Part 2 $\boxed{\text{Bogoliubov theory. Non-condensed fraction}}$

I) Considered regime

- Homogeneous system, $d=3$ dimensions (as in Part 1: box, Volume $V = L_1 L_2 L_3$, periodic boundary conditions)
- Contact model for the interaction: $V(\vec{r}) = g\,\delta^3(\vec{r})$, $g = 4\pi a$
- Weakly interacting regime: $\rho a^3 \ll 1$
- $a > 0$

Rem: It would be more rigorous to use the lattice model (defined in 2.3.3 of the Lectures), with $a \ll l \ll \rho^{-\frac{1}{3}}$. The calculation would become slightly more heavy, and the result for the non-condensed fraction f_{nc} would be the same (in the limit $a \ll l \ll \rho^{-1/3}$). Therefore we use simply the contact model.

- We will take the thermodynamic limit (t.l.) at the end of the calculations
- We only assume $T < T_c$ where $T_c \underset{(\rho a^3 \ll 1)}{\simeq} T_c^{ideal}$:

$$T_c^{ideal} = 2\pi \left[\frac{\rho}{\zeta(3/2)}\right]^{2/3} \quad \text{(see G. Batrouni's Lectures)}$$

Indeed:

- In principle, Bogoliubov theory requires $T \ll T_c$ (see 3.1 in the Lectures)
- But even for $\begin{cases} T \sim T_c \\ \text{and} \\ T < T_c \end{cases}$, we expect that

the Bogoliubov result for the non-condensed fraction, f_{nc}^{Bog}, is very close to the exact one, f_{nc}^{exact}.

Indeed, for $T \sim T_c$, $f_{nc}^{ideal} = \left(\frac{T}{T_c}\right)^{3/2} \sim 1$.

Therefore we expect for $\rho a^3 \ll 1$:

$$\begin{cases} f_{nc}^{exact}(T/T_c, \rho a^3) \simeq f_{nc}^{exact}(T/T_c, 0) = f_{nc}^{ideal}(T/T_c) \\ f_{nc}^{Bog}(T/T_c, \rho a^3) \simeq f_{nc}^{Bog}(T/T_c, 0) \end{cases}$$

We will see that, for $a=0$, Bogoliubov theory becomes equivalent to the Approximation of a Never Empty Condensate (ANEC): $f_{nc}^{Bog}\left(\frac{T}{T_c}, 0\right) = f_{nc}^{ideal, ANEC}\left(\frac{T}{T_c}\right) \simeq f_{nc}^{ideal}\left(\frac{T}{T_c}\right)$

[1.2 in the Lectures].

So, $f_{nc}^{exact} \simeq f_{nc}^{Bog} \simeq f_{nc}^{ideal}$ $\left[\text{for } \rho a^3 \ll 1 \text{ and } \begin{cases} T \sim T_c \\ T < T_c \end{cases}\right]$

II Bogoliubov Hamiltonian in the homogeneous case.

We first have to determine the condensate wavefunction ϕ (3.3 in the Lectures). The answer is $\boxed{\phi = \frac{1}{\sqrt{V}}}$. We will prove it in Part 3, I.5.a (in a slightly more general case).

Thus, $\boxed{E(\phi) = \frac{N}{2}\rho g}$ and $\mu = \frac{d}{dN}E(\phi) \Rightarrow \boxed{\mu = \rho g}$

(One can then check that the Gross-Pitaevski equation is satisfied.)

The Bogoliubov Hamiltonian (3.5 in the Lectures) then becomes:

$$\boxed{H_{Bog} = \frac{N\rho g}{2} + \int d\vec{r}\left[\Lambda^+(\vec{r})\left(-\frac{1}{2}\Delta_{\vec{r}} + \rho g\right)\Lambda(\vec{r}) + \frac{\rho g}{2}\left(\Lambda^+(\vec{r})^2 + \Lambda(\vec{r})^2\right)\right]} \quad (1)$$

where we remind that $\Lambda^+(\vec{r}) = a^+_{Q|\vec{r}\rangle}\hat{A}_\phi$, $Q = \mathbb{1} - |\phi\rangle\langle\phi|$, and A_ϕ removes one particle from the condensate,

$$A_\phi \left|(n_{\vec{k}})_{\vec{k}}\right\rangle = \begin{cases} 0 & \text{if } n_{\vec{0}} = 0 \\ \left|(n_{\vec{k}} - \delta_{\vec{k},\vec{0}})_{\vec{k}}\right\rangle & \text{if } n_{\vec{0}} \geq 1. \end{cases}$$

In the present homogeneous case, it is convenient to go to momentum space:

Def $\boxed{\Lambda_{\vec{k}} \equiv a_{\vec{k}} A_\phi^+}$, for $\vec{k} \neq \vec{0}$

Rem $\Rightarrow \Lambda^+_{\vec{k}} = A_\phi a^+_{\vec{k}} = a^+_{\vec{k}} A_\phi$.

