Some TTbar deformed mathematics

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The Art of Mathematical Physics: Saleur at 60 $+ \epsilon$

'The Last Time I Saw Paris'

“One should treat mathematical physics through the rectangle, the annulus, and the torus”

This work came out of trying to show that certain objects of “TTbar”-deformed 2d CFT, which should retain their modular invariance/covariance properties, in fact do so, and then realizing that the proof had nothing to do with CFT but applies to many of the modular and Jacobi forms of 19th C mathematics.

We shall proceed by considering ‘TTbar”-deformed CFT in a rectangle, torus and annulus as exemplars of these.

A well known theta function identity attributed to Jacobi but known to Gauss:

$$\vartheta_3(0; \mathrm{i}\delta) \equiv \sum_{n\in\mathbb{Z}} e^{-\pi n^2 \delta} = \delta^{-1/2} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/\delta}$$

- easily proved using Poisson sum formula
- ‘modular form’ of weight $\frac{1}{2}$ under $S : \tau \to -1/\tau$ with $\tau = \mathrm{i}\delta$

A deformed theta function:

$$\vartheta_3^\beta(0;\mathrm{i}\delta) \equiv \sum_{n\in\mathbb{Z}} \frac{\sqrt{(1+\sqrt{1+2\beta n^2})/2}}{\sqrt{1+2\beta n^2}}\, e^{-\pi\delta\frac{\sqrt{1+2\beta n^2}-1}{\beta}}$$

satisfies the identity

$$\vartheta_3^\beta(0;\mathrm{i}\delta) = \delta^{-1/2}\,\vartheta_3^{\beta/\delta^2}(0;\mathrm{i}/\delta)$$

The “$T\overline{T}$” deformation of a 2d QFT

A family of non-local field theories $\mathcal{T}^\lambda$ where the infinitesimal flow $\mathcal{T}^\lambda \to \mathcal{T}^{\lambda+\delta\lambda}$ corresponds to adding a term

$$(\delta\lambda)\int \det T^\lambda(x) d^2x = \tfrac{1}{2}(\delta\lambda)\epsilon^{ik}\epsilon^{jl}\int T^\lambda_{ij}(x) T^\lambda_{kl}(x) d^2x$$

to the action, where T^λ_{ij} is the stress tensor of the deformed theory. Induces left-right scattering in the UV.

“Solvable” because:

- factorization $T^\lambda_{ij}(x)T^\lambda_{kl}(x) \to T^\lambda_{ij}(x)T^\lambda_{kl}(x+y)$ [Zam 2004]
- $\equiv$ coupling to random (flat) metric [Dubovsky *et al* 2018; JC 2018]
- $\det T^\lambda$ is a total derivative of a semi-local field [JC 2019]

- $\Rightarrow$ "state-dependent" change of coordinates: [Conti *et al* 2018; JC 2019]

$$\tfrac{1}{2}(\delta\lambda)\epsilon^{ik}\epsilon^{jl}\int T^{\lambda}_{ij}(x)T^{\lambda}_{kl}(x)d^2x = \int T^{\lambda}_{ij}(x)\delta g^{ij}(x)d^2x$$

$$\text{where} \qquad \delta g^{ij} = \tfrac{1}{2}(\delta\lambda)\epsilon^{ik}\epsilon^{jl}T^{\lambda}_{kl}$$

Symmetry and conservation of T^{λ}_{kl} imply

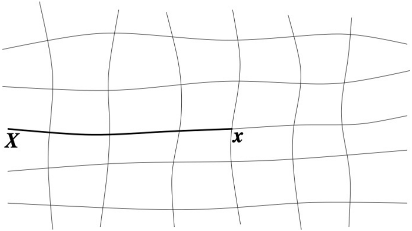
$$\delta g^{00,1} = \delta g^{01,0} \qquad \delta g^{11,0} = \delta g^{10,1}$$

so that

$$\delta g^{ij}(x) = (\delta x)^{i,j} + (\delta x)^{j,i} \equiv \text{diffeomorphism: } x \to x + \delta x(x)$$

$T\overline{T}$ deformation $\equiv$ coordinate change $x \to x^\lambda(x)$ where

$$\partial_\lambda x^\lambda(x)^i = -\int_X^x \epsilon^{ik}\epsilon^{jl}T^\lambda_{kl}(y)dy_j = -\epsilon^{ik} \times \text{flux of } T^\lambda_{k.} \text{ across } (X, x)$$

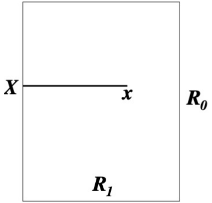


$$\partial_\lambda x^\lambda(x)^0 = -\text{momentum flux across } (X, x)$$

$$\partial_\lambda x^\lambda(x)^1 = \text{energy flux across } (X, x)$$

WHAT DOES ANY OF THIS MEAN?

