Some recent results in spin glass theory

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Hubert Saleur Fest, September 22, 2021

Outline:

1) Ising spin glasses: introduction, motivation
2) Newman and Stein’s metastate concept
3) The metastate in replica symmetry breaking
4) Complexity as information and upper bounds

N. Read, Phys. Rev. E **90**, 032142 (2014) [arXiv:1407.4136];
Phys. Rev. E **97**, 012134 (2018) [arXiv:1709.05270];
M.A. Moore and N.R., Phys. Rev. Lett. **120**, 130602 (2018) [arXiv:1801.09779];
J. Höller and N.R., Phys. Rev. E **101**, 042114 (2020) [arXiv:1909.03284];
S. Jensen, N.R., and A.P. Young, Phys. Rev. E **104**, 034105 (2021) [arXiv:2106.13191];
N. Read, arXiv:2108.11849.

Ising spin glasses

Ising version of Edwards-Anderson model:

$$H(s) = -\sum_{(i,j)} J_{ij} s_i s_j, \quad s_i = \pm 1; \quad s = (s_i)$$

(i,j) denotes $i,\ j$ are nearest neighbors on hypercubic lattice, d dimensions

$J_{ij} = J_{ji}$ are independent, identically distributed, mean zero (e.g. Gaussian);
[. . .] or $\mathbf{E}\ \cdots$ will denote expectation over J_{ij} .
Can allow ints beyond nearest neighbors.

EA found that in mean field theory there is a phase transition, below which

$$[\langle s_i \rangle] = 0, \quad \left[\langle s_i \rangle^2\right] \neq 0$$

--- i.e. spins order (spontaneously break spin flip symmetry),
but randomly: “spin glass”

Sherrington and Kirkpatrick proposed an “exactly soluble” infinite-range model

Study of SK model led to Parisi’s “mean field solution”, with “breaking of replica symmetry” (**RSB**) below transition --- a surprisingly complex structure

Various aspects of the Parisi solution have by now been *rigorously* proved correct ---Guerra, Talagrand, Panchenko

For the original EA model---short-range, finite-dimensional---the Parisi school expects similar behavior as in SK.

But others (McMillan, Bray-Moore, D. Fisher-Huse) proposed a scenario with a much simpler structure --- albeit capable of e.g. complex dynamical behavior (i.e. glassy dynamics) --- known as the **scaling-droplet** picture

The controversy is still not resolved.

The goal of this work is to shed a little light on it.

A feature of the Parisi RSB picture is the claim that there are many ordered or “pure” states [or “free-energy valleys”] (and further that somehow they are arranged hierarchically or ultrametrically).

In the scaling-droplet picture, it is *assumed* (implicitly or explicitly) that (in zero magnetic field) there is only one pure (ordered) state and its spin flip --- as in an Ising ferromagnet.

This difference is the central point, but is subtle: the notion of a pure state, or of spontaneously breaking spin flip symmetry, only makes sense in an infinite system. (We will later see a further distinction between pictures.)

In a finite system, the Gibbs probability distribution for the spins (in canonical ensemble, for fixed J_{ij} s) is uniquely determined by the usual formulas

$$\Gamma(s) = p_H(s) = \frac{e^{-\beta H(s)}}{Z}, \quad Z = \sum_{\substack{\{s_i=\pm 1:\\ i=1,\ldots,N\}}} e^{-\beta H(s)}$$

and there can be no phase transitions. $(\beta = 1/T)$

These formulas don’t make sense in an infinite system (the sums don’t converge).

Infinite-size Gibbs states and pure (or extremal) states at T>0:

A **Gibbs (DLR) state** is defined by specifying its conditional distributions for the spins $s|_\Lambda$ in any finite region Λ, conditioned on the spins outside:

$$\Gamma(s|_\Lambda \mid s|_{\Lambda^c}) = p_{H'}(s|_\Lambda), \qquad H'(s|_\Lambda) = -\sum_{\{i,j\}:i,j\in\Lambda} J_{ij}s_is_j - \sum_{i\in\Lambda, j\in\Lambda^c} J_{ij}s_is_j$$

where H' is finite (with probability one) for all s if

$$\sum_j \mathbf{E}|J_{ij}| < \infty$$

(for all i, by trans invariance)

(short-range case).
This does not determine the behavior “at infinity”; there could be many Gibbs states for same H.