Ex 1

S.t. $$H_{Bog} = \frac{N\rho g}{2} + \sum_{\vec{k}\neq\vec{0}}\left[\left(\frac{k^2}{2} + \rho g\right)\Lambda^+_{\vec{k}}\Lambda_{\vec{k}} + \frac{\rho g}{2}\left(\Lambda^+_{\vec{k}}\Lambda^+_{-\vec{k}} + \Lambda_{-\vec{k}}\Lambda_{\vec{k}}\right)\right].$$

Hint: express $\Lambda^+(\vec{r})$ in terms of the $\Lambda^+_{\vec{k}}$'s and use (1)

<u>Solution</u> : $Q|\vec{r}\rangle = \sum_{\vec{k}} |\vec{k}\rangle\langle\vec{k}|Q|\vec{r}\rangle = \sum_{\vec{k}\neq\vec{0}} |\vec{k}\rangle\langle\vec{k}|\vec{r}\rangle$,

$$\Lambda^+(\vec{r}) = a^+_{Q|\vec{r}\rangle}\, A_\phi = \sum_{\vec{k}\neq\vec{0}} \langle\vec{k}|\vec{r}\rangle\, a^+_{\vec{k}}\, A_\phi$$

$$\Lambda^+(\vec{r}) = \sum_{\vec{k}\neq\vec{0}} \langle\vec{k}|\vec{r}\rangle\, \Lambda^+_{\vec{k}} \; ; \quad \Lambda(\vec{r}) = \sum_{\vec{k}\neq\vec{0}} \langle\vec{r}|\vec{k}\rangle\, \Lambda_{\vec{k}} .$$

- $$\int d\vec{r}\; \Lambda^+(\vec{r})\left(-\frac{1}{2}\Delta_{\vec{r}} + \rho g\right)\Lambda(\vec{r}) = \sum_{\substack{\vec{k}\neq\vec{0}\\ \vec{k}'\neq\vec{0}}} \left(\frac{k'^2}{2}+\rho g\right)\Lambda^+_{\vec{k}}\Lambda_{\vec{k}'}\int d\vec{r}\,\langle\vec{k}|\vec{r}\rangle\cdot\langle\vec{r}|\vec{k}'\rangle$$
$$= \sum_{\vec{k}\neq\vec{0}} \left(\frac{k^2}{2}+\rho g\right)\Lambda^+_{\vec{k}}\Lambda_{\vec{k}}$$
- $$\int d\vec{r}\; \Lambda^+(\vec{r})^2 = \sum_{\substack{\vec{k}\neq\vec{0}\\ \vec{k}'\neq\vec{0}}} \Lambda^+_{\vec{k}}\Lambda^+_{\vec{k}'}\int d\vec{r}\,\langle\vec{k}|\vec{r}\rangle\underbrace{\langle\vec{k}'|\vec{r}\rangle}_{=\langle\vec{r}|-\vec{k}'\rangle}$$
$$= \sum_{\vec{k}\neq\vec{0}} \Lambda^+_{\vec{k}}\Lambda^+_{-\vec{k}}$$

The result follows. $\square$

We can then check that the $\Lambda_{\vec{k}}$'s are "almost bosonic" operators

Ex 2 | S.t.: • $[\Lambda_{\vec{q}}, \Lambda_{\vec{q}'}] = 0$

• $[\Lambda_{\vec{q}}, \Lambda^+_{\vec{q}'}]\,|(n_{\vec{k}})\rangle = \delta_{\vec{q},\vec{q}'}\,|(n_{\vec{k}})\rangle$ if $n_{\vec{0}} \geq 1$

<u>Solution</u> :

- $[\Lambda_{\vec{q}}, \Lambda_{\vec{q}'}] = [A^+_\phi a_{\vec{q}}, A^+_\phi a_{\vec{q}'}] = (A^+_\phi)^2[a_{\vec{q}}, a_{\vec{q}'}] = 0.$
- $$[\Lambda_{\vec{q}}, \Lambda^+_{\vec{q}'}] = \underbrace{A^+_\phi A_\phi}_{\mathbb{1}\text{ if acting on a state } |(n_{\vec{k}})\rangle \text{ with } n_{\vec{0}}\geq 1} a_{\vec{q}}\, a^+_{\vec{q}'} - \underbrace{A_\phi A^+_\phi}_{\mathbb{1}}\, a^+_{\vec{q}'}\, a_{\vec{q}} = [a_{\vec{q}}, a^+_{\vec{q}'}] = \delta_{\vec{q},\vec{q}'}$$

$\square$

In the following we will use

$$\boxed{[\Lambda_{\vec{q}}, \Lambda^+_{\vec{q}'}] = \delta_{\vec{q},\vec{q}'}}, \qquad (2)$$

"forgetting" about the restriction $n_{\vec{0}} \geq 1$, i.e. we do an ANEC.

<u>Note</u> : We now present an equivalent but more formal way to perform this ANEC. This is not very important in practise, and the reader may skip this note. We go back to the general case of an arbitrary ONB $(|u_\alpha\rangle)_\alpha$ of the single-particle Hilbert space, and we

choose this basis such that $|u_0\rangle = |\phi\rangle$ (in the homogeneous case, one can take $\alpha = \vec{k}$, $|u_\alpha\rangle = |\vec{k}\rangle$, $|u_0\rangle = |\vec{k}=\vec{0}\rangle$).