Example: $R_1 \times R_0$ rectangle (conformal boundary conditions)



Taking $x_1 = R_1$,

$\partial_\lambda R_1^\lambda = N_1^\lambda =$ normal stress across $x_0 =$ const. (= energy in 1+1 dim.)

Similarly $\partial_\lambda R_0^\lambda = N_0^\lambda$. In the fixed stress ensemble, evolution is *linear* $R_i^\lambda = R_i^0 + \lambda N_i$

A 19th C digression: Cauchy, Lagrange, Euler and others meet at the Académie

Cauchy: "All this talk about stress – why not think of $x \to x^\lambda(x)$ as the deformation of an elastic solid, for which I have a marvelous theory of stress and strain?"

JC: "Well, yes, but this TTbar solid has infinite Poisson's ratio"

"Non, ce n'est pas possible! Ces physiciens du 21ème siècle sont tous fous"

[walks off muttering]

Lagrange steps forward: "But these are just the equations of a 2d fluid in my particle picture with $\vec{N} =$ velocity, $\lambda =$ time."

Euler interrupts: "But the fixed strain ensemble then corresponds to MY picture $\partial_\lambda \vec{N} = -(\vec{N}.\vec{\nabla})\vec{N}$"

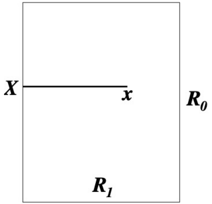
Burgers (from the 20th C): "But that's my equation too, and for $\lambda > 0$ the initial conditions with $N \propto 1/R$ in your CFTs will lead to shock formation!"

21st C theorists: "Zamolodchikov!! This must be a 'Hagedorn' singularity – a maximum temperature!"

Carnot et al.:

"Non, c'est pas possible, ces physiciens du 21ème siècle sont tous fous..."

Back to the rectangle:



$$Z^{\mathrm{CFT}}(R_0, R_1) = R_1^{c/4}\eta(q)^{-c/2} \quad \text{where } q = e^{-2\pi R_0/R_1} = e^{-2\pi\delta}$$

where $\eta(q) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m)$ [Kleban, Vassileva 1991]

Modular S-symmetry $Z^{\mathrm{CFT}}(R_0, R_1) = Z^{\mathrm{CFT}}(R_1, R_0) \Leftrightarrow$

$$\eta(1/\delta) = \delta^{1/2}\eta(\delta) \qquad \eta \text{ is a modular form of weight } \tfrac{1}{2}$$

[T-symmetry under $\delta \to \delta + \mathrm{i} \Rightarrow$ exact quantum recurrences in the CFT]

$$Z^{\mathrm{CFT}}(R_0, R_1) = R_1^{c/4} \sum_{n=0} a_n(c) q^{-\frac{c}{48}+n}$$

$$= \sum_n |B_n^{\mathrm{CFT}}(R_1)|^2 e^{-E_n^{\mathrm{CFT}}(R_1)R_0} \equiv \int e^{-N_0 R_0}\, \rho^{\mathrm{CFT}}(N_0, R_1)\, dN_0$$

[Fixed strain (R_0, R_1) ensemble $\to$ mixed (N_0, R_1) ensemble]

from which we conjecture that the deformed partition function is, at least formally,

$$Z^{\lambda}(R_0, R_1) = \int e^{-N_0 R_0}\, \rho^{\mathrm{CFT}}(N_0, R_1 + \lambda N_0)\, dN_0$$

If so, it should be that $Z^{\lambda}(R_0, R_1) = Z^{\lambda}(R_1, R_0)$.

Not obvious, but note that, formally,

$$\partial_\lambda Z^{\lambda}(R_0, R_1) = -\partial_{R_0} \partial_{R_1} Z^{\lambda}(R_0, R_1)$$

which respects the symmetry.

HOW TO MAKE MATHEMATICAL SENSE OF THIS?

Theorem.

Suppose that $q = e^{-2\pi\delta}$ *and* $F^0(\delta) = \sum_{n=0}^{\infty} a_n q^{\Delta+n}$ *converges for* $|q| < 1$ *and satisfies* $F^0(1/\delta) = \delta^k F^0(\delta)$.

