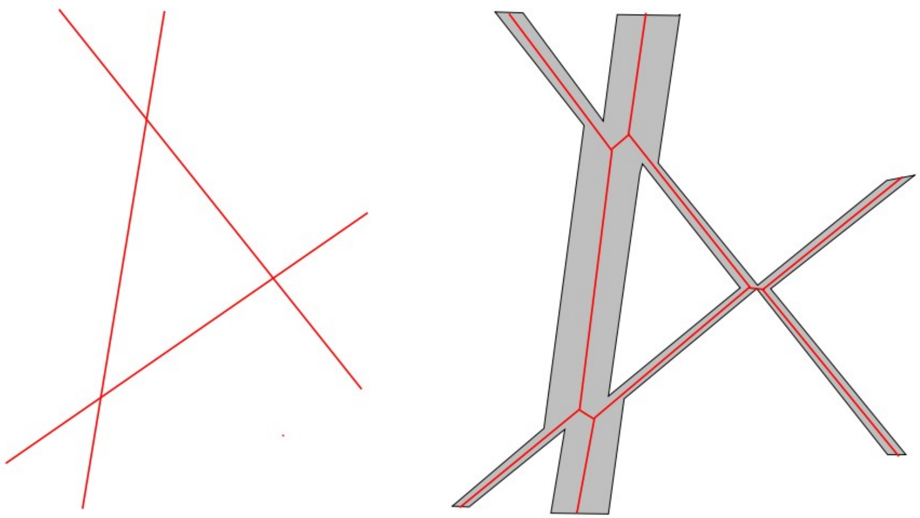




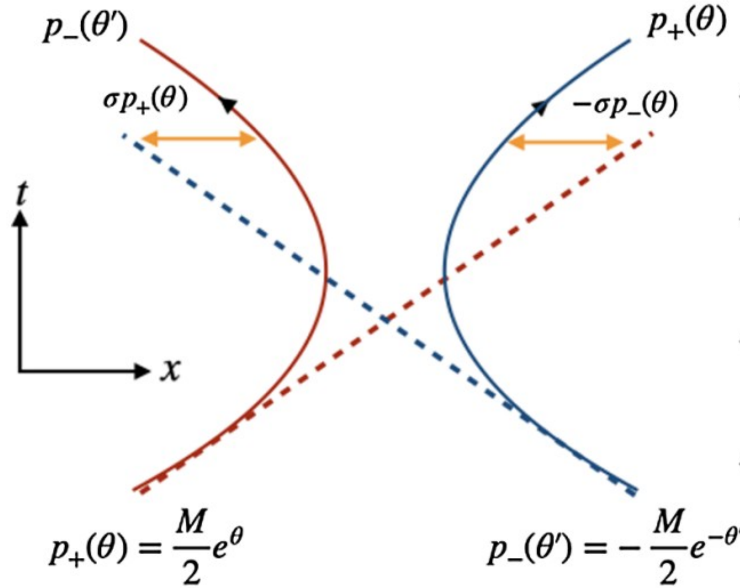
Bobenko, Bordemann, Gunn & Pinkall (1993)
Many works by Prosen *et al*
Friedman Gopalakrishnan Vasseur PRL (2019)

# Motivation 2 : $T\bar{T}$ deformations of integrable QFT

- Recent interpretation of $T\bar{T}$ deformations as giving a finite (momentum dependent) width to particles



Cardy & Doyon, arXiv:2010.15733 (2020)




Medenjak-Policastro-Yoshimura, PRD 103 (2021)

- While $T\bar{T}$ based on conserved energy and momentum, other family of deformations based on momentum and particle number, and where the S matrix gets modified as

$$S(p_1, p_2) \to S(p_1, p_2)e^{i\kappa(p_1 - p_2)}$$

Cardy & Doyon, arXiv:2010.15733 (2020)
Yiang arXiv:2011.00637 (2020)

# Hard rod deformations : state of the art

Several models around, but no clear systematic construction / understanding of integrability

- Bariev model Bariev J. Phys. A (1991)

$$H = \sum_j \left(\sigma_j^- \sigma_{j+2}^+ + \sigma_j^+ \sigma_{j+2}^-\right) \frac{1 - U\sigma_{j+1}^z}{2}$$

- Hard-core fermion models Fendley Nienhuis Schoutens J. Phys. A (2003)

- Constrained XXZ Alcaraz Lazo JSTAT (2007)

$$H_\ell = \sum_{j=1}^{L} \mathcal{P}_\ell \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+\ell+1}^z\right] \mathcal{P}_\ell$$

no two down spins
less than $\ell$ sites apart

- “Phase model” Pozsgay Eisler JSTAT (2016)

# Plan of the talk

## 1. The folded XXZ model

Pozsgay, Gombor, Hutsalyuk, Jiang, Pristyák, EV arXiv:2105.02252

See also Zadnik Fagotti, SciPost Phys Core 4 (2021)
Zadnik Bidzhiev Fagotti SciPost Phys 10 (2021)

Yang, Liu, Gorshkov & Iadecola PRL 124 (2020)