We define $\quad \Lambda_\alpha^+ \equiv a_\alpha^+ A_\phi \quad (\Lambda_\alpha = a_\alpha A_\phi^+)$. (for $\alpha \neq 0$)

We recall that the physical N-particle Hilbert space $\mathcal{E}_N$ is generated by the Fock-state ONB:

$$\left\{ |(n_\alpha)_\alpha\rangle \,/\, \forall \alpha,\ n_\alpha \in \mathbb{N} \text{ and } \sum_\alpha n_\alpha = N \right\}.$$

We define an extended Hilbert space $\tilde{\mathcal{E}}_N$ which includes additional states having a negative n_0: an ONB of $\tilde{\mathcal{E}}_N$ is $\left\{ |(n_\alpha)\rangle \,/\, \forall \alpha \neq 0,\ n_\alpha \in \mathbb{N} \,;\ n_0 \in \mathbb{Z} \,;\ \text{and} \sum_\alpha n_\alpha = N \right\}$.

[Of course, these additional states are unphysical, but they are convenient in order to formalize the ANEC (and thus simplify calculations) as we shall see; their presence is not a problem as long as they only appear with a negligible probability and do not affect the result (e.g. for f_{nc}).]

A natural definition of an extended operator $\hat{A}_\phi$ acting on $\tilde{\mathcal{E}}_N$ is: $\quad \hat{A}_\phi |(n_\alpha)\rangle = |(n_\alpha - \delta_{\alpha,0})\rangle$

$$\left(\Rightarrow \quad \hat{A}_\phi^+ |(n_\alpha)\rangle = |(n_\alpha + \delta_{\alpha,0})\rangle \right)$$

Let us call $\mathcal{E}_N^{n_0 \geq 1}$ the subspace of $\mathcal{E}_N$ where the condensate is not empty, i.e. an ONB of $\mathcal{E}_N^{n_0\geq 1}$ is

$$\left\{ |(n_\alpha)\rangle \,/\, \forall \alpha,\ n_\alpha \in \mathbb{N} \text{ and } \sum_\alpha n_\alpha = N \text{ and } n_0 \geq 1 \right\}.$$

Then clearly:
$$\begin{cases} \hat{A}_\phi |\psi\rangle = A_\phi |\psi\rangle & \text{if } |\psi\rangle \in \mathcal{E}_N^{n_0 \geq 1} \\ \hat{A}_\phi^+ |\psi\rangle = A_\phi^+ |\psi\rangle & \text{if } |\psi\rangle \in \mathcal{E}_N. \end{cases}$$

Natural extended operators $\hat{\Lambda}_\alpha$ are then defined by:

$$\hat{\Lambda}_\alpha = a_\alpha \hat{A}_\phi^+ \qquad (\text{for } \alpha \neq 0)$$

$$\left[\Rightarrow \quad \hat{\Lambda}_\alpha^+ = a_\alpha^+ \hat{A}_\phi \right]$$

Clearly, $\begin{cases} \hat{\Lambda}_\alpha^+ = \Lambda_\alpha^+ \text{ on } \mathcal{E}_N^{n_0 \geq 1} \ (\text{i.e. } \hat{\Lambda}_\alpha^+|\psi\rangle = \Lambda_\alpha^+|\psi\rangle \text{ if } |\psi\rangle \in \mathcal{E}_N^{n_0\geq 1}) \\ \hat{\Lambda}_\alpha = \Lambda_\alpha \text{ on } \mathcal{E}_N. \end{cases}$

Now, these $\hat{\Lambda}_\alpha$'s have the advantage of being "exactly bosonic" operators :
$$\begin{cases} [\hat{\Lambda}_\alpha, \hat{\Lambda}_{\alpha'}] = 0 \\ [\hat{\Lambda}_\alpha, \hat{\Lambda}^+_{\alpha'}] = \delta_{\alpha\alpha'} . \end{cases}$$

Defining
$$\begin{cases} \hat{n}_\alpha = \hat{\Lambda}^+_\alpha \hat{\Lambda}_\alpha \quad \text{if} \quad \alpha \neq 0 \\ \hat{n}_0 = N - \sum_{\alpha\neq 0} \hat{\Lambda}^+_\alpha \hat{\Lambda}_\alpha , \end{cases}$$

we have $\hat{n}_\alpha = n_\alpha$ on $\tilde{\mathcal{E}}_N$.

Our physical Hamiltonian, H_{Bog}, is a combination of Λ_α's and Λ^+_α's (see Ex 1 for the homogeneous case). Thus we can define an extended Hamiltonian $\tilde{H}_{Bog}$, by replacing Λ_α by $\hat{\Lambda}_\alpha$ and Λ^+_α by $\hat{\Lambda}^+_\alpha$ in H_{Bog}. The ANEC then consists in working with $\tilde{H}_{Bog}$, $\tilde{\mathcal{E}}_N$ and the $\hat{n}_\alpha$'s (instead of H_{Bog}, $\mathcal{E}_N$ and the n_α's).