Convex combinations or mixtures of Gibbs states are still Gibbs.
Define **pure states** to be extremal Gibbs states, i.e. those that cannot be expressed as a convex combo (most generally, an integral) of other Gibbs states.

Any Gibbs state has a *unique* decomposition as a convex combo of pure states:

$$\Gamma(s) = \sum_\alpha w_\alpha(\Gamma)\Gamma_\alpha(s),$$

where w_α are non-negative weights, and $\Gamma_\alpha(s)$ are pure states.

Why not just take the limit of the state in a sequence of finite size systems?

The problem is that the $L \to \infty$ limit of the finite-size Gibbs states---hence of correlation functions---may not exist.
(Here $J = (J_{ij})$ for all edges fixed once and for all; bc's can be free, periodic, etc; as size increases, new spins and new bonds are added at the edge.)

--- **Chaotic size dependence** (CSD)
--- associated with the existence of many pure states (Newman-Stein 1992)
--- NS proposed the concept of a **metastate** to handle this issue (NS 1996, 1997); a related construction was given earlier by Aizenman-Wehr (1990).

A state is a probability distribution on the spins (for fixed J_{ij} s). A metastate is a probability distribution on (infinite-size) Gibbs states, also for fixed J_{ij} s.

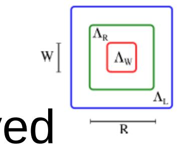
Essentially, constructed by first taking the limit of the joint probability distribution of bonds and states in finite size. (Can show such a limit in distribution exists, but it may be necessary to use a subsequence, and then limit may not be unique.)
Then the conditional distribution on (infinite-size) states for given J is the metastate $\kappa_J(\Gamma)$; it can be shown the states are Gibbs (so Gibbs states exist!).

This (AW) metastate contains info on dependence of state on bonds asymptotically far away; the NS metastate instead records the dependence on system size, in the limit.

Alternate view:

Aizenman-Wehr (AW) metastate: Let $L >> R >> W, \quad \Lambda_W \subset \Lambda_R \subset \Lambda_L$.

First, find prob dist $\kappa_J(\Gamma(s|_W))$ on Gibbs states in window Λ_W, induced by disorder (i.e. bonds with at least one end) in "outer region" $\Lambda_L - \Lambda_R$ for fixed disorder in Λ_R. Then take limits $L \to \infty,\ R \to \infty,\ W \to \infty$ (possibly need a subsequence). Result is an AW (ground state) metastate $\kappa_J(S)$.



Thus an AW metastate tells us, for a large system, how the state observed in a window depends on the set of bonds in the outer region far away.

Likewise a NS metastate tells us how the state in the window depends on the system size chosen, at asymptotically large sizes.

--- under some conditions, NS and AW metastates can be shown to be the same

Characterizing the metastate in SG phase is a second key question about EA model

--- is it trivial (supported on a single Gibbs state), or not?

In scaling-droplet picture, it is assumed the metastate is trivial and unique, and the unique Gibbs state contains single pair of pure states, or single pure state for non-zero magnetic field.

According to RSB, at $T > 0$ the Gibbs state contains many pure states, hierarchically. Then a metastate *must* be non-trivial (NS 1996)---there is CSD---and is responsible for so-called non-self-averaging.

Notation: $\langle \cdots \rangle_\Gamma$ = thermal average in a Gibbs state

$[\cdots]_{\kappa_J}$ = expectation over metastate at fixed J

$[\cdots]_{\nu(J)}$ = expectation over dist of J (e.g. Gaussian)

Useful definition: let $\rho_J(S) = [\Gamma(S)]_{\kappa_J}$ be the “metastate averaged state” (MAS).
--- it too is a Gibbs state

Thus a thermal average in the MAS is given by $\langle \cdots \rangle_{\rho_J} = [\langle \cdots \rangle_\Gamma]_{\kappa_J}$

In AW metastate, the averages have simple meanings in terms of finite size:

$$\langle\cdots\rangle_\Gamma \leftrightarrow \langle\cdots\rangle$$ (i.e. finite-size thermal average)

$$[\cdots]_{\kappa_J} \leftrightarrow [\cdots]_>$$ (disorder average in outer region)

$$[\cdots]_{\nu(J)} \leftrightarrow [\cdots]_<$$ (disorder average in inner region)

after which limits should be taken.