## 2. Geometric construction of hard-rod deformations ?

Work in progress

# 1. The folded XXZ model

Pozsgay, Gombor, Hutsalyuk, Jiang, Pristyák, EV arXiv:2105.02252

See also Zadnik Fagotti, SciPost Phys Core 4 (2021)
Zadnik Bidzhiev Fagotti SciPost Phys 10 (2021)

Yang, Liu, Gorshkov & Iadecola PRL 124 (2020)

Start from the XXZ Hamiltonian and conserved charges, which are all polynomial in $\Delta$

$$Q_2 = \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta(\sigma_j^z \sigma_{j+1}^z - 1)$$

$$Q_3 = \sum_{j=1}^{L} \sigma_{j+1}^z (\sigma_j^+ \sigma_{j+2}^- - \sigma_j^- \sigma_{j+2}^+) + \Delta(\sigma_j^z + \sigma_{j+3}^z)(\sigma_{j+1}^+ \sigma_{j+2}^- - \sigma_{j+1}^- \sigma_{j+2}^+)$$

$$Q_4 = \sum_{j=1}^{L} \ldots + \Delta^2 (1 + \sigma_j^z \sigma_{j+3}^z)(\sigma_{j+1}^+ \sigma_{j+2}^- + \sigma_{j+1}^- \sigma_{j+2}^+)$$

Start from the XXZ Hamiltonian and conserved charges, which are all polynomial in $\Delta$

$$Q_2 = \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \boxed{\Delta(\sigma_j^z \sigma_{j+1}^z - 1)}$$

$$Q_3 = \sum_{j=1}^{L} \sigma_{j+1}^z (\sigma_j^+ \sigma_{j+2}^- - \sigma_j^- \sigma_{j+2}^+) + \boxed{\Delta(\sigma_j^z + \sigma_{j+3}^z)(\sigma_{j+1}^+ \sigma_{j+2}^- - \sigma_{j+1}^- \sigma_{j+2}^+)}$$

$$Q_4 = \sum_{j=1}^{L} \dots + \boxed{\Delta^2 (1 + \sigma_j^z \sigma_{j+3}^z)(\sigma_{j+1}^+ \sigma_{j+2}^- + \sigma_{j+1}^- \sigma_{j+2}^+)}$$

**Leading terms $\tilde{Q}_n$ in $\Delta$ still form a mutually commuting family**

→ **"Folded XXZ" Hamiltonian :**

$$\boxed{\tilde{Q}_4 = \sum_{j=1}^{L} (1 + \sigma_j^z \sigma_{j+3}^z)(\sigma_{j+1}^+ \sigma_{j+2}^- + \sigma_{j+1}^- \sigma_{j+2}^+)}$$

$$|\uparrow \downarrow\uparrow \uparrow\rangle \longleftrightarrow |\uparrow \uparrow\downarrow \uparrow\rangle$$

$$|\downarrow \downarrow\uparrow \downarrow\rangle \longleftrightarrow |\downarrow \uparrow\downarrow \downarrow\rangle$$

# Bond-site transformation

Degrees of freedom on bonds, depending on relative orientations of adjacent spins :
empty if same orientation, occupied if opposite orientations