Going back to the homogeneous case,
$$\tilde{H}_{Bog} = \frac{N\rho g}{2} + \sum_{\vec{k}\neq\vec{0}} \left[\left(\frac{k^2}{2} + \rho g\right) \hat{\Lambda}^+_{\vec{k}} \hat{\Lambda}_{\vec{k}} + \frac{\rho g}{2} \left(\hat{\Lambda}^+_{\vec{k}} \hat{\Lambda}^+_{-\vec{k}} + \hat{\Lambda}_{-\vec{k}} \hat{\Lambda}_{\vec{k}} \right) \right].$$

For $g = 0$, the density operator is
$$\tilde{\sigma} \propto e^{-\beta \tilde{H}_{Bog}} \propto e^{-\beta \sum_{\vec{k}\neq\vec{0}} \frac{k^2}{2} \hat{\Lambda}^+_{\vec{k}} \hat{\Lambda}_{\vec{k}}}$$

and we see that our extended formalism is equivalent to an ideal gas, for the $\vec{k} \neq \vec{0}$ modes only, in the grand-canonical ensemble, with a "chemical potential" zero; this is equivalent with the ANEC for the ideal gas, 1.2 in the Lectures.

In the following, we will abusively write $\Lambda_{\vec{k}}$ instead of $\hat{\Lambda}_{\vec{k}}$.

III Diagonalisation of the Bogoliubov Hamiltonian

Instead of applying a general diagonalisation procedure for quadratic Hamiltonians (which is useful in the trapped case, see Y. Castin, Lecture Notes from Les Houches

1999, edited in 2000 by Springer, available on cond-mat/0105058), we will take advantage of the simplicity of the homogeneous case to "guess" the form of the result, before checking it (similarly to G. Batrouni's Lecture).

Let us introduce parameters: $u_k \in \mathbb{R}^+$, $v_k \in \mathbb{R}$, $\varepsilon_k \in \mathbb{R}^+$, $E_0 \in \mathbb{R}$, and define the operators

$$\left\{ \begin{array}{l} H_B \equiv E_0 + \sum_{\vec{k} \neq \vec{0}} \varepsilon_k \, b^+_{\vec{k}} b_{\vec{k}} \\ b^+_{\vec{k}} \equiv u_k \, \Lambda^+_{\vec{k}} + v_k \, \Lambda_{-\vec{k}} \end{array} \right.$$

$\boxed{\text{Ex 3}}$ s.t. $\boxed{[b_{\vec{k}}, b_{\vec{k}'}] = 0 \qquad (3)}$

Solution: $[b_{\vec{k}}, b_{\vec{k}'}] = u_k v_{k'} [\Lambda_{\vec{k}}, \Lambda^+_{-\vec{k}'}] + v_k u_{k'} [\Lambda^+_{-\vec{k}}, \Lambda_{-\vec{k}'}]$

$$= u_k v_k \left(\delta_{\vec{k},-\vec{k}'} - \delta_{\vec{k},-\vec{k}'} \right) = 0 .$$

$\boxed{\text{Ex 4}}$ Consider the equations:

$$(i) \left\{ \begin{array}{ll} [b_{\vec{k}}, b^+_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} & (4) \\ H_B = H_{Bog} & \end{array} \right.$$

$$(ii) \left\{ \begin{array}{ll} \varepsilon_k = \sqrt{\frac{k^2}{2}\left(\frac{k^2}{2} + 2\rho g\right)} & (ii_1) \\ u_k = \sqrt{\frac{1}{2}\left(\frac{k^2/2 + \rho g}{\varepsilon_k} + 1\right)} & (ii_2) \\ v_k = \sqrt{\frac{1}{2}\left(\frac{k^2/2 + \rho g}{\varepsilon_k} - 1\right)} & (ii_3) \\ E_0 = \frac{N\rho g}{2} - \sum_{\vec{k} \neq \vec{0}} \varepsilon_k v_k^2 & (ii_4) \end{array} \right.$$

- Show that $(i) \Rightarrow (ii)$
- Check that $(ii) \Rightarrow (i)$

Solution: Using the bosonic commutation relations obeyed by the $\Lambda_{\vec{k}}$'s, we get that

$$[b_{\vec{k}}, b^+_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} \iff \boxed{u_k^2 - v_k^2 = 1 \qquad (i'a)}$$

Similarly,

$$H_B = E_0 + \sum_{\vec{k}\neq\vec{0}} \varepsilon_k \left(u_k \Lambda_k^+ + v_k \Lambda_{-k}\right)\left(u_k \Lambda_k + v_k \Lambda_{-k}^+\right)$$

$$= E_0 + \sum_{\vec{k}\neq\vec{0}} \varepsilon_k \left[(u_k^2 + v_k^2)\Lambda_k^+ \Lambda_k + v_k^2 + u_k v_k \left(\Lambda_k^+ \Lambda_{-k}^+ + \Lambda_{-k}\Lambda_k\right)\right]$$

Thus, using the H_{Bog} found in Ex 1,

$$H_B = H_{Bog} \iff (i'b) \begin{cases} \frac{N\rho g}{2} = E_0 + \sum_{\vec{k}\neq\vec{0}} \varepsilon_k v_k^2 & (i'b_1) \\ \frac{k^2}{2} + \rho g = \varepsilon_k (u_k^2 + v_k^2) & (i'b_2) \\ \frac{\rho g}{2} = \varepsilon_k u_k v_k & (i'b_3) \end{cases}$$

In summary, $(i) \iff \begin{cases} (i'a) \\ \text{and} \\ (i'b) \end{cases}$.

