

Morphology of field reversals in turbulent dynamos

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Abstract – We show that the modes involved in the dynamics of reversals of the magnetic field generated by the flow of an electrically conducting fluid in a spherical domain, strongly depend on the magnetic Prandtl number P_m , *i.e.*, the ratio between viscous and magnetic diffusivities. For $P_m \sim 1$, the axial dipolar field reverses in the presence of a strong equatorial dipolar component, whereas for $P_m < 0.5$, field reversals mostly involve axisymmetric modes, axial magnetic dipole and quadrupole coupled through a broken symmetry of the flow. Using symmetry arguments, we write a dynamical system for these three modes that qualitatively reproduces the main features of the reversals observed in direct simulations for small P_m .

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Introduction. – The generation of magnetic field by the flow of an electrically conducting fluid, *i.e.*, the dynamo effect, has been mostly studied to understand the magnetic fields of planets and stars [1]. The Earth and the Sun provide the best documented examples: they both involve a spatially coherent large-scale component of magnetic field with well-characterized dynamics. The Earth's dipole is roughly stationary on time scales much larger than the ones related to the flow in the liquid core, but randomly flips. Field reversals also occur for the Sun but nearly periodically. It has been observed recently that the magnetic field generated by a von Karman flow of liquid sodium (VKS experiment) can display either periodic or random reversals [2]. The ability of all these very different dynamos to reverse polarity is their most striking property. This is obviously related to the $\mathbf{B} \rightarrow -\mathbf{B}$ symmetry of the equations of magnetohydrodynamics (MHD), implying that if a magnetic field \mathbf{B} is a solution for a given flow, $-\mathbf{B}$ is another solution with the same flow. However, this does not explain how these two solutions can be connected as time evolves. Direct simulations of the MHD equations for a convective flow in a rotating sphere have displayed several possible mechanisms: a magnetic field with a quadrupolar symmetry midway through the transition between the two opposite dipoles has been reported [3]. It has been found that this type of reversals are triggered by events that break the equatorial symmetry of the flow [4]. However, it has also been

observed in other simulations that each reversal “can differ greatly in various aspects from others” [5]. In contrast, reversals observed in the VKS experiment have been found to be very robust. Despite large turbulent fluctuations of the flow, successive reversal trajectories can be superimposed and thus display the same morphology [2]. A possible explanation for these different behaviors relies on the value of the magnetic Prandtl number $P_m = \mu_0 \sigma \nu$, where μ_0 is the magnetic permeability of vacuum, σ is the electrical conductivity of the fluid and ν is its kinematic viscosity. Thus, P_m is the ratio between viscous and magnetic diffusivities. The numerical simulations of the Earth's magnetic-field reversals have been performed so far for P_m of order one or larger. For liquid metals in laboratory experiments or Earth's core, $P_m \sim 10^{-5}$ – 10^{-6} cannot be achieved in numerical simulations owing to computational limitations. The purpose of this work is to show that even on the relatively narrow range of P_m accessible to direct simulations, the magnetic modes involved in field reversals, and thus the reversals' morphology, are indeed strongly modified when P_m is varied. To wit, we simulate a flow driven in a spherical geometry by volumic forces. We observe reversals of the generated magnetic field for a wide range of parameters. We show that reversals of the magnetic dipole involve a coupling with a quadrupole mode only for P_m small enough. In that case, we present a minimal three-mode model for the reversal dynamics. When P_m is larger, as in most previous numerical

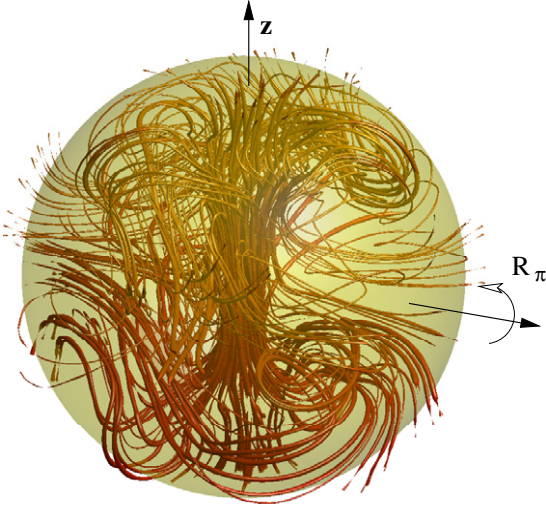


Fig. 1: Magnetic-field lines obtained with a symmetric forcing ($C=1$) for $R_m = 300$ and $P_m = 1$. Note that the field involves a dipolar component with its axis aligned with the axis z of rotation of the propellers.

simulations, many modes are involved in the dynamics and in particular an equatorial dipole.