Then

$$F^\alpha(\delta) \equiv \sum_{n=0}^{\infty} a_n \frac{[(1+\sqrt{1+4\pi\alpha(\Delta+n)\delta})/2]^{1-k}}{\sqrt{1+4\pi\alpha(\Delta+n)\delta}} e^{-\frac{1}{2\alpha}(\sqrt{1+4\pi\alpha(\Delta+n)\delta}-1)}$$

satisfies $F^\alpha(1/\delta) = \delta^k F^\alpha(\delta)$.

Notes

1. equivalent to $Z^\lambda(R_0, R_1) = Z^\lambda(R_1, R_0)$ with $\alpha = \lambda/(R_0 R_1)$, $\delta = R_0/R_1$, but now F^0 is **not necessarily a CFT object**
2. we lose symmetry under $T : \delta \to \delta - \mathrm{i}$ (LR scattering destroys recurrences) but see later
3. if $\Delta < 0$, rhs converges only for $\delta > 4\pi\alpha|\Delta|$ corresponding to ‘Hagedorn’ singularity in $n = 0$ term on lhs.

Outline of proof:

$$\text{Let} \qquad Z^0(R_0, R_1) \equiv R_1^{-k} F^0(\delta = R_0/R_1)$$

Laplace transform

$$\Omega^0(s, R_1) = \int_0^\infty e^{-sR_0'} Z^0(R_0', R_1) dR_0' = R_1^{1-k} \int_0^\infty e^{-s\delta' R_1} F^0(\delta') d\delta'$$

$$Z^0(R_0, R_1) = \int_C e^{sR_0} \Omega^0(s, R_1) \frac{ds}{2\pi \mathrm{i}} \quad \text{so } \rho(N_0, R_1) = 2\mathrm{Im}\, \Omega^0(s, R_1)|_{s=-N_0}$$

So **define**

$$Z^\lambda(R_0, R_1) \equiv \int_C e^{sR_0} \Omega^0(s, R_1 - \lambda s) \frac{ds}{2\pi \mathrm{i}}$$

$$= \int_C e^{s\delta R_1} [R_1 - \lambda s]^{1-k} \int_0^\infty e^{-s\delta'(R_1 - \lambda s)} F^0(\delta') d\delta' \frac{ds}{2\pi \mathrm{i}}$$

In terms of dimensionless quantities

$$F^{\alpha}(\delta) = \int_C e^{s\delta}[1-\alpha\delta s]^{1-k} \int_0^{\infty} e^{-s\delta'(1-\alpha\delta s)} F^0(\delta') d\delta' \frac{ds}{2\pi \mathrm{i}}$$

2 ways to manipulate this:

1. for each term $\propto e^{-2\pi(\Delta+n)\delta'}$ in $F^0(\delta')$, integrating over δ' gives

$$\frac{e^{s\delta}[1-\alpha\delta s]^{1-k}}{2\pi(\Delta+n)+s(1-\alpha\delta s)}$$

and picking up the pole at $s=-(1/2\alpha\delta)(\sqrt{1+8\pi\alpha\delta(\Delta+n)}-1)$ gives the shifted exponent and the prefactor.

2. completing the square in s gives

$$F^{\alpha}(\delta) = \int_0^{\infty} K^{\alpha}(\delta, \delta')(\delta'/\delta)^{k/2} F^0(\delta')(d\delta'/\delta')$$

where

$$K^{\alpha}(\delta, \delta') = e^{-(\delta-\delta')^2/4\alpha\delta\delta'} \int_{-\infty}^{\infty} [(\delta/\delta')^{1/2} + (\delta'/\delta)^{1/2} + \mathrm{i}t]^{1-k} e^{-\alpha t^2} dt$$

satisfies $K^{\alpha}(1/\delta, 1/\delta') = K^{\alpha}(\delta, \delta')$. This implies the theorem $F^{\alpha}(1/\delta) = \delta^k F^{\alpha}(\delta)$.

- a kind of Weierstrass transform, but strongly peaked as δ or $\delta' \to 0$ or ∞
- many choices of K^{α} have these properties, but only this one gives a discrete deformed spectrum for N_0

