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Ahead of the Fisher–KPP front

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Abstract

The solution h to the Fisher–KPP equation with a steep enough initial condition develops into a front moving at velocity 2, with logarithmic corrections to its position. In this paper we investigate the value $h(ct,t)$ of the solution ahead of the front, at time t and position ct, with $c>2$. That value goes to zero exponentially fast with time, with a well-known rate, but the prefactor depends in a non-trivial way of c, the initial condition and the nonlinearity in the equation. We compute an asymptotic expansion of that prefactor for velocities c close to 2. The expansion is surprisingly explicit and irregular. The main tool of this paper is the so-called 'magical expression' which relates the position of the front, the initial condition, and the quantity we investigate.

Keywords: partial differential equation, Fisher–KPP front, asymptotic behaviour

Mathematics Subject Classification numbers: 35K57, 35C20

1. Introduction

The study of fronts interpolating from a stable solution to an unstable solution is an important problem in mathematics, physics and biology; see for instance [AW75, McK75, DS88, Mur02, Saa03, Mun15]. The archetypal model is the Fisher–KPP equation [Fis37, KPP37]

$$\partial_t h = \partial_x^2 h + h - F(h), \tag{1}$$

where, throughout the paper, the nonlinearity $F(h)$ is assumed to satisfy the so-called 'Bramson's conditions' [Bra78, Bra83]:

$$\begin{aligned} &F\in C^1[0,1], \quad F(0)=0, \quad F(1)=1, \quad F'(h)\geqslant 0, \quad F(h)<h \text{ for } h\in(0,1), \\ &\qquad F'(h)=\mathcal{O}(h^p) \text{ for some } p>0 \text{ as } h\searrow 0. \end{aligned} \tag{2}$$

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(The choice $F(h) = h^2$ is often made.) One checks with these conditions that $h = 0$ is an unstable solution and $h = 1$ is a stable solution. We always assume implicitly that the initial condition h_0 satisfies $h_0 \in [0,1]$ and that $h_0(x)$ is not identically equal to 0 or to 1 for almost all x; this implies, by comparison, that $0 < h(x,t) < 1$ for all x and all $t > 0$. We also always assume for simplicity that $h_0(x) \to 1$ as $x \to -\infty$, but this could be significantly relaxed.

A famous result due to Bramson [Bra78, Bra83] (see also [HNRR13, Rob13]) states that,

$$h\left(2t - \frac{3}{2}\log t + z, t\right) \xrightarrow[t\to\infty]{} \omega(z-a) \qquad \text{iff} \int \mathrm{d}x\, h_0(x) x \mathrm{e}^x < \infty, \tag{3}$$

where $\omega(z)$, called the critical travelling wave, is a decreasing function interpolating from $\omega(-\infty) = 1$ to $\omega(+\infty) = 0$, and where the shift a depends on the initial condition. In words, if h_0 decays 'fast enough' at infinity, then the stable solution $h = 1$ on the left invades the unstable solution $h = 0$ on the right, and the position of the invasion front is $2t - \frac{3}{2}\log t + a$.

The critical travelling wave ω is the unique solution to

$$\omega'' + 2\omega' + \omega - F(\omega) = 0, \qquad \omega(-\infty) = 1, \qquad \omega(0) = \frac{1}{2}, \qquad \omega(+\infty) = 0, \tag{4}$$

and there exists $\tilde{\alpha} > 0$ and $\tilde{\beta} \in \mathbb{R}$ (depending on the choice of the nonlinearity $F(h)$) such that $\omega(z) = (\tilde{\alpha} z + \tilde{\beta})\mathrm{e}^{-z} + \mathcal{O}(\mathrm{e}^{-(1+q)z})$ as $z \to \infty$, where q is any number in $(0,p)$. We prefer to write the equivalent statement:

$$\omega(z-a) = (\alpha z + \beta)\mathrm{e}^{-z} + \mathcal{O}(\mathrm{e}^{-(1+q)z}) \quad \text{as } z \to \infty, \tag{5}$$

where $\alpha > 0$ and β now depend also on the initial condition h_0 through a and are given by $\alpha = \tilde{\alpha}\mathrm{e}^a$ and $\beta = (\tilde{\beta} - a\tilde{\alpha})\mathrm{e}^a$.

Let μ_t be the position where the front at time t has value $1/2$ (or the largest such position if there are more than one):

$$h(\mu_t, t) = \frac{1}{2}. \tag{6}$$

Bramson's result (3) implies that $\mu_t = 2t - \frac{3}{2}\log t + a + o(1)$ for large times if $\int \mathrm{d}x\, h_0(x) x \mathrm{e}^x < \infty$. Recent results indicate that a more precise estimate of μ_t can be given: if h_0 decays to zero 'fast enough' as $x \to \infty$, the position μ_t of the front is believed to satisfy:

$$\mu_t = 2t - \frac{3}{2}\log t + a - 3\frac{\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}[5 - 6\log 2]\frac{\log t}{t} + \mathcal{O}\left(\frac{1}{t}\right), \tag{7}$$

where we recall that a depends on the initial condition and on the choice of $F(h)$. The $1/\sqrt{t}$ correction is known as the Ebert–van Saarloos correction, from a non-rigorous physics paper [ES00]. This result was proved [NRR19] for $F(h) = h^2$ and h_0 a compact perturbation of the step function (*i.e.* h_0 differs from the step function $\mathbb{1}_{\{x<0\}}$ on a compact set); see also [BBHR16]. The $(\log t)/t$ correction was conjectured in [BBD17, BBD18] using universality argument and a implicit solution of a related model; it was proved in [Gra19] for $F(h) = h^2$ and h_0 a compact perturbation of the step function. Arguments given in [BBD17] suggest that the Ebert–van Saarloos term holds iff $\int \mathrm{d}x\, h_0(x) x^2 \mathrm{e}^x < \infty$ and that the $(\log t)/t$ terms holds iff $\int \mathrm{d}x\, h_0(x) x^3 \mathrm{e}^x < \infty$, for any choice of $F(h)$ satisfying (2).