$$Q_2^B = \sum_j P_j^\bullet$$

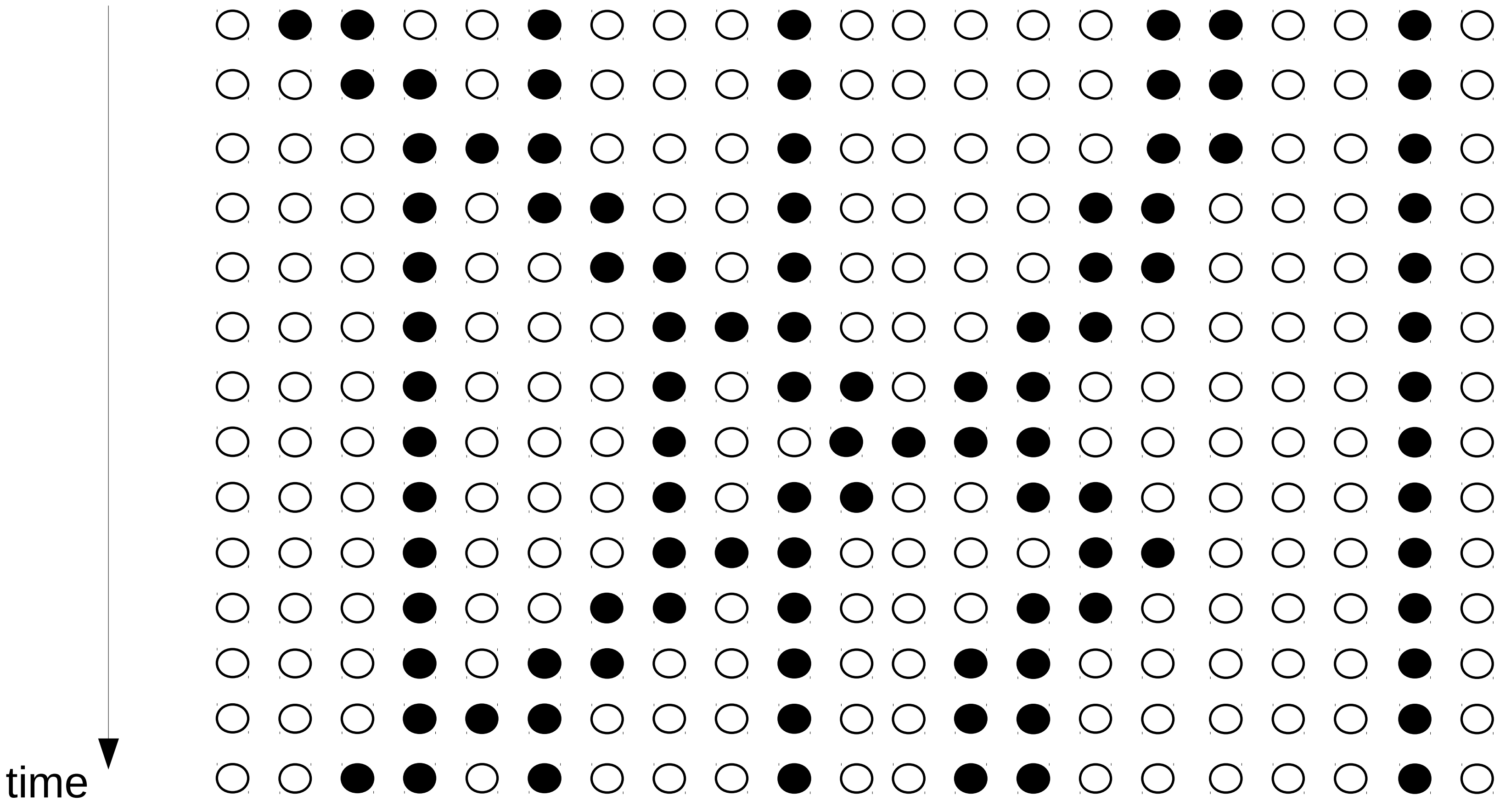
$$Q_3^B = i \sum_j \left( \sigma_j^+ P_{j+1}^\bullet \sigma_{j+2}^- - \sigma_j^- P_{j+1}^\bullet \sigma_{j+2}^+ \right)$$

$$Q_4^B = \sum_j \sigma_j^- P_{j+1}^\bullet \sigma_{j+2}^+ + \sigma_j^+ P_{j+1}^\bullet \sigma_{j+2}^-$$

⚠ Mapping exact in the bulk, but subtleties with periodic boundary conditions !

Dynamics under $Q_4^B$ :
pairs of domain walls can jump left or right, but isolated DWs are fixed

$$\circ \bullet \bullet \longrightarrow \bullet \bullet \circ$$
$$\bullet \bullet \circ \longrightarrow \circ \bullet \bullet$$



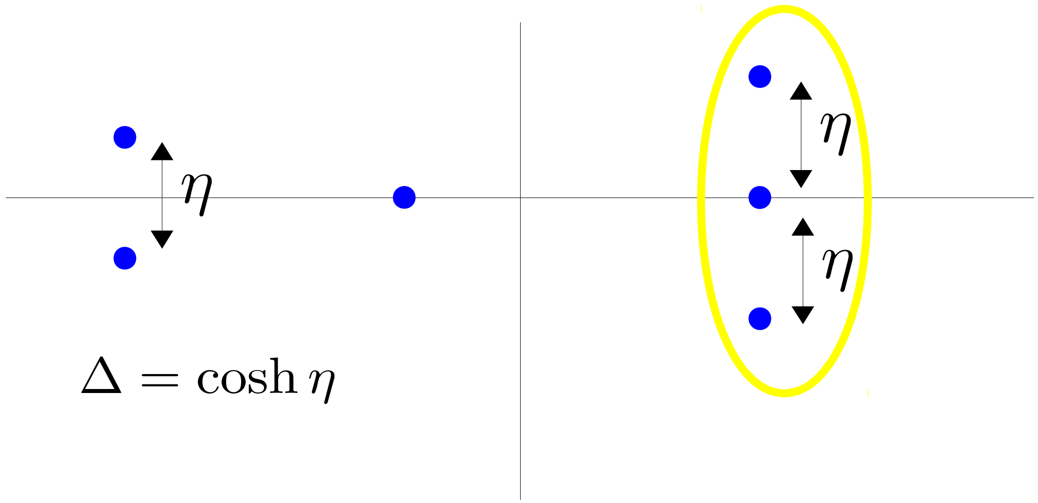

- Number of isolated DWs and of pairs of DWs are fixed

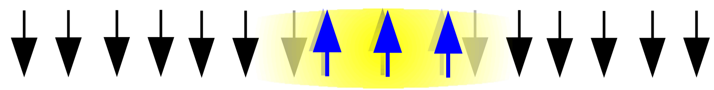
- In the sector without isolated DWs, model equivalent to the **constrained XX model**, obtained from XX model through the mapping $|\uparrow\rangle \to |\circ\rangle\,,\,|\downarrow\rangle \to |\bullet\bullet\rangle$

  More generally, the full model can be mapped onto the "SU(3) XX model" of Maasarani Mathieu, NPB (1998)