It remains to show that $\begin{cases} (i'a) \\ (i'b) \end{cases} \iff (ii)$.

$(\Longrightarrow)$: Let us assume $(i'a)$ and $(i'b)$, and show (ii).

$(i'a)$ and $(i'b_2)$ imply: $u_k^2 = \frac{1}{2}\left(\frac{k^2/2 + \rho g}{\varepsilon_k} + 1\right)$

$$v_k^2 = \frac{1}{2}\left(\frac{k^2/2 + \rho g}{\varepsilon_k} - 1\right).$$

Since $u_k \in \mathbb{R}_+$, we get (ii_2).

From $(i'b_3)$, $v_k > 0$, and (ii_3) follows.

Injecting (ii_2) and (ii_3) into $(i'b_3)$ gives (ii_1).

$(\Longleftarrow)$: From (ii_2) and (ii_3), we clearly satisfy $(i'a)$.

(ii) also implies: $\varepsilon_k u_k v_k = \frac{1}{2}\sqrt{\left(\frac{k^2}{2} + \rho g + \varepsilon_k\right)\left(\frac{k^2}{2} + \rho g - \varepsilon_k\right)}$

$$= \frac{1}{2}\sqrt{\left(\frac{k^2}{2} + \rho g\right)^2 - \varepsilon_k^2} = \frac{\rho g}{2}. \quad \boxtimes$$

In the following we take $\varepsilon_k, u_k, v_k, E_0$ as in (ii).

Thus (i) holds. In particular, $\boxed{H_{Bog} = E_0 + \sum_{\vec{k}\neq\vec{0}} \varepsilon_k b_k^+ b_k}$ $(*)$

<u>Rem</u>: Actually, (ii_4) gives a divergent result for E_0.
This pathology is due to the use of a contact model for the interaction. It does not affect the quantities we wish to calculate. If we would wish to calculate E_0,

we would use another model, such as the lattice model.

Ex 5 We define $|\Psi_0\rangle \equiv \mathcal{N} \exp\left[\frac{1}{2}\sum_{\vec{q}\neq\vec{0}} \alpha_q \Lambda^+_{\vec{q}} \Lambda^+_{-\vec{q}}\right] |N:\phi\rangle$, where $|N:\phi\rangle = \frac{1}{\sqrt{N!}} (a^+_{\vec{0}})^N |0\rangle$ is the pure condensate and $\mathcal{N}$ is a normalisation factor.
S.t. for a proper choice of the α_q's:
$$b_{\vec{k}} |\Psi_0\rangle = 0, \quad \forall \vec{k} \neq \vec{0}.$$

<u>Solution</u>: $\Lambda_{\vec{k}} |\Psi_0\rangle = \mathcal{N} \exp\left[\frac{1}{2}\sum_{\vec{q}\notin\{\vec{0},\vec{k},-\vec{k}\}} \alpha_q \Lambda^+_{\vec{q}} \Lambda^+_{-\vec{q}}\right]$
$$\cdot \Lambda_{\vec{k}} \exp\left[\alpha_k \Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}}\right] |N:\phi\rangle.$$

Using the relation $[A, BC] = [A,B]C + B[A,C]$, one finds
$$[\Lambda_{\vec{k}}, (\Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}})^n] = [\Lambda_{\vec{k}}, \Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}}](\Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}})^{n-1} + \{(n-1) \text{ similar terms}\}$$
$$= \Lambda^+_{-\vec{k}} (\Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}})^{n-1} \times n.$$

Then, $\Lambda_{\vec{k}} \exp\left[\alpha_k \Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}}\right] |N:\phi\rangle = [\Lambda_{\vec{k}}, \exp[\alpha_k \Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}}]] |N:\phi\rangle$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_k)^n}{n!} [\Lambda_{\vec{k}}, (\Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}})^n] |N:\phi\rangle$$
$$= \alpha_k \Lambda^+_{-\vec{k}} \sum_{n=1}^{\infty} \frac{(\alpha_k)^{n-1}}{(n-1)!} (\Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}})^{n-1} |N:\phi\rangle$$
$$= \alpha_k \Lambda^+_{-\vec{k}} \exp\left[\alpha_k \Lambda^+_{\vec{k}} \Lambda^+_{-\vec{k}}\right] |N:\phi\rangle.$$