Another quantity of interest is the value of $h(ct,t)$ for $c > 2$ and large t. For instance, recalling [McK75] that $h(x,t)$, for $F(h) = h^2$ and $h_0 = \mathbb{1}_{\{x<0\}}$, is the probability that the rightmost position at time t in a branching Brownian motion is located on the right of x, then $h(ct,t)$ would be the probability of a large deviation where this rightmost position sustains a velocity $c > 2$ some time t.

For a step initial condition, it is known [CR88, BH14, BH15, DMS16, BBCM22] that

$$h(ct,t) \sim \Phi(c)\frac{1}{\sqrt{4\pi t}}e^{(1-\frac{c^2}{4})t} \quad \text{as } t\to\infty, \text{ for } c>2, \tag{8}$$

for some continuous function $c\mapsto\Phi(c)$. This result holds for an arbitrary nonlinearity $F(h)$ [CR88]. (Note: the function $\Phi(c)$ in (8) is defined as in [DMS16]. The function $\tilde{C}(\sigma_e)$ in [BH15] and $C(\rho)$ in [BBCM22] are identical and related to $\Phi(c)$ by $\frac{1}{\sqrt{4\pi}}\Phi(c) = \frac{2}{c}C(\frac{c}{2})$.)

We show in proposition 3 below that (8) actually holds for any initial condition h_0 and any $c>2$ such that $\int dx\, h_0(x)e^{\frac{c}{2}x}<\infty$, with a function $\Phi(c)$ depending of course on h_0 and on $F(h)$.

The time dependence in (8) is not surprising: the solution h_{lin} to the linearised Fisher–KPP equation, *i.e.* (1) with $F(h)=0$, and with a step initial condition $h_0(x)=\mathbb{1}_{\{x<0\}}$ satisfies (8) with a prefactor $\Phi_{\text{lin}}(c)=2/c$. However, the dependence in c of the prefactor $\Phi(c)$ for the (nonlinear) Fisher–KPP equation is much more complicated. For $h_0(x)=\mathbb{1}_{\{x<0\}}$, it has been proved [BH15, BBCM22] that

$$\Phi(2)=0, \qquad \Phi(c)\sim\frac{2}{c} \quad \text{as } c\to\infty. \tag{9}$$

It is argued in [DMS16] that, for $h_0(x)=\mathbb{1}_{\{x<0\}}$ and $F(h)=h^2$,

$$\Phi(2+\epsilon)\sim 2\sqrt{\pi}\,\alpha\epsilon \quad \text{as } \epsilon\searrow 0, \qquad \Phi(c)\simeq\frac{2}{c}-\frac{8}{c^3}+\frac{6.818\dots}{c^5}+\cdots \quad \text{as } c\to\infty, \tag{10}$$

where α is the coefficient defined in (5).

The main result of this paper is an asymptotic expansion of the function Φ for c close to 2:

Theorem 1. *For the Fisher–KPP equation (1) with $F(h)=h^2$ and an initial condition h_0 which is a compact perturbation of the step function, one has*

$$\begin{aligned}\Phi(2+\epsilon) = \sqrt{\pi}\left(\alpha-\frac{\beta}{2}\epsilon\right)\bigg[&2\epsilon+3\epsilon^2\log\epsilon-3\left(1-\frac{\gamma_E}{2}\right)\epsilon^2+\frac{9}{4}\epsilon^3\log^2\epsilon\\ &+\frac{3}{4}(3\gamma_E-6\log 2-1)\epsilon^3\log\epsilon\bigg]+\mathcal{O}(\epsilon^3)\end{aligned} \tag{11}$$

where γ_E is Euler's constant, and where α and β are the coefficients defined in (5).

Theorem 2. *The expansion (11) actually holds for any choice of $F(h)$ and of h_0 such that*

1. *$\int dx\, h_0(x)e^{rx}<\infty$ for some $r>1$, (otherwise, $\Phi(c)$ would not be defined for $c>2$ and the expansion (11) would be meaningless)*
2. *The position μ_t of the front satisfies the expansion (7),*
3. *There exists $C>0$, $t_0\geqslant 0$ and a neighbourhood U of 1 such that*

$$\left|\int dz\,\big(F[h(\mu_t+z,t)]-F[\omega(z)]\big)e^{rz}\right|\leqslant\frac{C}{t} \qquad \text{for } t>t_0 \text{ and } r\in U. \tag{12}$$

As will be apparent in the proofs, the expansion (11) for $\Phi(2+\epsilon)$ is closely related to the expansion (7) for the position μ_t; in some sense, the $\epsilon^2\log\epsilon$ and $\epsilon^3\log\epsilon^2$ terms in (11) are connected to the $1/\sqrt{t}$ term in (7), and the $\epsilon^3\log\epsilon$ to the $(\log t)/t$ term.

We will also see in the proof that (7) cannot hold unless $\int dx\, h_0(x)x^3e^x<\infty$. As already mentioned, we expect the converse to be true.