Direct simulation of reversals of the magnetic field. – The MHD equations are integrated in a spherical geometry for the solenoidal velocity \mathbf{v} and magnetic \mathbf{B} fields,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \pi + \nu \Delta \mathbf{v} + \mathbf{f} + \frac{1}{\mu_0 \rho} (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \Delta \mathbf{B}. \quad (2)$$

In the above equations, ρ is the density, μ_0 is the magnetic permeability and σ is the electrical conductivity of the fluid. The forcing is $\mathbf{f} = f_0 \mathbf{F}$, where $F_\phi = s^2 \sin(\pi s b)$, $F_z = \varepsilon \sin(\pi s c)$, for $z > 0$, and opposite for $z < 0$. We use polar coordinates (s, ϕ, z) , normalized by the radius of the sphere a . F_ϕ generates counter-rotating flows in each hemisphere, while F_z enforces a strong poloidal circulation. The forcing is only applied in the region $0.25a < |z| < 0.65a$, $s < s_0$. In the simulations presented here, $s_0 = 0.4$, $b^{-1} = 2s_0$ and $c^{-1} = s_0$. This forcing has been used to model the Madison experiment [6]. It is invariant by rotation of an angle π along any axis in the mid-plane, hereafter called the \mathcal{R}_π symmetry. In order to break it, we consider in the present study a forcing of the form $C\mathbf{f}$, where $C=1$ for $z < 0$ but can be different from one for $z > 0$. This describes two propellers that counter-rotate at different frequencies. Although performed in a spherical geometry, this simulation involves a mean flow with a topology similar to that of the VKS experiment. We solve the above system of equations using the Parody numerical code [7]. This code was originally developed in the context of the geodynamo (spherical shell) and we have modified it to make it suitable for a full sphere.

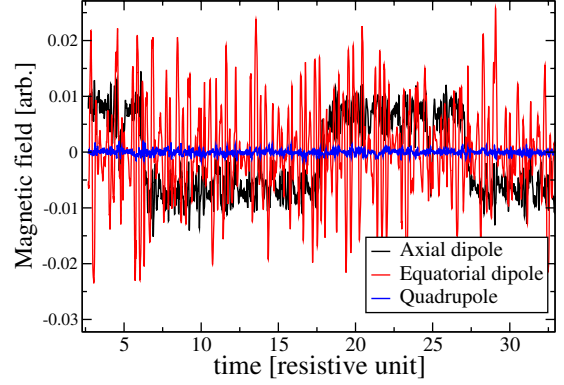


Fig. 2: Time recording of the axial dipolar magnetic mode (in black), the axial quadrupolar mode (in blue) and the equatorial dipole (in red) for $R_m = 300$, $P_m = 1$ and $C = 2$.

We use the same dimensionless numbers as in [6], the magnetic Reynolds number $R_m = \mu_0 \sigma a \max(|\mathbf{v}|)$, and the magnetic Prandtl number $P_m = \nu \mu_0 \sigma$. The kinetic Reynolds number is then $Re = R_m / P_m$. In contrast to [8] where the dynamo onset has been studied for laminar flows (small Re) for large enough P_m , all the simulations of the present study are performed for $Re > 300$, so that an axial dipole is obtained for symmetric forcing ($C=1$) as displayed in fig. 1. We observe that the magnetic field bifurcates supercritically. For R_m 50% above the threshold, the ratio of the magnetic to kinetic energy is 10^{-3} . This is in the same range as for the VKS experiment but significantly less than for the geodynamo.

We next break the \mathcal{R}_π symmetry of the forcing. Time recordings of some components of the magnetic field are displayed in fig. 2 for $R_m = 300$, $P_m = 1$ and $C = 2$, which means that one of the propellers is spinning twice as fast as the other one. We observe that the axial dipolar component (in black) randomly reverses sign. The phases with given polarity are an order of magnitude longer than the duration of a reversal that corresponds to an Ohmic diffusion time. The magnetic field strongly fluctuates during these phases because of hydrodynamic fluctuations. It also displays excursions or aborted reversals, *i.e.*, the dipolar component almost vanishes or even slightly changes sign but then grows again with its direction unchanged. All of these features are observed in paleomagnetic records of the Earth's magnetic field [9] and also in the VKS experiment [2]. Despite their different geometries, a common feature of the simulations and the VKS experiment is that time dependent dynamics are observed only when the two propellers counter-rotate at different frequencies. However, the simulation for $P_m = 1$ also displays strong differences with the VKS experiment. The equatorial dipole is the mode with the largest fluctuations whereas the axial quadrupolar component is an order of magnitude smaller than the dipolar modes. In addition, it does not seem to be coupled to the axial dipolar component. In contrast, the axial dipolar and quadrupolar modes are the dominant ones