# Bethe ansatz – reminder on XXZ

Bethe roots organize themselves into **strings**



n-string = wavepacket of n bound up spins in a sea of down spins

- **energy/higher charges carried by an n-string**

$$q_{2,n}(\lambda) = \coth(-n\eta/2 + i\lambda) - \coth(n\eta/2 + i\lambda).$$

$$q_{4,n}(\lambda) = -\frac{\partial^2 q_{2,n}(\lambda)}{\partial \lambda^2}$$

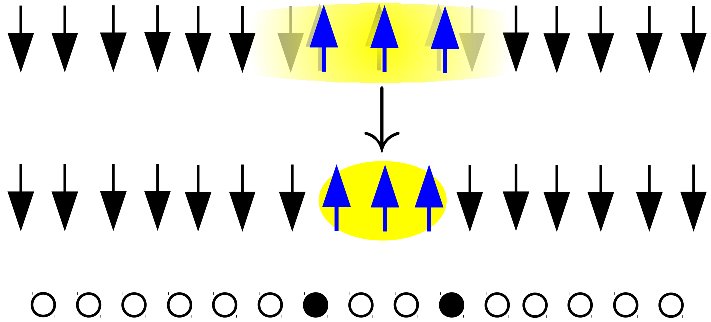
- **scattering between n-string and m-string**

$$S_{n,m} = \begin{cases} \Theta_{n-m}\Theta^2_{n-m+2}\ldots\Theta_{n+m} & \text{for } n \neq m \\ \Theta^2_2 \ldots \Theta^2_{2n-2}\Theta_{2n} & \text{for } n = m \end{cases} \qquad \Theta_n(\lambda) = \frac{\sin(\lambda + in\eta)}{\sin(\lambda - in\eta)}$$

# Bethe ansatz – $\Delta \to \infty$ limit (see also Bogoliubov & Malyshev 2011)

Strings become more and more bound :
n-string = n consecutive down spins

Bond picture : 2 domain walls at distance n



- **charges carried by strings**

$$q_{2,n}(\lambda) \to 2 \qquad \text{density of domain walls}$$

$$q_{4,n}(\lambda) \to \begin{cases} -\cos(2\lambda) & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{cases}$$

only pairs of neighbouring DWs contribute to the energy

- **scattering between strings**

$$S_{m,n} = \begin{cases} e^{-2mip} & \text{for } n > m \\ -e^{-2nip} & \text{for } n = m \end{cases}$$

detailed info on $n \geq 2$-strings disappears from BAE, only total number of strings and total momentum matter

# Further aspects of the model

- **DWs move only by pairs = fracton-like dynamics**

- **Solvable quench dynamics**

Evolution of local observables can be computed exactly from certain initial states

$$|\Psi_0\rangle = \quad \downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow \qquad \langle \Psi_0 | \mathbf{p} \rangle = \det_{jk} e^{ip_j(2k+1)}$$

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observable : emptiness formation probability : $\mathbb{E}_\ell(x) = \prod_{j=1}^{\ell} \frac{1+\sigma^z_{x-1+j}}{2}$

$$\langle \Psi(t)|\mathbb{E}_3|\Psi(t)\rangle = \frac{1}{6} - \frac{1}{6}\left(\int_0^\pi \frac{dz}{\pi}\cos(2\cos(z)t)\right)^2 - \frac{1}{6}\left|\int_0^\pi \frac{dz}{\pi}\sin(2\cos(z)t)e^{iz}\right|^2$$

- **Hilbert space fragmentation** (as in Fabian’s talk)

Exponentially growing number of disconnected sectors corresponding to configurations of isolated DWs at given distance from one another.

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Exact degeneracies exponential in system size

## 2. Geometric construction of hard-rod deformations ?

Work in progress

# The hard-rod XXZ model Gombor Pozsgay 2021, Pozsgay Gombor Hutsalyuk 2021

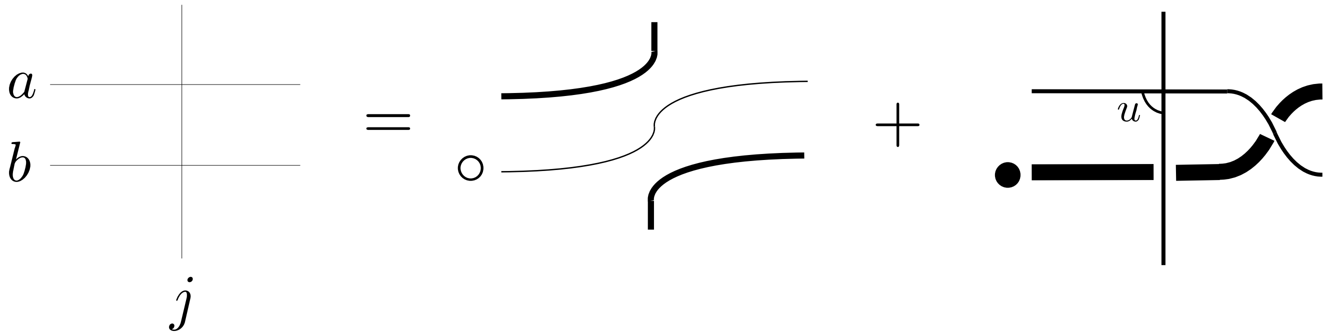
**See also talk by B. Pozsgay**

$$H=\sum_j \left[\sigma_j^- P_{j+1}^{\bullet}\sigma_{j+2}^+ + \sigma_j^+ P_{j+1}^{\bullet}\sigma_{j+2}^- - \Delta\left(P_j^{\circ}P_{j+1}^{\bullet}P_{j+2}^{\bullet} + P_j^{\bullet}P_{j+1}^{\bullet}P_{j+2}^{\circ}\right)\right]$$