The technical condition (12) should not be surprising: the quantity $\delta(z,t):=h(\mu_t+z,t)-\omega(z)$ goes to zero as $t\to\infty$. Moreover, it satisfies $\partial_t\delta=\partial_x^2\delta+\dot{\mu}_t\partial_x\delta+\delta-F(\omega+\delta)+F(\omega)+$

$(\dot{\mu}_t - 2)\omega'$. For large times, one can expect from (7) that $\dot{\mu}_t - 2 \sim -\frac{3}{2t}$ and $\partial_t\delta \simeq \partial_x^2\delta + 2\partial_x\delta + \delta - F'(\omega)\delta - \frac{3}{2t}\omega'$. Then, it seems likely that $\delta(z,t) \sim \frac{1}{t}\psi(z)$ with ψ a solution to $\psi'' + 2\psi' + \psi - F'(\omega)\psi = \frac{3}{2}\omega'$. (This is actually a result of [Gra19] in the case $F(h) = h^2$.) This leads to, $F[h(\mu_t + z,t)] - F[\omega(z)] \sim \delta(z,t)F'[\omega(z)] \sim \frac{1}{t}\psi(z)F'[\omega(z)]$, of order $1/t$. Furthermore, (ignoring polynomial prefactors), $\psi(z)$ decreases as e^{-z} for large z and $F'[\omega(z)]$ should roughly decrease as e^{-pz}, see (2), so that the integral in (12) should converge quickly for r around 1 for $z \to \pm\infty$, and give a result of order $1/t$.

In terms of the function $C(\rho)$ defined in [BBCM22], our result can be written as

$$C(1+\epsilon) = (\alpha - \beta\epsilon)\left[2\epsilon + 6\epsilon^2\log\epsilon + (3\gamma_E + 6\log 2 - 4)\epsilon^2 + 9\epsilon^3\log^2\epsilon + 3(3\gamma_E + 1)\epsilon^3\log\epsilon\right] + \mathcal{O}(\epsilon^3). \quad (13)$$

Note that the authors write $C(\rho) \sim \alpha(\rho - 1)$ as $\rho \searrow 1$ (bottom of p 2095), but their α is twice ours.

Theorem 1 is the direct consequence of theorem 2 and of the following result:

Proposition 1 (mostly Cole Graham 2019 [Gra19]). *For the Fisher–KPP equation (1) with $F(h) = h^2$ and an initial condition h_0 which is a compact perturbation of the step function, (7) and (12) hold.*

The fact that (7) holds under the hypotheses of proposition 1 is the main result of [Gra19]. The proofs of [Gra19] contain the hard parts in showing that (12) also holds.

The main tool used in this paper is the so-called magical relation, which gives a relation between the initial condition, the position μ_t of the front, and the nonlinear part of the equation. Introduce

$$\gamma := \sup\left\{r > 0; \int \mathrm{d}x\, h_0(x)e^{rx} < \infty\right\}, \quad (14)$$

and

$$\varphi(\epsilon,t) := \int \mathrm{d}z\, F[h(\mu_t + z,t)]e^{(1+\epsilon)z}, \qquad \hat{\varphi}(\epsilon) := \int \mathrm{d}z\, F[\omega(z)]e^{(1+\epsilon)z}. \quad (15)$$

(With these quantities, the condition (12) can be written $|\varphi(\epsilon,t) - \hat{\varphi}(\epsilon)| \leqslant C/t$ for all $t > t_0$ and all ϵ in some neighbourhood of 0.) Then

Proposition 2 (magical relation). *For any $\epsilon \in (-1, \gamma - 1)$ the following relation holds*

$$\int_0^\infty \mathrm{d}t\, \varphi(\epsilon,t)e^{-\epsilon^2 t + (1+\epsilon)(\mu_t - 2t)} = \int \mathrm{d}x\, h_0(x)e^{(1+\epsilon)x} - \mathbb{1}_{\{\epsilon>0\}}\Phi(2+2\epsilon). \quad (16)$$

Furthermore, if $\gamma > 1$ and (12) holds, one has

$$\hat{\varphi}(\epsilon)\int_0^\infty \mathrm{d}t\, e^{-\epsilon^2 t + (1+\epsilon)(\mu_t - 2t)} = \int \mathrm{d}x\, h_0(x)e^{(1+\epsilon)x} - \mathbb{1}_{\{\epsilon>0\}}\Phi(2+2\epsilon) + \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3) \quad (17)$$

where $\mathcal{P}(\epsilon)$ is some polynomial in ϵ.

The second form (17) gives a relation between μ_t and h_0 which does not involve the front $h(x,t)$ at any finite time. Notice also that the nonlinear term $F(h)$ only appears in $\hat{\varphi}(\epsilon)$.

The magical relation was introduced in [BD15, BBD17, BBD18], but only for $\epsilon < 0$. It allowed (non-rigorously) to compute the asymptotic expansion of the position of the front for an arbitrary initial condition, and in particular to obtain (7). The basic idea is the following: for $\epsilon < 0$, the whole right hand side of (17) can be written as $\mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3)$ for some polynomial $\mathcal{P}(\epsilon)$ if h_0 goes to zero fast enough. (Specifically, it can be shown that the necessary and sufficient condition is $\int \mathrm{d}x\, h_0(x)x^3e^x < \infty$.) However, the left hand side produces very easily some singular terms of ϵ in a small ϵ expansion; it turns out that μ_t should satisfy (7) in order to avoid all the singular terms up to order ϵ^3.

In this paper, by considering both sides $\epsilon < 0$ and $\epsilon > 0$, we can eliminate the unknown polynomial $\mathcal{P}(\epsilon)$ in (17) and obtain (11).

The rest of the paper is organised as follow; in section 2, we show that the function $\Phi(c)$ is well defined, and we give a useful representation. In section 3, we prove the first part of proposition 2, *i.e.* (16). We state and prove some technical lemmas in section 4, which allow us to finish the proof of proposition 2 and to prove theorem 2 in section 5. In section 6, we prove proposition 1. Finally, a technical lemma is proved in the appendix.