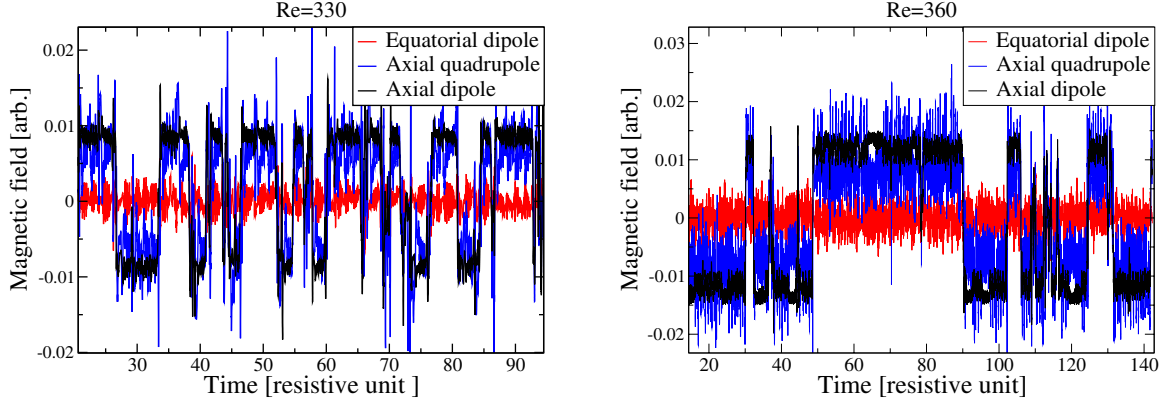


Fig. 3: Time recordings of the axial dipole (black), the axial quadrupole (blue) and the equatorial dipole (red). Left: $R_m = 165$, $P_m = 0.5$ and $C = 1.5$. Right: $R_m = 180$, $P_m = 0.5$ and $C = 2$.

and strongly coupled in the reversals observed in the VKS experiment [10].

We now turn to simulations using smaller values of P_m , thus introducing a distinction between the viscous and ohmic timescales. The time evolution of the magnetic modes for $R_m = 165$, $P_m = 0.5$ and $C = 1.5$ is represented in fig. 3 (left). It differs significantly from the previous case ($P_m = 1$). First of all, the quadrupole is now a significant part of the field, and reverses together with the axial dipole. The equatorial dipole remains comparatively very weak and unessential to the dynamics. One can argue that R_m has also been modified when changing P_m from 1 to 0.5. However, for $P_m = 1$, we have observed the same dynamics of reversals when R_m has been decreased down to $R_m = 220$ below which reversals are not observed any more.

The high amount of fluctuations observed in these signals is related to hydrodynamic fluctuations. One could be tempted to speculate that a higher degree of hydrodynamic fluctuations necessarily yields a larger reversal rate. Such is in fact not the case. A more sensible approach could be to try to relate the rate of reversals to the amount of fluctuations of the magnetic modes in a phase with given polarity. Increasing R_m from 165 to 180 does yield larger fluctuations as shown in fig. 3 (right). However the reversal rate is in fact lowered because C was modified to $C = 2$. This clearly shows that the asymmetry parameter C plays an important role in addition to the fluctuations of the magnetic field. For $P_m = 0.5$, reversals occur only in a restricted region, $1.1 < C < 2.5$, which is also a feature of the VKS experiment. The reversal rate strongly depends on the value of C with respect to these borders, in good agreement with the model presented in [11]. Thus, the transition from a stationary regime to a reversing one is not generated by an increase of hydrodynamic fluctuations.

Let us now investigate the details of a polarity reversal (fig. 4). Interestingly the dipolar and quadrupolar components do not vanish simultaneously. Instead the decrease of the dipole is associated with a sudden increase of the

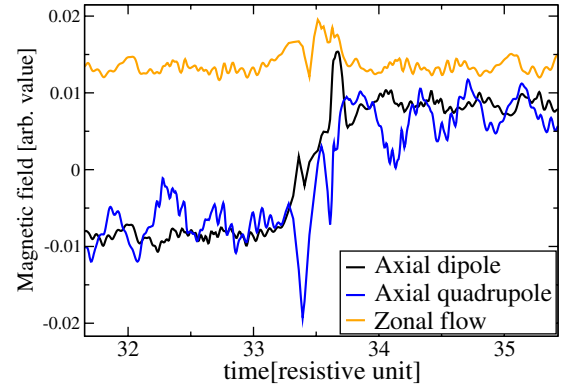


Fig. 4: Time recordings of the axial dipole (black), the axial quadrupole (blue) and zonal velocity (yellow) during a reversal. $R_m = 165$, $P_m = 0.5$ and $C = 1.5$.