So, $\Lambda_{\vec{k}} |\Psi_0\rangle = \alpha_k \Lambda^+_{-\vec{k}} |\Psi_0\rangle$ and
$$b_{\vec{k}} |\Psi_0\rangle = 0 \iff (u_k \Lambda_{\vec{k}} + v_k \Lambda^+_{-\vec{k}}) |\Psi_0\rangle = 0$$
$$\Leftarrow u_k \alpha_k + v_k = 0 \Leftarrow \boxed{\alpha_k = -\frac{v_k}{u_k}} \quad \square$$

<u>Def</u> Given an integer $n_{\vec{k}} \in \mathbb{N}$ for each $\vec{k} \neq \vec{0}$, we set
$$|\{n_{\vec{k}}\}\rangle \equiv \prod_{\vec{k}\neq\vec{0}} \frac{1}{\sqrt{n_{\vec{k}}!}} (b^+_{\vec{k}})^{n_{\vec{k}}} |\Psi_0\rangle \tag{5}$$

<u>Note</u>: This is an ONB of $\tilde{\mathcal{E}}_N$.

$\boxed{\text{Ex 6}}$ S.t.: $\begin{cases} H_{Bog} |\{n_{\vec{k}}\}\rangle = \left[E_0 + \sum_{\vec{k}\neq\vec{0}} \varepsilon_k n_{\vec{k}}\right] |\{n_{\vec{k}}\}\rangle & (6) \\ \hat{\vec{P}} \; |\{n_{\vec{k}}\}\rangle = \left[\sum_{\vec{k}\neq\vec{0}} \vec{k}\, n_{\vec{k}}\right] |\{n_{\vec{k}}\}\rangle & (7) \end{cases}$ **2.9**

where $\hat{\vec{P}}$ is the total momentum operator of the gas
$\left(\hat{\vec{P}} = \sum_{\vec{k}} \vec{k}\, a^+_{\vec{k}} a_{\vec{k}} \right)$

<u>Solution</u>: • From Ex. 5, $H_{Bog} |\psi_0\rangle = E_0 |\psi_0\rangle$.
Since moreover the $b_{\vec{k}}$'s have bosonic commutation relations,
$b^+_{\vec{q}} b_{\vec{q}} |\{n_{\vec{k}}\}\rangle = n_{\vec{q}} |\{n_{\vec{k}}\}\rangle$, and (6) follows [from (*)].
• We first check that $\hat{\vec{P}} |\psi_0\rangle = \vec{0}$. Then we use the fact that $b^+_{\vec{k}}$ increases the total momentum by $\vec{k}$, because it is a linear combination of $\Lambda^+_{\vec{k}}$ and $\Lambda_{-\vec{k}}$. (This is actually a way to "guess" that $b^+_{\vec{k}}$ is a <u>linear combination</u> of $\Lambda^+_{\vec{k}}$ and $\Lambda_{-\vec{k}}$.) Then, (7) follows. ☑

From (6) [and the fact that $(|\{n_{\vec{k}}\}\rangle)$ is an ONB] it follows that $|\psi_0\rangle$ is the ground state of H_{Bog}.
At thermodynamic equilibrium in the canonical ensemble, the density operator of the gas is $\hat{\sigma}_c \propto e^{-\beta H}$, and within Bogoliubov theory $\hat{\sigma}_c^{Bog} \propto e^{-\beta H_{Bog}}$, i.e. from (*)

$$\hat{\sigma}_c^{Bog} \propto \exp\left[-\beta \sum_{\vec{k}\neq\vec{0}} \varepsilon_k b^+_{\vec{k}} b_{\vec{k}}\right] \quad (8)$$

<u>Important observation</u>:
• (3,4) $\Rightarrow$ we can interpret $b^+_{\vec{k}}$ as the creation operator of a bosonic particle, which is called quasi-particle
• (5) and Ex 5 $\Rightarrow$ The (extended) Hilbert space of the physical gas can be interpreted as a Fock state of quasi-particles (restricted to $\vec{k}\neq\vec{0}$ modes), $|\psi_0\rangle$ being the vacuum for the quasi-particles.

• (6,7) ⟹ Each quasi-particle has energy ε_k and momentum $\vec{k}$. The quasi-particles do not interact. (2.10)

[For example, the state $b^+_{\vec{k}} |\Psi_0\rangle$ is a single-quasi-particle state; it is an eigenstate of H_{Bog} and $\hat{\vec{P}}$ with eigenvalues ε_k and $\vec{k}$.]

• (8) ⟹ The quasi-particle gas is at thermodynamic equilibrium in the grand-canonical ensemble, with a chemical potential $\mu_{\text{quasi-particles}} = 0$.

<u>Consequence</u>: We can use the results obtained for the ideal gas in the grand-canonical ensemble:

- $\langle b^+_{\vec{k}} b_{\vec{k}'} \rangle = \dfrac{\delta_{\vec{k},\vec{k}'}}{e^{\beta\varepsilon_k} - 1}$
- $\langle b_{\vec{k}} b_{\vec{k}'} \rangle = 0$
- Wick's theorem applies, and allows to calculate $\langle c_1 \dots c_k \rangle$ if c_j's are linear combinations of $b_{\vec{k}}$'s and $b^+_{\vec{k}}$'s.