2. The function $\Phi(c)$

Proposition 3. *For a given initial condition h_0 such that $\int \mathrm{d}x\, h_0(x)\mathrm{e}^x < \infty$, let $h(x,t)$ be the solution to (1). For $c \geqslant 2$, the following (finite or infinite) limits exist and are equal:*

$$\Phi(c) := \lim_{t\to\infty} \sqrt{4\pi t}\, h(ct,t)\mathrm{e}^{\left(\frac{c^2}{4}-1\right)t} = \lim_{t\to\infty} \mathrm{e}^{-t\left(1+\frac{c^2}{4}\right)} \int \mathrm{d}x\, h(x,t)\mathrm{e}^{\frac{c}{2}x} \in [0,\infty]. \tag{18}$$

Furthermore, $\Phi(2) = 0$, $\Phi(c) > 0$ for $c > 2$ and

$$\Phi(c) < \infty \quad \Longleftrightarrow \quad \int \mathrm{d}x\, h_0(x)\mathrm{e}^{\frac{c}{2}x} < \infty. \tag{19}$$

The function $c \mapsto \Phi(c)$ is continuous in the domain where it is finite.

Remark. The condition $\int \mathrm{d}x\, h_0(x)\mathrm{e}^x < \infty$ implies, in particular, that the front has a velocity 2.

Before doing a rigorous proof, here is a quick and dirty argument to show that the second limit in (18) is equal to the first: starting from the integral in that limit, make the change of variable $x = vt$ (with v being the new variable) and boldly replace $h(vt,t)$ under the integral sign using the equivalent implied by the first limit to obtain

$$\begin{aligned}\int \mathrm{d}x\, h(x,t)\mathrm{e}^{\frac{c}{2}x} = t\int \mathrm{d}v\, h(vt,t)\mathrm{e}^{\frac{c}{2}vt} &\simeq \frac{t}{\sqrt{4\pi t}}\int \mathrm{d}v\,\Phi(v)\mathrm{e}^{(1-\frac{v^2}{4}+\frac{c}{2}v)t}\\ &= \mathrm{e}^{(1+\frac{c^2}{4})t}\frac{\sqrt{t}}{\sqrt{4\pi}}\int \mathrm{d}v\,\Phi(v)\mathrm{e}^{-\frac{1}{4}(v-c)^2 t}.\end{aligned} \tag{20}$$

(The fact that the substitution only makes sense for $v \geqslant 2$ is not a problem since, clearly, the part of the integral for $v < 2$ does not contribute significantly.) The remaining integral is dominated by v close to c in the large time limit. Replacing $\Phi(v)$ by $\Phi(c)$ and computing the remaining Gaussian integral gives the second limit.

We will need in the proof a bound on how $h(x,t)$ decreases for large x: for $r > 0$, introduce

$$g(r,t) := \int \mathrm{d}x\, h(x,t)\mathrm{e}^{rx}. \tag{21}$$

Lemma 1. *For all x, all $t > 0$, and all $r > 0$ such that $g(r,0) = \int \mathrm{d}x\, h_0(x)\mathrm{e}^{rx} < \infty$,*

$$h(x,t) \leqslant \frac{\mathrm{e}^{(1+r^2)t}}{\sqrt{4\pi t}} g(r,0)\mathrm{e}^{-rx}, \qquad g(r,t) \leqslant \mathrm{e}^{(1+r^2)t} g(r,0). \tag{22}$$

Proof. Using the comparison principle, one obtains that $h(x,t) \leqslant h_{\text{lin}}(x,t)$, where $h_{\text{lin}}(x,t)$ is the solution to $\partial_t h_{\text{lin}} = \partial_x^2 h_{\text{lin}} + h_{\text{lin}}$ with initial condition h_0. Solving for h_{lin}, we get

$$h(x,t) \leqslant \frac{\mathrm{e}^t}{\sqrt{4\pi t}} \int \mathrm{d}y\, h_0(y) \mathrm{e}^{-\frac{(x-y)^2}{4t}}, \tag{23}$$

and then,

$$h(x,t)\mathrm{e}^{rx} \leqslant \frac{\mathrm{e}^t}{\sqrt{4\pi t}} \int \mathrm{d}y\, h_0(y)\mathrm{e}^{ry} \times \mathrm{e}^{r(x-y)-\frac{(x-y)^2}{4t}} = \frac{\mathrm{e}^{(1+r^2)t}}{\sqrt{4\pi t}} \int \mathrm{d}y\, h_0(y)\mathrm{e}^{ry} \times \mathrm{e}^{-\frac{(x-y-2rt)^2}{4t}}. \tag{24}$$

Both inequalities in (22) are obtained from that last relation, respectively by writing that the Gaussian term is smaller than 1, or by integrating over x.

□

Proof of proposition 3. We write the nonlinearity in (1) as $F(h) = h \times G(h)$. From (2), the function G, defined on $[0,1]$, is continuous, satisfies $0 \leqslant G(h) \leqslant 1$, $G(h) = \mathcal{O}(h^p)$ for some $p > 0$ as $h \to 0$ and $G(1) = 1$. To avoid parentheses, we will write $G(h(x,t))$ as $G \circ h(x,t)$ using the composition operator $\circ$. We write the solution $h(x,t)$ of (1) using the Feynman–Kac representation (see [Fri75, theorem 5.3 p 148] or, for a short proof, [BBP19, proposition 3.1]):

$$h(x,t) = \mathrm{e}^t \mathbb{E}_x\Big[h_0(B_t)\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h(B_s, t-s)}\Big], \tag{25}$$

where under $\mathbb{E}_x$, B is a Brownian with diffusivity $\sqrt{2}$ started from x. (So that $\mathbb{E}_x(B_t^2) = x^2 + 2t$.)