- This extends folded XXZ (in bond representation) to $\Delta \neq 0$
- In the sector with no isolated DWs, this is the constrained XXZ model of Alcaraz Lazo JSTAT (2007)

**Lax operator :** based on a pair of 2-dimensional aux spaces

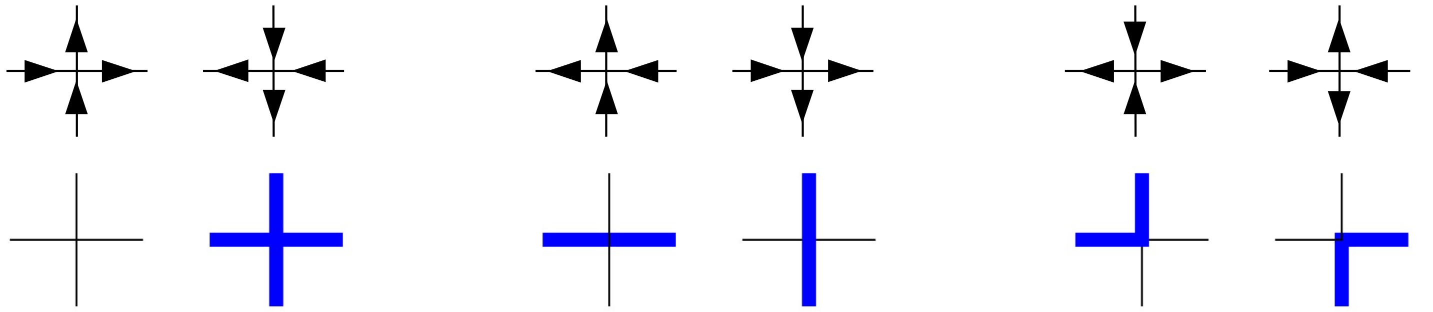
$$\mathcal{L}_{a,b,j}(u)=\mathcal{P}_{a,j}\mathcal{P}_{b,j}P_b^{\circ}+\mathcal{L}_{a,j}^{\mathrm{XXZ}}(u)\mathcal{P}_{a,b}P_b^{\bullet}$$



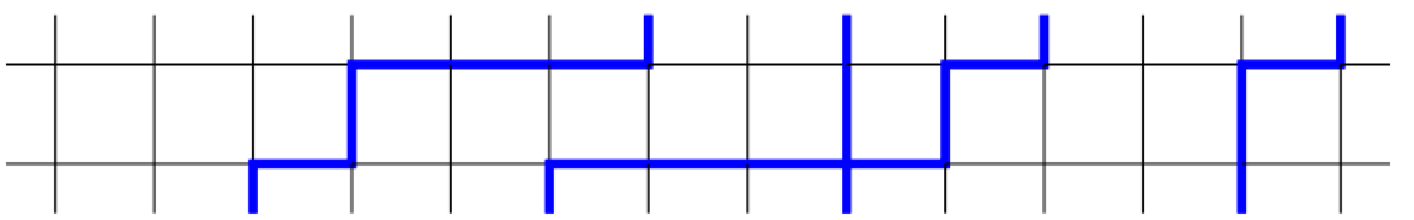

**RLL equation** with trigonometric R matrix, not of difference form

# Geometric construction

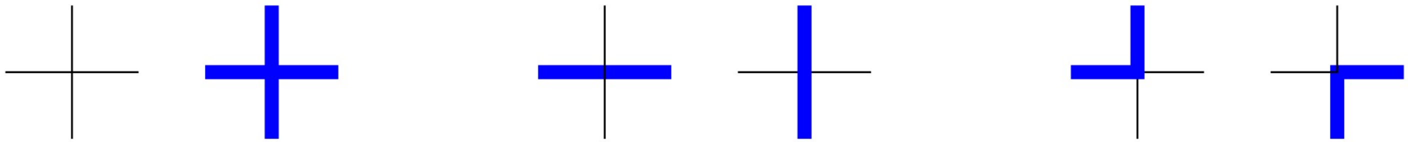
Six-vertex model :