Restoring the symmetry under $T : \delta \to \delta - \mathrm{i}$

- as a model, consider a 1-point function $\langle \Phi \rangle(\vec{R}_0, \vec{R}_1)$ on a torus $\mathbb{C}/\mathbb{L}(\vec{R}_0, \vec{R}_1)$

$$\langle \Phi \rangle(\vec{R}_0, \vec{R}_1)^{\mathrm{CFT}} = |\vec{R}_1|^{-k} F^0(\delta) \qquad (k = h_\Phi)$$

  where $\tau = \mathrm{i}\delta = \mathrm{i}(\delta_0 + \mathrm{i}\delta_1)$ is the modular parameter.
- S-invariance $\langle \Phi \rangle(\vec{R}_0, \vec{R}_1)^{\mathrm{CFT}} = \langle \Phi \rangle(\vec{R}_1, -\vec{R}_0)^{\mathrm{CFT}}$ implies

$$F^0(1/\delta) = |\delta|^k F^0(\delta)$$

- T-invariance $\langle \Phi \rangle(\vec{R}_0, \vec{R}_1)^{\mathrm{CFT}} = \langle \Phi \rangle(\vec{R}_0, \vec{R}_1 + \vec{R}_0)^{\mathrm{CFT}}$ implies $F^0(\delta_0, \delta_1 + 1) = F^0(\delta_0, \delta_1)$ so

$$F^0(\delta) = \sum_{p \in \mathbb{Z}} F^0_p(\delta_0) e^{2\pi \mathrm{i} p \delta_1}$$

A similar construction now shows that $F^{\alpha}(1/\delta) = |\delta|^{k} F^{\alpha}(\delta)$ but with p-dependent modified exponents

$$\sqrt{1 + 4\pi\alpha(\Delta + n)\delta} \to \sqrt{1 + 4\pi\alpha(\Delta + n)\delta_0 + 4\pi^2\alpha^2 p^2 \delta_0^2}$$

Note that a purely (anti-)holomorphic form with $\Delta + n = \pm p$ does not deform

Deformed Virasoro Characters

Annulus = rectangle with periodic bc around x^0.
Partition function

$$Z^{\mathrm{CFT}}(R_0, R_1) = \sum_a n_a \chi_a(q = e^{-2\pi\delta}) = \sum_a n_a \sum_b S_a^b \chi_b(e^{-2\pi/\delta})$$

where $\delta = R_0/2R_1$.

But now it is Z on annulus with a marked point X which satisfies a PDE:

$$\partial_\lambda\big(Z^\lambda(R_0, R_1)/R_0\big) = -\partial_{R_1}\partial_{R_2}\big(Z^\lambda(R_0, R_1)/R_0\big)$$

Modifies the deformation to

$$\chi_a^\alpha(\delta) = \delta \int_C e^{s\delta} \int_0^\infty e^{-s\delta'(1-\alpha\delta s)} \chi_a(\delta') \frac{d\delta'}{\delta'} \frac{ds}{2\pi i}$$

The integral over δ' leads to $\log(s - s_-)(s - s_+)$ and wrapping the s-contour around $\log(s - s_-)$ gives a term $e^{s_- \delta}$.
Only the exponents are deformed, the integer coefficients remain the same, as expected.

$$\chi_a^\alpha(\delta) = \int_0^\infty (\pi/\alpha\delta\delta')^{1/2} e^{-(\delta-\delta')^2/4\alpha\delta\delta'} (\delta/\delta') \chi_a(\delta') d\delta'$$

These extra factors complicate the S-transformation rule, so $\chi_a^\alpha(\delta) \neq \sum_b S_a^b \chi_b^\alpha(1/\delta)$ (deformed boundary states no longer Ishibashi)

However, on the torus [Datta, Jiang 2020]

$$[\chi_a \chi_{\bar{a}}]^\alpha(\delta) = \int_{\mathbb{H}} (\pi/\alpha) e^{-|\delta-\delta'|^2/4\alpha\delta_0\delta_0'} [\chi_a \chi_{\bar{a}}](\delta') \frac{d^2\delta'}{{\delta_0'}^2}$$

and then

$$[\chi_a \chi_{\bar{a}}]^\alpha(\delta) = \sum_{b,\bar{b}} S_a^b S_{\bar{a}}^{\bar{b}} [\chi_a \chi_{\bar{a}}]^\alpha(1/\delta)$$

The proof of this serves as a model for identities on deformed products of modular and Jacobi forms.

“Theta functions obey a bewildering number and variety of identities.” (*Elliptic Curves* [Mckean & Moll 1997])

Some mathematics/physics consequences and questions

- generalizes modular forms to functions with irrational power spectrum
- yields new(?) relations for arithmetic functions, e.g. partitions $P(N)$
- new (integrable?) lattice models with weights involving deformed theta functions
- other ‘solvable’ deformations?
- it’s all (mainly French) early 19th C mathematical physics...

HAPPY BIRTHDAY HUBERT!!

A birthday present....

Recently discovered work allegedly by Cézanne on his little known visit to Santa Barbara