In (25), we condition the Brownian to end at $B_t = y$ and we integrate over y:

$$h(x,t) = \mathrm{e}^t \int \mathrm{d}y \frac{\mathrm{e}^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} h_0(y) \mathbb{E}_{t:x\to y}\Big[\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h(B_s, t-s)}\Big] \tag{26}$$

where, under $\mathbb{E}_{t:x\to y}$, B is a Brownian bridge going from x to y in a time t, with a diffusivity $\sqrt{2}$. We reverse time and remove the linear part from the bridge:

$$h(x,t) = \mathrm{e}^t \int \mathrm{d}y \frac{\mathrm{e}^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} h_0(y) \mathbb{E}_{t:y\to x}\Big[\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h(B_s, s)}\Big] \tag{27}$$

$$= \int \mathrm{d}y \frac{\mathrm{e}^{t-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} h_0(y) \mathbb{E}_{t:0\to 0}\Big[\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h\big(B_s + (x-y)\frac{s}{t} + y, s\big)}\Big]. \tag{28}$$

Then, at $x = ct$,

$$h(ct,t) = \frac{\mathrm{e}^{(1-\frac{c^2}{4})t}}{\sqrt{4\pi t}} \int \mathrm{d}y\, \mathrm{e}^{\frac{c}{2}y - \frac{y^2}{4t}} h_0(y) \mathbb{E}_{t:0\to 0}\Big[\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h\big(B_s + cs - y\frac{s}{t} + y, s\big)}\Big]. \tag{29}$$

We move the prefactors to the left hand side and write the Brownian bridge as a time-changed Brownian path

$$\sqrt{4\pi t}\,\mathrm{e}^{(\frac{c^2}{4}-1)t} h(ct,t) = \int \mathrm{d}y\, \mathrm{e}^{\frac{c}{2}y - \frac{y^2}{4t}} h_0(y) \mathbb{E}_0\Big[\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h\big(\frac{t-s}{t} B_{\frac{ts}{t-s}} + cs - y\frac{s}{t} + y, s\big)}\Big]. \tag{30}$$

For any y, any $c \geqslant 2$, and almost all Brownian path B, one has

$$\int_0^t \mathrm{d}s\, G\circ h\Big(\frac{t-s}{t} B_{\frac{ts}{t-s}} + cs - y\frac{s}{t} + y, s\Big) \to \int_0^\infty \mathrm{d}s\, G\circ h\,(B_s + cs + y, s) \qquad \text{as } t\to\infty. \tag{31}$$

Indeed, first consider the case $c>2$, pick $\tilde{c}\in(2,c)$ and t_0 such that $c-y/t_0>\tilde{c}$. Recall that, for almost all path B, there exists a constant A (depending on B) such that $|B_u|\leqslant A(1+u^{0.51})$ for all u; this implies that $\frac{t-s}{t}\left|B_{\frac{ts}{t-s}}\right|\leqslant A(1+s^{0.51})$ for all t and all $s<t$. Then

$$\frac{t-s}{t}B_{\frac{ts}{t-s}}+cs-y\frac{s}{t}+y\geqslant\tilde{c}s+C+y\quad\text{for all } t>t_0 \text{ and all } s\in(0,t),\tag{32}$$

where C is some constant depending on B. (Indeed, the function $s\mapsto cs-ys/t-\tilde{c}s-As^{0.51}$ is uniformly bounded from below for $t>t_0$.) Using (22) for $r=1$, we obtain that

$$h\left(\frac{t-s}{t}B_{\frac{ts}{t-s}}+cs-y\frac{s}{t}+y,s\right)\leqslant\frac{C}{\sqrt{s}}\mathrm{e}^{-(\tilde{c}-2)s-y}\quad\text{for all } t>t_0 \text{ and all } s\in(0,t),\tag{33}$$

with C another constant depending on B. As $G(h)=\mathcal{O}(h^p)$ for some $p>0$ as $h\to0$ and $G(1)=1$, there exists a constant C such that $G(h)\leqslant Ch^{\min(1,p)}$. Then, we see by dominated convergence that (31) holds for $c>2$, and furthermore we see that the right hand side is smaller than $Ce^{-\min(1,p)y}$ for some constant C depending on B.

For $c=2$, the right hand side of (31) is $+\infty$. Indeed, B_s+cs+y is infinitely often smaller than $2s-\sqrt{s}$, where the front h is close to 1. Then, noticing that (31) with the upper limits of both integrals replaced by some $T>0$ clearly holds by dominated convergence, and that, by choosing T large enough, the right hand side is arbitrarily large, we see that the left hand side of (31) must diverge as $t\to\infty$.

From (31), we immediately obtain by dominated convergence

$$\mathbb{E}_0\left[\mathrm{e}^{-\int_0^t\mathrm{d}s\,G\circ h\left(\frac{t-s}{t}B_{\frac{ts}{t-s}}+cs-y\frac{s}{t}+y,s\right)}\right]\to\mathbb{E}_0\left[\mathrm{e}^{-\int_0^\infty\mathrm{d}s\,G\circ h(B_s+cs+y,s)}\right]\qquad\text{as }\ t\to\infty,\tag{34}$$

where the right hand side is 0 if $c=2$ and positive if $c>2$. (As we have shown, the integral in the exponential is almost surely infinite if $c=2$, and almost surely finite if $c>2$.) Furthermore, for $c>2$, the right hand side converges to 1 as $y\to\infty$. (Recall that, for $c>2$, the integral in the exponential is smaller than $Ce^{-\min(1,p)y}$.)

If $\int h_0(y)\mathrm{e}^{cy/2}\mathrm{d}y<\infty$, then a last application of dominated convergence in (30) shows that the first limit defining $\Phi(c)$ in (18) does exist and is given by:

$$\Phi(c):=\lim_{t\to\infty}\sqrt{4\pi}\,t\,\mathrm{e}^{(\frac{c^2}{4}-1)t}h(ct,t)=\int\mathrm{d}y\,\mathrm{e}^{\frac{c}{2}y}h_0(y)\mathbb{E}_0\left[\mathrm{e}^{-\int_0^\infty\mathrm{d}s\,G\circ h(B_s+cs+y,s)}\right]<\infty,\tag{35}$$

and furthermore $\Phi(2)=0$ and $\Phi(c)>0$ for $c>2$. Note that [BBCM22] gives a similar expression.