Following correlation function should reveal non-trivial AW metastate:

$$C(\mathbf{r}_i - \mathbf{r}_j) = \left[\langle s_i s_j\rangle^2_{\rho_J}\right]_{\nu(J)} = \lim \left[[\langle s_i s_j\rangle]^2_>\right]_<$$

In SG phase, $\langle s_i s_j\rangle^2 \rightarrow$ constant at large distances. If metastate is trivial, same will be true for $C(\mathbf{r}_i - \mathbf{r}_j)$.

But for nontrivial metastate, average over outer disorder will produce decay of $C(\mathbf{r}_i - \mathbf{r}_j)$ --- even at T = 0! --- to a smaller value, or zero.

Expect power-law (even at zero temp):

$$C(\mathbf{r}_i - \mathbf{r}_j) \sim \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|^{d-\zeta}} \quad \text{as} \quad |\mathbf{r}_i - \mathbf{r}_j| \to \infty$$

Clearly $\zeta \leq d$.

Can interpret ζ for T=0 as something like the fractal dimension of each "cluster" that flips when ground state changes due to change in disorder far away.

Further, expect that the number of distinct pure states that can be distinguished by observing the window of size W would be

$$\exp cW^{d-\zeta'}$$

and I also argued that $\zeta' = \zeta$.
(Hence (NS) expect lower bound $\zeta \geq 1$ in the EA model --- return to this later.)

Thus, if $\zeta < d$, the total number of pure states in the MAS is *uncountable* (NS 2007), but *subextensive* in EA model.

(*Much larger* than the number of pure states contributing to a single Γ drawn from the metastate, which is countable according to RSB.)

For AW metastate, can use RSB field theory to calculate correlator above, making use of results from De Dominicis et al, 1998.

Find

$$C(\mathbf{r}_i - \mathbf{r}_j) \sim \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|^{d-4}} \quad \text{as} \quad |\mathbf{r}_i - \mathbf{r}_j| \to \infty$$

so according to RSB (in zero magnetic field) (NR, 2014)

$$\zeta = 4 \qquad \text{for} \qquad d > 6\,,$$

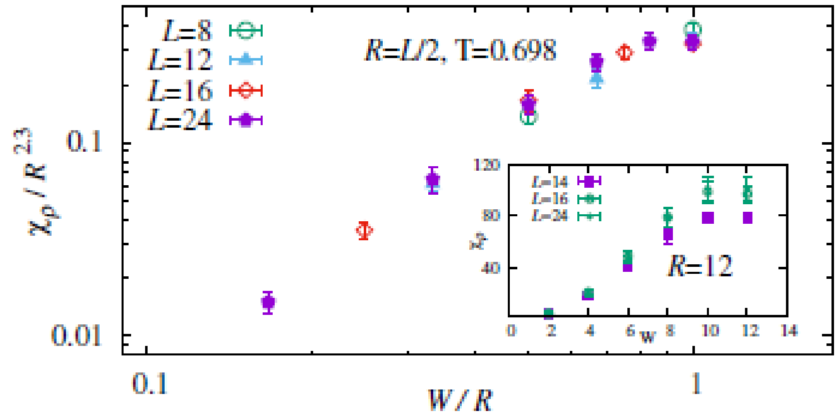
where loop corrections can be neglected.

For $d < 6$, replica calculations are difficult; we expect ζ decreases and reaches $\zeta = d_\ell$ just at the lower critical dimension $d = d_\ell$ (only a guess!); it is believed that $d_\ell \simeq 2.5$.

Studying various correlations or overlap distributions similarly, find that RSB is consistent with all rigorous requirements from NS metastate analysis.
Can also find overlap distribution in MAS.