quadrupolar component, related with a burst of activity in the axisymmetric zonal velocity mode breaking the \mathcal{R}_π symmetry (*i.e.*, linearly coupling the dipolar and quadrupolar families). The quadrupole quickly decays as the dipole recovers with a reversed polarity. Immediately after the reversal the dipole systematically overshoots its mean value during a polarity interval. This behavior of the magnetic modes is typical of reversals obtained with this value of P_m and is in excellent agreement with the model presented in [11].

A three-mode model of magnetic-field reversals in the limit of small P_m . – These direct numerical simulations illustrate the role of the magnetic Prandtl number in the dynamics of reversals. When P_m is of order one, the magnetic perturbations due to the advection of magnetic-field lines by the velocity field, evolve with a time scale similar to the one of the velocity fluctuations. We thus expect these two fields to be strongly coupled. Modification of the magnetic-field lines due to their advection by a local fluctuation of the flow can then trigger a reversal of the field [12]. This type of scenario has been observed in some direct numerical simulations

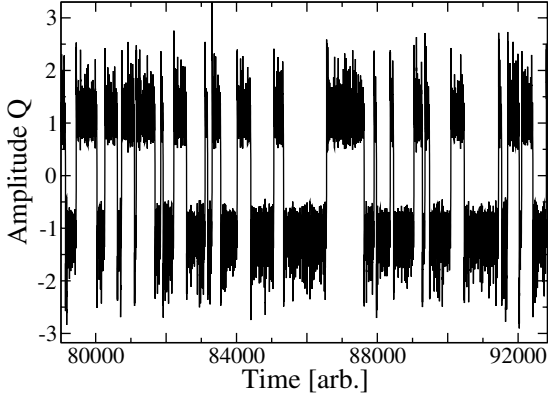


Fig. 5: Numerical integration of the amplitude equations (3)–(5). Time recording of the amplitude of the quadrupolar mode for $\mu = 0.119$, $\nu = 0.1$ and $\Gamma = 0.9$.

of the geodynamo, usually performed with P_m larger than one [13]. When P_m is small, magnetic perturbations decay much faster and we expect only the largest-scale magnetic modes to govern the dynamics. To illustrate this argument in a more quantitative way, we have computed the correlation r of the axial dipole and quadrupole for $Re = 330$ and $0.3 < P_m < 1$. r is obtained by dividing the covariance of the real parts of both modes by the product of their standard deviations. When P_m is decreased from 1 to 0.3, r increases from 0.4 to 0.9.

We now write the simplest dynamical system that involves the three modes that display correlation in the low P_m simulations: the dipole D , the quadrupole Q , and the zonal velocity mode V that breaks the \mathcal{R}_π symmetry. These modes transform as $D \rightarrow -D$, $Q \rightarrow Q$ and $V \rightarrow -V$ under the \mathcal{R}_π symmetry. Keeping nonlinear terms up to quadratic order, we get

$$\dot{D} = \mu D - VQ, \quad (3)$$

$$\dot{Q} = -\nu Q + VD, \quad (4)$$

$$\dot{V} = \Gamma - V + QD. \quad (5)$$

A nonzero value of Γ is related to a forcing that breaks the \mathcal{R}_π symmetry, *i.e.* propellers rotating at different speeds. The dynamical system (3)–(5) with $\Gamma = 0$ occurs in different hydrodynamic problems and has been analyzed in detail [14]. The relative signs of the coefficients of the nonlinear terms have been taken such that the solutions do not diverge when $\mu > 0$ and $\nu < 0$. Their modulus can be taken equal to one by appropriate scalings of the amplitudes. The velocity mode is linearly damped and its coefficient can be taken equal to -1 by an appropriate choice of the time scale. Note that similar equations were obtained with a drastic truncation of the linear modes of MHD equations [15]. However, in that context μ should be negative and the damping of the velocity mode was discarded, thus strongly modifying the dynamics.