We now assume that $\int h_0(y)\mathrm{e}^{cy/2}\mathrm{d}y=\infty$ and show that the limit of (30) diverges. Notice that we must be in the $c>2$ case since we also assumed that $\int h_0(y)\mathrm{e}^{y}\,\mathrm{d}y<\infty$. Cutting the integral in (30) at some arbitrary value A and then sending $t\to\infty$ gives

$$\liminf_{t\to\infty}\sqrt{4\pi}\,t\,\mathrm{e}^{(\frac{c^2}{4}-1)t}h(ct,t)\geqslant\int_{-\infty}^{A}\mathrm{d}y\,\mathrm{e}^{\frac{c}{2}y}h_0(y)\mathbb{E}_0\left[\mathrm{e}^{-\int_0^\infty\mathrm{d}s\,G\circ h(B_s+cs+y,s)}\right].\tag{36}$$

As the expectation appearing in the integral goes to 1 as $y\to\infty$, the hypothesis $\int h_0(y)\mathrm{e}^{cy/2}\mathrm{d}y=\infty$ implies that the right hand side diverges as A to ∞, and then that $\Phi(c)$ exists and is infinite.

Using the same methods, one can show from (35) that $\Phi(c)$ is a continuous function (in the range of c where Φ is finite) by first showing that $\int_0^\infty\mathrm{d}s\,G\circ h(B_s+c_ns+y,s)\to\int_0^\infty\mathrm{d}s\,G\circ h(B_s+cs+y,s)$ if $c_n\to c$, by dominated convergence, using the same bounds as above (specifically that the integrands are uniformly bounded by an exponentially decreasing function of s if $c>2$ and that the result is infinity if $c=2$.)

To show that the second expression in (18) is equal to the first, start again from (27):

$$h(x,t) = \mathrm{e}^t \int \mathrm{d}y \frac{\mathrm{e}^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} h_0(y) \mathbb{E}_{t:y\to x}\Big[\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h(B_s,s)}\Big]$$
$$= \mathrm{e}^t \int \mathrm{d}y\, h_0(y) \mathbb{E}_y\Big[\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h(B_s,s)} \delta(B_t - x)\Big].$$

Then

$$\mathrm{e}^{-t\left(1+\frac{c^2}{4}\right)} \int \mathrm{d}x\, h(x,t) \mathrm{e}^{\frac{c}{2}x} = \mathrm{e}^{-\frac{c^2}{4}t} \int \mathrm{d}y\, h_0(y) \mathbb{E}_y\Big[\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h(B_s,s)} \mathrm{e}^{\frac{c}{2}B_t}\Big]$$
$$= \int \mathrm{d}y\, e^{\frac{c}{2}y} h_0(y) \mathbb{E}_y\Big[\mathrm{e}^{-\int_0^t \mathrm{d}s\, G\circ h(B_s+cs,s)}\Big] \qquad (37)$$

where the last transform is through Girsanov's theorem (or a change of probability of the Brownian). Taking the limit $t \to \infty$ is immediate and gives back the expression of $\Phi(c)$ written in (35).

□

3. Magical relation

Proposition 2 (the magical relation) can be split into two lemmas:

Lemma 2. *For any $\epsilon \in (-1, \gamma - 1)$ the following relation holds*

$$\int_0^\infty \mathrm{d}t\, \varphi(\epsilon,t) \mathrm{e}^{-\epsilon^2 t + (1+\epsilon)(\mu_t - 2t)} = \int \mathrm{d}x\, h_0(x) \mathrm{e}^{(1+\epsilon)x} - \mathbb{1}_{\{\epsilon>0\}} \Phi(2+2\epsilon). \quad (38)$$

Lemma 3. *Furthermore, if $\gamma > 1$ and (12) holds, one has*

$$\hat{\varphi}(\epsilon) \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\epsilon^2 t + (1+\epsilon)(\mu_t - 2t)} = \int \mathrm{d}x\, h_0(x) \mathrm{e}^{(1+\epsilon)x} - \mathbb{1}_{\{\epsilon>0\}} \Phi(2+2\epsilon) + \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3), \quad (39)$$

where $\mathcal{P}(\epsilon)$ is some polynomial in ϵ.

In this section, we prove lemma 2. The proof of lemma 3 is delayed to section 5 because it requires some technical lemmas stated in section 4.

Proof of lemma 2. (Many of the arguments in this proof were already in [BBD18] for the case $\epsilon < 0$.)

Recall the definitions (14) of γ and (21) of $g(r,t)$:

$$\gamma := \sup\Big\{r; \int \mathrm{d}x\, h_0(x) \mathrm{e}^{rx} < \infty\Big\}, \qquad g(r,t) := \int \mathrm{d}x\, h(x,t) \mathrm{e}^{rx}. \quad (40)$$

According to lemma 1,

$$g(r,t) \leqslant \mathrm{e}^{(1+r^2)t} g(r,0) < \infty \qquad \text{for } r \in (0,\gamma) \text{ and } t \geqslant 0. \quad (41)$$

We wish to write an expression for $\partial_t g(r,t)$, and the first step is to justify that we can differentiate under the integral sign:

$$\partial_t g(r,t) = \int \mathrm{d}x\, \partial_t h(x,t) \mathrm{e}^{rx} = \int \mathrm{d}x \left[\partial_x^2 h(x,t) + h(x,t) - F[h(x,t)]\right] \mathrm{e}^{rx} \qquad \text{for } 0 < r < \gamma, \quad (42)$$

and then (still assuming $0 < r < \gamma$) that we can integrate twice by parts the $\partial_x^2 h$ term:

$$\partial_t g(r,t) = \int \mathrm{d}x \left[(r^2 h(x,t) + h(x,t) - F[h(x,t)]\right] \mathrm{e}^{rx} = (1+r^2) g(r,t) - \int \mathrm{d}x\, F[h(x,t)] \mathrm{e}^{rx}. \quad (43)$$