Recent numerical study (Billoire, . . . , Parisi, . . ., PRL 119, 037203 (2017) [arXiv:1704.01390]) studies precisely these quantities in d=3 EA model!
Scaling form:

$$\chi_\rho = \sum_{\mathbf{r}\in\Lambda_W} C(\mathbf{r}), \qquad \chi_\rho(W,R) = R^\zeta f(W/R) = W^\zeta g(W/R)$$



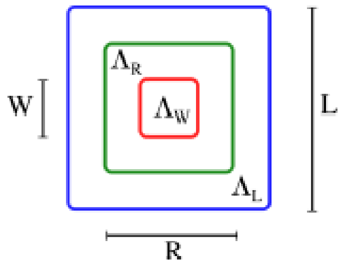


FIG. 3. Main plot: Collapse of MAS susceptibility data measured with $R = L/2$ at $T = 0.698 \simeq 0.64T_c$. Inset: Deviations from the asymptotic behavior are evident only for $R/L > 3/4$.

They find asymptopia reached when $L/R, R/W > 4/3$; L up to 24.
Scaling plots give

$$\zeta = 2.3(3) \qquad (d = 3)$$

They also studied finite size version of overlap distribution in MAS, $P_\rho(q)$ (different quantity from Parisi $P(q)$; nonetheless consistent with RSB):

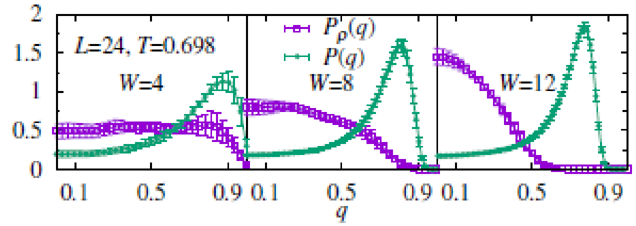


FIG. 4. $P_\rho(q)$ and $P(q)$ for $L = 24$, $R = L/2$, $T = 0.698$ and different values of the measuring window size, $W = 4, 8, 12$.

Difference from usual $P(q)$, and difference of ζ from 4, are evidence for non-trivial metastate in d=3 EA model. Trends:

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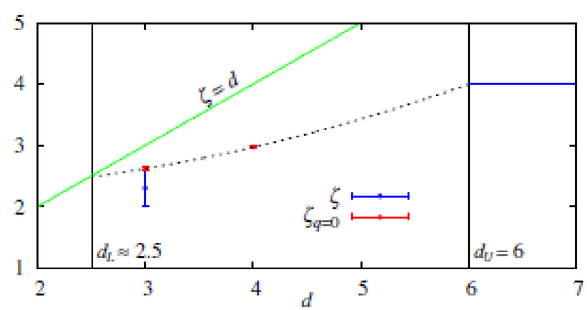

FIG. 5. The exponent ζ as a function of d.

(Red points from earlier work on quantity I expect to be the same.)

These results are evidence of CSD, and some form of replica symmetry breaking, in three dimensions.

Another earlier study by Wittman and Young, arXiv:1504.07709, used the 1D long-range model with $\mathrm{Var}\, J_{ij} \propto 1/|i-j|^{2\sigma}$, and chose $\sigma = 5/8$, which should correspond to $d = 8$ in the EA model.

In this regime $(1/2 < \sigma < 2/3)$ the RSB calculation predicts $\zeta = 4\sigma - 2$.

They studied correlations after a quench from high temp, arguing that this "*dynamic* metastate" simulates the *equilibrium* MAS correlation function.

The results are in excellent agreement with the (static) RSB prediction at $\sigma = 5/8$.

Extended to more values of σ in Jensen, N.R., and Young (2021).

Complexity of states and metastates as information

Palmer (1982) suggested name "complexity" for the entropy of the weights w_α in the pure state decomposition $\Gamma(s) = \sum_\alpha w_\alpha(\Gamma)\Gamma_\alpha(s)$:

$$-\sum_\alpha w_\alpha \ln w_\alpha$$

and that it might be extensive (proportional to volume) for some Γ.