<u>Two limiting regimes</u>:

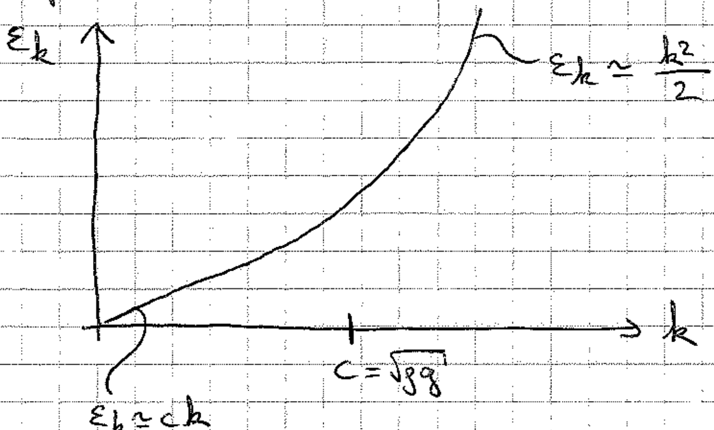
• <u>$k \gg \sqrt{\rho g}$</u>: $\varepsilon_k \simeq \frac{k^2}{2}$, $u_k \simeq 1$, $v_k \ll 1$, $b^+_{\vec{k}} \simeq \Lambda^+_{\vec{k}}$.

⟹ Everything becomes close to the ideal gas case. Interpretation: a high-energy particle is nearly insensitive to the interactions. = "<u>Free-particle regime</u>".

• <u>$k \ll \sqrt{\rho g}$</u>: $\varepsilon_k \underset{(T=0)}{\simeq} \sqrt{\rho g}\cdot k$. Actually, $c = \sqrt{\rho g}$ is the speed of sound in hydrodynamic theory (see e.g. Y. Castin's lecture, Les Houches 1999, cond-mat/0105058 for a justification of classical hydrodynamics from the Gross-Pitaevskii equation) since at $T=0$, $E \simeq \rho g \frac{N}{2} \Rightarrow P = -\frac{\partial E}{\partial V} = \frac{\rho^2 g}{2} \Rightarrow c^2 = \frac{\partial P}{\partial \rho} = \rho g$.

In this "<u>phonon regime</u>", $u_k \simeq v_k \gg 1$: the quasi-particles are not simply free particles (and $b^+_{\vec{k}} \not\simeq \Lambda^+_{\vec{k}}$).

The spectrum thus looks like :



IV Calculation of the non-condensed fraction

It is defined as $f_{nc} \equiv \langle \hat{\delta N} \rangle / N$, where $\hat{\delta N}$ is the number of non-condensed particles,

$$\hat{\delta N} = \sum_{\vec{k} \neq \vec{0}} a^+_{\vec{k}}\, a_{\vec{k}} = \sum_{\vec{k} \neq \vec{0}} \Lambda^+_{\vec{k}}\, \Lambda_{\vec{k}} \qquad (9)$$

[Note : in our formal version of the ANEC, $\hat{\delta N}$ is approximated by $\tilde{\delta N} = \sum_{\vec{k} \neq \vec{0}} \tilde{\Lambda}^+_{\vec{k}}\, \tilde{\Lambda}_{\vec{k}}$.]

We remember that :
$$\begin{pmatrix} b^+_{\vec{k}} \\ b_{-\vec{k}} \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} \Lambda^+_{\vec{k}} \\ \Lambda_{-\vec{k}} \end{pmatrix}.$$

Inverting this matrix,
$$\begin{pmatrix} \Lambda^+_{\vec{k}} \\ \Lambda_{-\vec{k}} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} b^+_{\vec{k}} \\ b_{-\vec{k}} \end{pmatrix} \qquad (10).$$

Inserting (10) into (9) gives :
$$\langle \hat{\delta N} \rangle = \sum_{\vec{k} \neq \vec{0}} \left[v_k^2 + (u_k^2 + v_k^2) \langle b^+_{\vec{k}}\, b_{\vec{k}} \rangle \right].$$

Thus,
$$\boxed{ f_{nc} \underset{t.l.}{=} f_{nc,0} + f_{nc,1}(T) \quad \text{where} \quad \begin{cases} f_{nc,0} = \dfrac{1}{\rho} \displaystyle\int \frac{d\vec{k}}{(2\pi)^3}\, v_k^2 \\ f_{nc,1}(T) = \dfrac{1}{\rho} \displaystyle\int \frac{d\vec{k}}{(2\pi)^3} (u_k^2 + v_k^2) \frac{1}{e^{\beta \varepsilon_k} - 1} \end{cases} } \qquad (11)$$

At $T=0$, $\langle b^+_{\vec{k}}\, b_{\vec{k}} \rangle = 0$ (no quasi-particle)
and $f_{nc}(T=0) = f_{nc,0}$.