Both steps (42) and (43) are justified by using bounding functions provided by the following lemma with β chosen in (r,γ):

Lemma 4. *Let $\beta \in (0,\gamma)$. For $t>0$, the quantities $h(x,t)$, $|\partial_x h(x,t)|$, $|\partial_x^2 h(x,t)|$ and $|\partial_t h(x,t)|$ are bounded by $A(t)\max(1,\mathrm{e}^{-\beta x})$ for some locally bounded function A.*

Lemma 4 follows from general results of parabolic regularity theory; however, for completeness, a proof is given in the appendix.

Recall the definition (15) of φ:

$$\varphi(\epsilon,t) := \int \mathrm{d}z\, F[h(\mu_t+z,t)]\mathrm{e}^{(1+\epsilon)z}; \tag{44}$$

we have

$$\int \mathrm{d}x\, F[h(x,t)]\mathrm{e}^{rx} = \mathrm{e}^{r\mu_t}\int \mathrm{d}z\, F[h(\mu_t+z,t)]\mathrm{e}^{rz} = \mathrm{e}^{r\mu_t}\varphi(r-1,t), \tag{45}$$

and so, in (43),

$$\partial_t g(r,t) = (1+r^2)g(r,t) - \mathrm{e}^{r\mu_t}\varphi(r-1,t). \tag{46}$$

Integrating, we obtain

$$g(r,t)\mathrm{e}^{-(1+r^2)t} = g(r,0) - \int_0^t \mathrm{d}s\, \varphi(r-1,s)\mathrm{e}^{r\mu_s-(1+r^2)s}. \tag{47}$$

We now send $t\to\infty$ in (47), distinguishing two cases

- If $\gamma>1$ and $1\leqslant r<\gamma$; notice that the left hand side is the expression appearing in the second limit in (18) with $c=2r$. This implies that

$$\Phi(2r) = g(r,0) - \int_0^\infty \mathrm{d}s\, \varphi(r-1,s)\mathrm{e}^{r\mu_s-(1+r^2)s} \qquad \text{if } 1\leqslant r<\gamma. \tag{48}$$

- If $0<r<\min(1,\gamma)$; we claim that the left hand side of (47) goes to 0 as $t\to\infty$, and so:

$$0 = g(r,0) - \int_0^\infty \mathrm{d}s\, \varphi(r-1,s)\mathrm{e}^{r\mu_s-(1+r^2)s} \qquad \text{if } 0<r<\min(1,\gamma). \tag{49}$$

Indeed, take $\beta\in(r,\min(1,\gamma))$. Applying (22) with β instead of r, we have

$$h(x,t) \leqslant \min\left[1, \frac{\mathrm{e}^{(1+\beta^2)t}}{\sqrt{4\pi t}} g(\beta,0)\mathrm{e}^{-\beta x}\right]. \tag{50}$$

For t given, let X be the point where both expressions inside the min are equal:

$$\mathrm{e}^{\beta X} = \frac{\mathrm{e}^{(1+\beta^2)t}}{\sqrt{4\pi t}} g(\beta,0). \tag{51}$$

We obtain from (50)

$$\begin{aligned} g(r,t) = \int \mathrm{d}x\, h(x,t)\mathrm{e}^{rx} &\leqslant \frac{\mathrm{e}^{rX}}{r} + \frac{\mathrm{e}^{(1+\beta^2)t}}{\sqrt{4\pi t}} g(\beta,0)\frac{\mathrm{e}^{-(\beta-r)X}}{\beta-r} \\ &= \left(\frac{1}{r}+\frac{1}{\beta-r}\right)\mathrm{e}^{rX} = C\frac{\mathrm{e}^{r(\beta^{-1}+\beta)t}}{t^{\frac{r}{2\beta}}}, \end{aligned} \tag{52}$$

where C is some quantity depending on r and β, but independent of time. As $\beta > r$ and as the function $\beta \to \beta^{-1} + \beta$ is decreasing for $\beta < 1$, we obtain $r(\beta^{-1} + \beta) < r(r^{-1} + r) = 1 + r^2$. We conclude that, indeed, the left hand side of (47) goes to zero as $r \to \infty$.

Combining (48) and (49), we have shown that

$$\int_0^\infty \mathrm{d}s\, \varphi(r-1,s) \mathrm{e}^{r\mu_s - (1+r^2)s} = g(r,0) - \mathbb{1}_{\{r>1\}} \Phi(2r) \qquad \text{for } r \in (0,\gamma). \tag{53}$$

(Recall that $\Phi(2) = 0$, hence the right hand side is continuous at $r = 1$.) Writing now that $r\mu_s - (1+r^2)s = r(\mu_s - 2s) - (1-r)^2 s$, and taking $r = 1+\epsilon$ and $s = t$ in (53), we obtain

$$\int_0^\infty \mathrm{d}t\, \varphi(\epsilon,t) \mathrm{e}^{-\epsilon^2 t + (1+\epsilon)(\mu_t - 2t)} = g(1+\epsilon,0) - \mathbb{1}_{\{\epsilon>0\}} \Phi(2+2\epsilon) \quad \text{for } \epsilon \in (-1,\gamma-1), \tag{54}$$

which completes the proof of lemma 2, the first part of proposition 2. □

To prove lemma 3 (the second part of proposition 2), we need to show that, for some polynomial $\mathcal{P}(\epsilon)$,

$$\int_0^\infty \mathrm{d}t \left[\varphi(\epsilon,t) - \hat{\varphi}(\epsilon)\right] \mathrm{e}^{-\epsilon^2 t + (1+\epsilon)(\mu_t - 2t)} = \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3), \tag{55}$$

if $\left|\varphi(\epsilon,t) - \hat{\varphi}(\epsilon)\right| \leqslant \frac{C}{t}$ for t large enough and ϵ in some real neighbourhood of 0 and if $\gamma > 1$, which implies that $\mu_t = 2t - \frac{3}{2}\log t + a + o(1)$ (by Bramson's result). To do so, we need several technical lemmas. We thus take a pause in the proof of proposition 2 to state and prove these lemmas, and we resume in section 5.