But pure states are defined only in infinite size, so this is problematic.

Recently, Holler and N.R. (2020) proposed instead using the mutual information between the spin configuration $\mathcal{S}_\Lambda$ in Λ, taking values $\mathcal{S}_\Lambda = s|_\Lambda$, and the pure state $\mathcal{A}$, taking values $\mathcal{A} = \alpha$, to define complexity relative to the window Λ:

$$K(\Lambda) = I(\mathcal{S}_\Lambda; \mathcal{A})_\Gamma = \sum_{s|_\Lambda, \alpha} w_\alpha \Gamma_\alpha(s|_\Lambda) \ln \frac{w_\alpha \Gamma_\alpha(s|_\Lambda)}{w_\alpha \Gamma(s|_\Lambda)}$$

which is well-defined and finite. Then one can study the limit $\Lambda \to \infty$.

This mutual information represents the info gained about which pure state one is in from an observation of the random variable $\mathcal{S}_\Lambda$ (inc due to spont. symm. breaking).

In the metastate construction, we can define the random variable $\mathcal{G}$ that takes values $\mathcal{G} = \Gamma$ with distribution $\kappa(\Gamma)$, take the expectation over Γ, and call it

$$K_\Gamma(\Lambda) = I(\mathcal{S}_\Lambda; \mathcal{A}|\mathcal{G}) = \sum_\Gamma \kappa(\Gamma) \sum_{s|_\Lambda, \alpha} w_\alpha(\Gamma)\Gamma_\alpha(s|_\Lambda) \ln \frac{w_\alpha(\Gamma)\Gamma_\alpha(s|_\Lambda)}{w_\alpha(\Gamma)\Gamma(s|_\Lambda)}$$

(all for given J), the complexity of a typical Gibbs state. N.R., 2021

Similarly, we can define the complexity of the metastate to be the mutual information of spins with $\mathcal{G}$,

$$K_\kappa(\Lambda) = I(\mathcal{S}_\Lambda; \mathcal{G}),$$

the info about the Gibbs state gained from an observation. (Zero for trivial metastate.)

Finally, we can also define the complexity of the MAS to be the mutual information in ρ rather than a Γ,

$$K_\rho(\Lambda) = I(\mathcal{S}_\Lambda; \mathcal{A})_\rho.$$

One can show that $K_\rho(\Lambda) = K_\Gamma(\Lambda) + K_\kappa(\Lambda)$. N.R., 2021

Bounds on (expected) complexities:

All three obey same upper bounds (window a hypercube Λ_W of side W):

1) For Ising spins, nearest neighbor interactions,

$$K_\Gamma(W) \leq 2dW^{d-1} \ln 2$$

for all $T \geq 0$ (counting boundary conditions, essentially).

(Holler, N.R., 2020)

2) Ising or vector spins, arbitrary interactions

$$\mathbf{E}\, K_\Gamma(W) \leq \frac{1}{T^2} \sum_{i \in \Lambda_W, j \in \Lambda_W^c} \mathrm{Var}\, J_{ij}$$

for all $T > 0$. Again $\propto 2dW^{d-1}$ for short-range interactions, but also gives interesting forms for long-range, e.g. power-law.

These give bounds on ζ'; $\zeta' \geq 1$ for short-range. (N.R., 2021)

Conclusion:

--- The metastate is an additional layer of structure in spin glass theory: a probability distribution on Gibbs states, derived from finite size

--- The metastate concept translates into measurable correlations, with quantitative predictions for an exponent in finite size

--- This is a basic, direct, way to assess existence of many pure states in MAS. More direct than trying to establish properties of overlap distributions, ultrametricity, etc. Works at T=0 also.

--- "Complexity", i.e. multiplicity of pure states, can be defined more precisely as mutual information, and bounds on the growth exponent can be proved.

--- Widely believed (not proved) that RSB is correct in EA at least for d > 6

--- Still needed is proof of triviality or nontriviality in lower dimensions
--- hard problem!
--- some results in one dimension, long-range N.R., 2018
(also partial results in EA in 2D: Arguin, Damron, Newman, Stein, 2010)