$$\frac{\sin(u+\gamma)}{\sin\gamma} \qquad \frac{\sin u}{\sin\gamma} \qquad 1$$

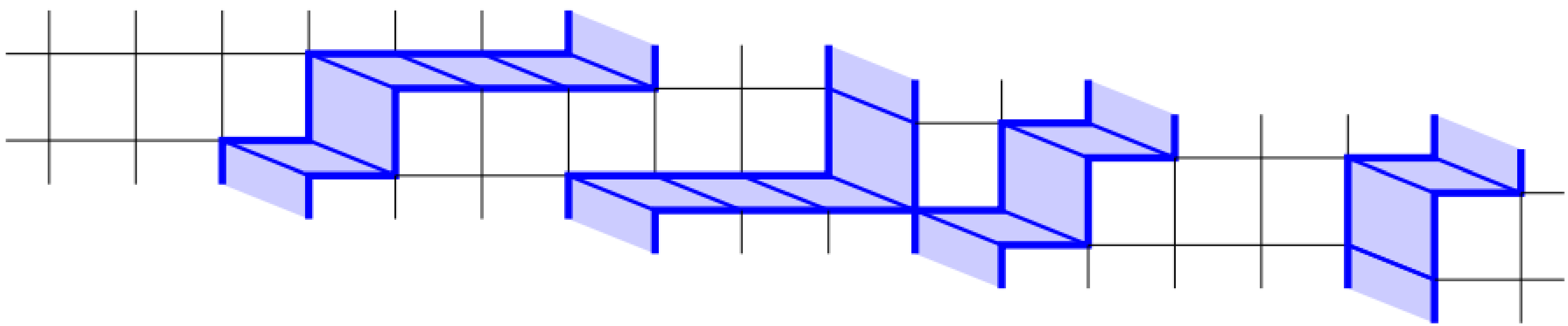


# Geometric construction

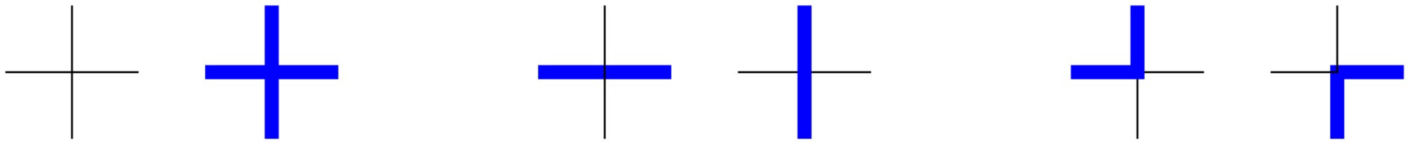


$$\frac{\sin(u+\gamma)}{\sin\gamma} \qquad \frac{\sin u}{\sin\gamma} \qquad 1$$

interpret lines as jumps between heights on a 3D surface



# Geometric construction

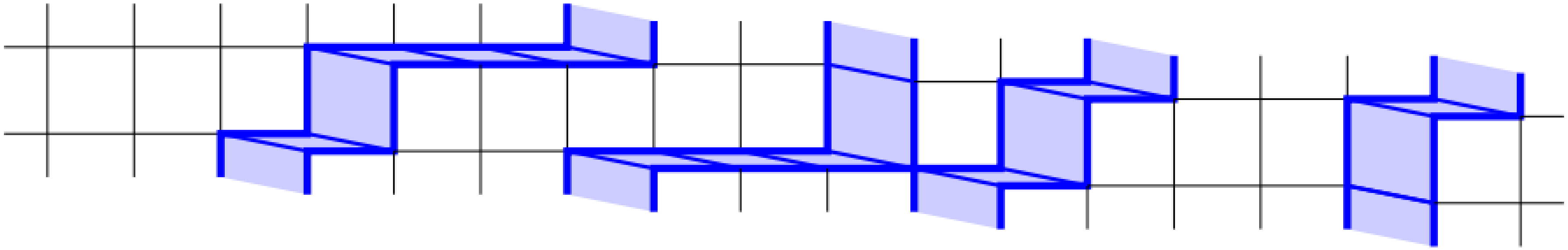


$$\frac{\sin(u+\gamma)}{\sin\gamma}$$

$$\frac{\sin u}{\sin\gamma}$$

$$1$$

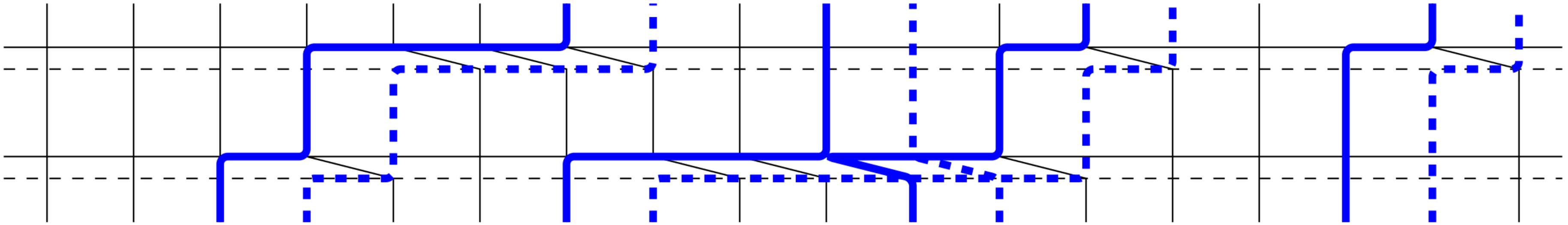
Very anisotropic limit (here $\ell = 2$ )