Ex 7 | S.t. $f_{nc}(T=0) = \dfrac{8}{3\sqrt{\pi}} \sqrt{\rho a^3}$

Solution : $f_{nc}(T=0) = f_{nc,0} = \frac{4\pi}{\rho(2\pi)^3 \cdot 2} \int_0^\infty dk\, k^2 \cdot \left(\frac{k^2/2 + \rho g}{\varepsilon_k} - 1 \right)$. **(2.12)**

Setting $k = \sqrt{2\rho g}\, q$:

$$f_{nc,0} = \frac{4\pi}{\rho(2\pi)^3 \cdot 2} (2\rho g)^{3/2} \underbrace{\int_0^\infty dq\, q^2 \left(\frac{q^2+1}{q\sqrt{q^2+2}} - 1 \right)}_{= \frac{\sqrt{2}}{3}}$$

(e.g. using Maple if we are lazy!)

and the result follows. $\square$

Ex 8 S.t.

- **For $T \ll \rho g$** : the integral in (11) is dominated by the phonon regime, and: $f_{nc,1}(T) \simeq f_{nc}(T=0) \cdot \left(\frac{\pi}{2} \cdot \frac{T}{\rho g} \right)^2$

[Thus $f_{nc}(T) \simeq f_{nc}(T=0)$]

- **For $T \gg \rho g$** : the integral in (11) is dominated by the free particle regime, and

$$f_{nc}(T) \simeq f_{nc,1}(T) \simeq \frac{\zeta(3/2)}{\rho \lambda^3} = \left(\frac{T}{T_c^{ideal}} \right)^{\frac{3}{2}} = f_{nc}^{ideal, ANEC}(T)$$

Solution : From (11),

$$f_{nc,1}(T) = \frac{4\pi}{\rho(2\pi)^3} \int_0^\infty dk\, k^2 \cdot \frac{\frac{k^2}{2} + \rho g}{\varepsilon_k} \frac{1}{e^{\beta \varepsilon_k} - 1} . \quad (**)$$

For $T \ll \rho g$: Setting $k = \sqrt{2\rho g}\, q$,

$$f_{nc,1} = \frac{4\pi}{\rho(2\pi)^3} (2\rho g)^{3/2} \int_0^\infty dq \cdot q^2 \cdot \frac{q^2+1}{q\sqrt{q^2+2}} \frac{1}{e^{\beta \rho g\, q\sqrt{q^2+2}} - 1}$$

Since $\beta \rho g \gg 1$, the integrand is negligible as soon as $q \gtrsim 1$, and only the region $q \ll 1$ contributes (i.e. $k \ll \sqrt{\rho g}$: phonon regime). Thus

$$f_{nc,1} \simeq \sqrt{\rho a^3} \cdot \frac{4\pi (2 \cdot 4\pi)^{3/2}}{(2\pi)^3} \int_0^\infty dq\, q^2 \cdot \frac{1}{q\sqrt{2}} \frac{1}{e^{\beta \rho g\, q \sqrt{2}} - 1} .$$

Setting $x = \beta \rho g \sqrt{2}\, q$:

$$f_{nc,1} \simeq \sqrt{\rho a^3} \frac{2^{7/2}}{\sqrt{\pi}\sqrt{2}} (\beta \rho g \sqrt{2})^{-2} \underbrace{\int_0^\infty dx \cdot \frac{x}{e^x - 1}}_{= \pi^2/6}$$

$$= \sqrt{\rho a^3} \cdot \frac{4}{\sqrt{\pi}} \cdot \left(\frac{T}{\rho g} \right)^2 \cdot \frac{\pi^2}{6} = f_{nc,0} \cdot \left(\frac{T}{\rho g} \frac{\pi}{2} \right)^2 .$$

For $T \gg \rho g$: Setting $k = \sqrt{\frac{2}{\beta}}\, p$, $(**)$ becomes : (2.13)

$$f_{nc,1} = \frac{4\pi}{\rho\,(2\pi)^3} \left(\frac{2}{\beta}\right)^{3/2} \int_0^\infty dp \cdot p^2 \cdot \frac{p^2 + \beta\rho g}{p\sqrt{p^2 + 2\beta\rho g}} \; \frac{1}{e^{p\sqrt{p^2+2\beta\rho g}} - 1}$$

$$\underset{(\beta\rho g \ll 1)}{\simeq} \frac{\sqrt{2}}{\pi^2} \frac{T^{3/2}}{\rho} \underbrace{\int_0^\infty dp \cdot \frac{p^2}{e^{p^2} - 1}}_{= \frac{\sqrt{\pi}}{4} \zeta(3/2)}$$

$$f_{nc,1} \simeq \frac{\zeta(3/2)}{\rho} \sqrt{\frac{T^3}{2^3 \pi^3}} = \frac{\zeta(3/2)}{\rho \lambda^3} = f_{nc}^{ideal, ANEC}(T)$$

$$\left[f_{nc,1}(T) \underset{(T \gg \rho g)}{\gg} \sqrt{\rho a^3} \sim f_{nc,0} \right]$$

☑