This system displays reversals of the magnetic modes D and Q for a wide range of parameters. A time recording

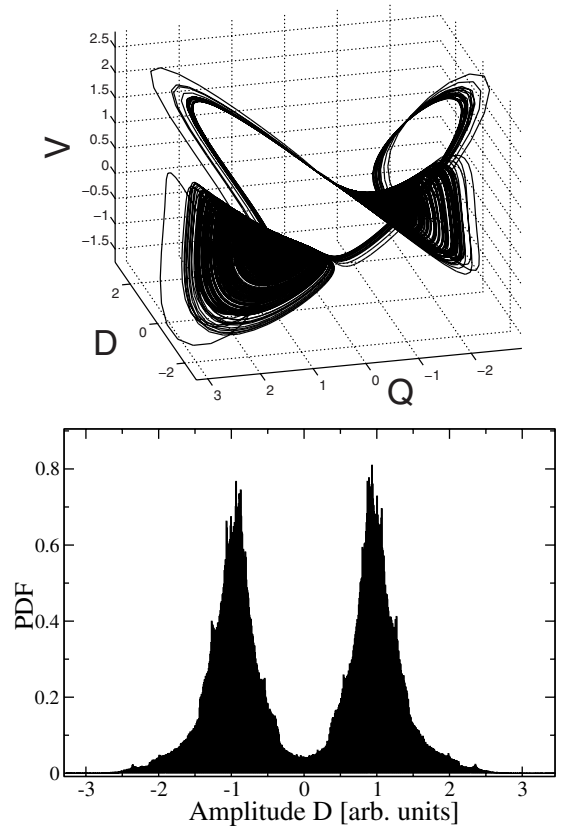


Fig. 6: Numerical integration of the amplitude equations (3)–(5). Top: three-dimensional phase space. Bottom: probability density function of D ($\mu = 0.119$, $\nu = 0.1$ and $\Gamma = 0.9$).

is shown in fig. 5. The mechanism for these reversals results from the interaction of the modes D and Q coupled by the broken \mathcal{R}_π symmetry when $V \neq 0$. It is thus similar to the one described in [11] but also involves an important difference: keeping the damped velocity mode into the system generates chaotic fluctuations. It is thus not necessary to add external noise to obtain random reversals. This system is fully deterministic as opposed to the one of reference [11]. The phase space displayed in fig. 6 (top) shows the existence of chaotic attractors in the vicinity of the $\pm \mathbf{B}$ quasi-stationary states. When these symmetric attractors are disjoint, the magnetic field fluctuates in the vicinity of one of the two states $\pm \mathbf{B}$ and the dynamo is statistically stationary. When μ is varied, these two attractors can get connected through a crisis mechanism, thus generating a regime with random reversals [16].

We do not claim that this minimal low order system fully describes the direct simulations presented here. For instance, in the case of exact counter-rotation ($C = 1$, *i.e.* $\Gamma = 0$), eqs. (3)–(5) do not have a stable stationary state with a dominant axial dipole. The different solutions obtained when μ is increased cannot capture all the dynamo regimes of the VKS experiment or of the direct simulations when R_m is increased away from the

threshold. Taking into account cubic nonlinearities provides a better description of the numerical results for $P_m = 0.5$. However, this three-mode system with only quadratic nonlinearities involves the basic ingredients of the reversals observed in the present numerical simulations for low enough values of the magnetic Prandtl number. Geomagnetic reversals have been modeled since a long time using low-dimensional dynamical systems [15,17] or equations involving a noisy forcing [11,18]. The above model (3)–(5) does not rely on an external noise source to generate random reversals. Compared to previous deterministic models [15,17], it displays dynamical and statistical properties that are much closer to the ones of our direct simulations at low P_m or of the VKS experiment. For instance, the direct recordings of D or Q do not involve the growing oscillations characteristic of reversals displayed by the Rikitake or Lorenz systems [17] but absent in dynamo experiments or in direct simulations. Correspondingly, the probability density function of D displayed in fig. 6 (bottom) is also much closer to the one obtained in experiments or direct simulations than the one of previous deterministic models [16].

Conclusion. – We have shown that different types of random reversals of a dipolar magnetic field can be obtained by varying the magnetic Prandtl number in a rather small range around $P_m = 1$. This may be of interest for simulations of the magnetic field of the Earth that have been mostly restricted to values of P_m larger than one.

We have observed that axisymmetric dipolar and quadrupolar modes decouple from the other magnetic modes while getting coupled together when P_m is decreased. Although we do not claim to have reached an asymptotic low P_m regime which is out of reach of the present computing power, we observe that dominant axial dipole and quadrupole are also observed in the VKS experiment for which $P_m \sim 10^{-5}$.

Finally, using symmetry arguments, we have written a three-mode dynamical system for the dipolar and quadrupolar magnetic modes coupled together with a zonal velocity mode. This model differs from the one in [11] because the random field reversals are not induced by external noise but by deterministic chaos. It also differs from the previous low-dimensional models, the shape of the reversals and their probability density function being much closer to the experimental observations.

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