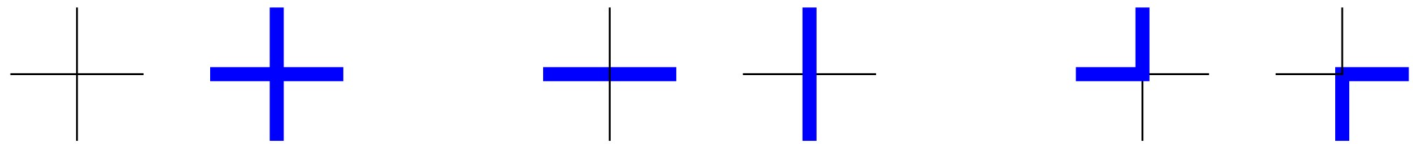
# Geometric construction

$$\frac{\sin(u+\gamma)}{\sin\gamma} \qquad \frac{\sin u}{\sin\gamma} \qquad 1$$

Generated from transfer matrices with two rows

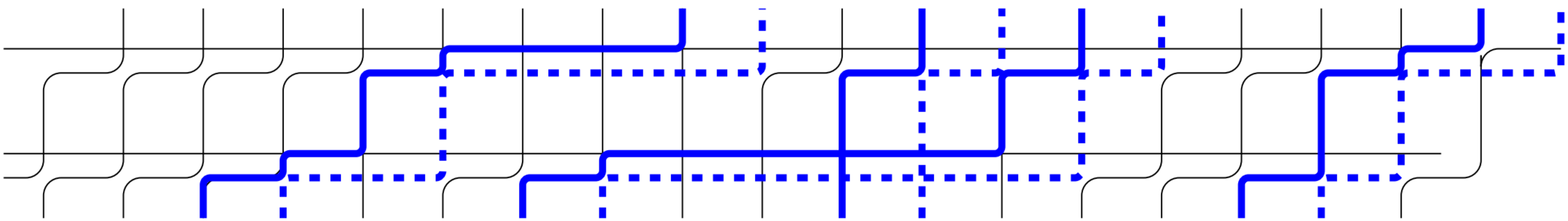


# Geometric construction



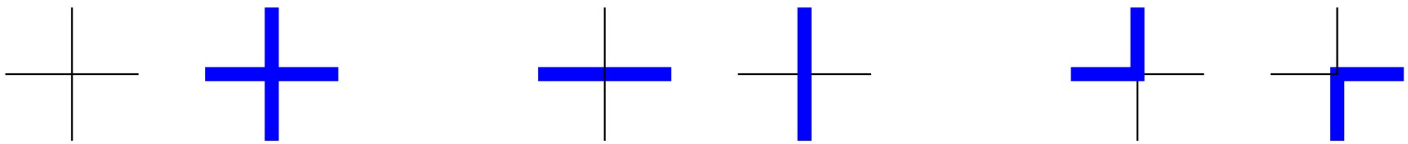
$\frac{\sin(u+\gamma)}{\sin\gamma}$ $\qquad$ $\frac{\sin u}{\sin\gamma}$ $\qquad$ $1$

Compose each transfer matrix with a one-site translation

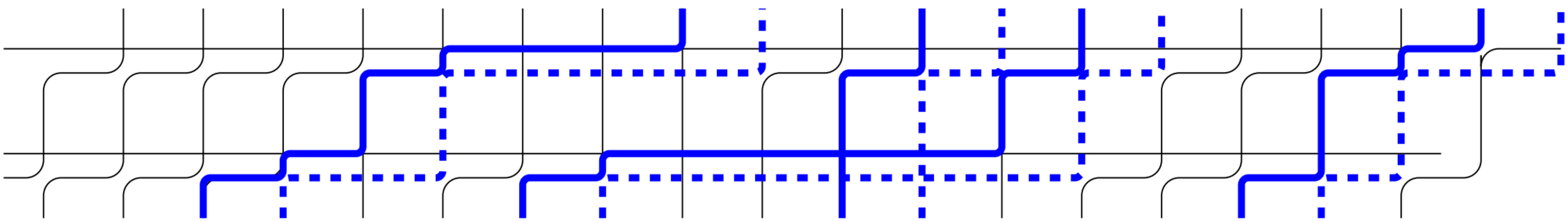


**Rules :** interaction weights carried by the full lines – dashed lines just pass through

# Geometric construction



$\frac{\sin(u+\gamma)}{\sin\gamma}$ $\qquad$ $\frac{\sin u}{\sin\gamma}$ $\qquad$ $1$

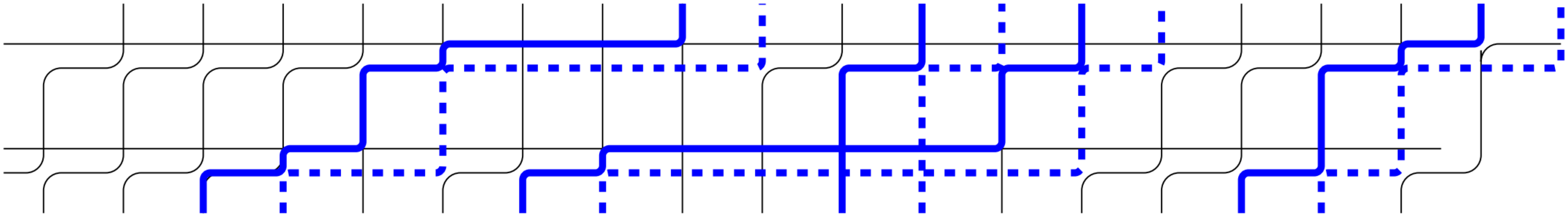


**Rules :** interaction weights carried by the full lines – dashed lines just pass through

$$L_j(u) = \frac{\sin(\gamma+u)}{\sin\gamma} E^{00} \otimes E_j^{00} + \frac{\sin u}{\sin\gamma} E^{33} \otimes E_j^{00} + \frac{\sin u}{\sin\gamma} E^{12} \otimes E_j^{21} + \frac{\sin(\gamma+u)}{\sin\gamma} E^{33} \otimes E_j^{11}$$

$$+ E^{33} \otimes E_j^{22} + E^{21} \otimes E_j^{12} + \frac{\sin(\gamma+u)}{\sin\gamma} E^{01} \otimes E_j^{10} + E^{13} \otimes E_j^{20} + E^{32} \otimes E_j^{01} + E^{20} \otimes E_j^{02}$$

# Geometric construction

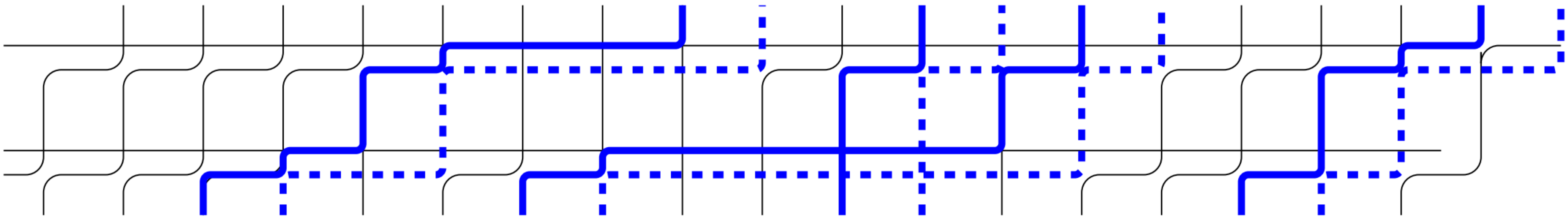


$$L_j(u) = \frac{\sin(\gamma+u)}{\sin\gamma} E^{00}\otimes E_j^{00} + \frac{\sin u}{\sin\gamma} E^{33}\otimes E_j^{00} + \frac{\sin u}{\sin\gamma} E^{12}\otimes E_j^{21} + \frac{\sin(\gamma+u)}{\sin\gamma} E^{33}\otimes E_j^{11}$$
$$+ E^{33}\otimes E_j^{22} + E^{21}\otimes E_j^{12} + \frac{\sin(\gamma+u)}{\sin\gamma} E^{01}\otimes E_j^{10} + E^{13}\otimes E_j^{20} + E^{32}\otimes E_j^{01} + E^{20}\otimes E_j^{02}$$

- In the sector with no isolated DWs, recovers the Lax operator of Pozsgay Gombor Hutsalyuk 2021

- RLL equation with R matrix much similar to that of Pozsgay Gombor Hutsalyuk 2021

# Geometric construction



$$L_j(u) = \frac{\sin(\gamma+u)}{\sin\gamma} E^{00} \otimes E_j^{00} + \frac{\sin u}{\sin\gamma} E^{33} \otimes E_j^{00} + \frac{\sin u}{\sin\gamma} E^{12} \otimes E_j^{21} + \frac{\sin(\gamma+u)}{\sin\gamma} E^{33} \otimes E_j^{11}$$
$$+ E^{33} \otimes E_j^{22} + E^{21} \otimes E_j^{12} + \frac{\sin(\gamma+u)}{\sin\gamma} E^{01} \otimes E_j^{10} + E^{13} \otimes E_j^{20} + E^{32} \otimes E_j^{01} + E^{20} \otimes E_j^{02}$$

- In the sector with no isolated DWs, recovers the Lax operator of Pozsgay Gombor Hutsalyuk 2021

- RLL equation with R matrix much similar to that of Pozsgay Gombor Hutsalyuk 2021

- interpretation as a two-line x two line R matrix (product of 4 R matrices all of difference form ?)

- Model with isolated DWs as a “decoration” of the present one ?

# Conclusions

- **Folded XXZ**

  appears naturally in the $\Delta \to \infty$ limit of XXZ, and has many nice features

- **Hard rod deformations**

  - systematic geometric construction

  - models with $\ell > 0$ coincide with the constrained XXZ of Alcaraz Lazo JSTAT (2007), but $\ell < 0$ models are new

  - much yet to be understood : R matrix, etc…, even though a lesson from Paul's talk would be to stop being obsessed with writing everything in the RTT=TTR form

**Happy Birthday Hubert…. and Jesper !**