4. Some technical lemmas

We begin by recalling a classical result on the analyticity of functions defined by an integral:

Lemma 5. *Let $f(\epsilon,t)$ be a family of functions such that*

- *$\epsilon \mapsto f(\epsilon,t)$ is analytic on some simply connected open domain U of $\mathbb{C}$ (independent of t) for almost all $t \in \mathbb{R}$,*
- *$|f(\epsilon,t)| \leqslant g(t)$ for all $\epsilon \in U$, where g is some integrable function: $\int g(t)\,\mathrm{d}t < \infty$.*

Then, $\epsilon \mapsto F(\epsilon) := \int \mathrm{d}t f(\epsilon,t)$ is analytic on U.

Proof. On any closed path γ in U, one has with Fubini

$$\oint_\gamma F(\epsilon)\,\mathrm{d}\epsilon = \int \mathrm{d}t \oint_\gamma \mathrm{d}\epsilon f(\epsilon,t). \tag{56}$$

This last integral is 0 since $\epsilon \mapsto f(\epsilon,t)$ is analytic and U is simply connected. Then, by Morera's theorem, F is analytic. □

The next lemma states that some functions of ϵ which are variations on the incomplete gamma functions have small ϵ expansions with only one or two singular terms.

Lemma 6. *Let α, β be real numbers such that either $\alpha \notin \{1,2,3,\ldots\}$, or $\beta \neq 0$. There exist functions $\epsilon \mapsto \mathcal{A}_{\alpha,\beta}(\epsilon)$ and $\epsilon \mapsto \tilde{\mathcal{A}}_{\alpha,\beta}(\epsilon)$ which are analytic around $\epsilon = 0$, such that for ϵ real, non-zero and $|\epsilon|$ small enough,*

$$\int_1^\infty \mathrm{d}t\, \mathrm{e}^{-\epsilon^2 t} \frac{1}{t^{\alpha+\beta\epsilon}} = |\epsilon|^{2\alpha-2+2\beta\epsilon} \Gamma(1-\alpha-\beta\epsilon) + \frac{\mathbb{1}_{\{\alpha=1\}}}{\beta\epsilon} + \mathcal{A}_{\alpha,\beta}(\epsilon), \tag{57}$$

$$\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{\log t}{t^{\alpha+\beta\epsilon}} = |\epsilon|^{2\alpha-2+2\beta\epsilon}\Big[-2\log|\epsilon|\Gamma(1-\alpha-\beta\epsilon)+\Gamma'(1-\alpha-\beta\epsilon)\Big] + \frac{\mathbb{1}_{\{\alpha=1\}}}{(\beta\epsilon)^2}+\tilde{\mathcal{A}}_{\alpha,\beta}(\epsilon). \tag{58}$$

Remark. The condition $\alpha\notin\{1,2,3,\ldots\}$ or $\beta\neq 0$ ensures that the gamma functions appearing in the result are well defined for $\epsilon\neq 0$ small enough. For $\alpha=n\in\{1,2,3,\ldots\}$ and $\beta=0$ one would have

$$\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{1}{t^n} = \frac{2(-1)^n}{(n-1)!}\epsilon^{2n-2}\log|\epsilon| + \mathcal{A}_{n,0}(\epsilon),$$

but we do not need this result in the present paper, and we skip the proof. For convenience, we give the results we actually use, writing simply $\mathcal{A}(\epsilon)$ for the analytic functions:

$$\begin{aligned}
&\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{1}{t^{\frac{3}{2}+\frac{3}{2}\epsilon}} = |\epsilon|^{1+3\epsilon}\Gamma\left(-\frac{1}{2}-\frac{3}{2}\epsilon\right)+\mathcal{A}(\epsilon),\\
&\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{1}{t^{2+\frac{3}{2}\epsilon}} = |\epsilon|^{2+3\epsilon}\Gamma\left(-1-\frac{3}{2}\epsilon\right)+\mathcal{A}(\epsilon),\\
&\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{\log t}{t^{\frac{5}{2}+\frac{3}{2}\epsilon}} = |\epsilon|^{3+3\epsilon}\left[-2\log|\epsilon|\Gamma\left(-\frac{3}{2}-\frac{3}{2}\epsilon\right)+\Gamma'\left(-\frac{3}{2}-\frac{3}{2}\epsilon\right)\right]+\mathcal{A}(\epsilon).
\end{aligned} \tag{59}$$

Proof of lemma 6. Fix α and β such that either $\alpha\notin\{1,2,3,\ldots\}$ or $\beta\neq 0$. We restrict ϵ to be real, non-zero and $|\epsilon|$ to be small enough so that $\alpha+\beta\epsilon\notin\{1,2,3,\ldots\}$. This ensures that the Γ function and its derivative in (57) and (58) are defined, and we define $A_{\alpha,\beta}(\epsilon)$ and $\tilde{A}_{\alpha,\beta}(\epsilon)$ by (respectively) (57) and (58). We now show that the functions thus defined can be extended into analytic functions around $\epsilon=0$.

We first consider $\alpha<1$. Note that by our restriction on the range of allowed ϵ, one also has $\alpha+\beta\epsilon<1$, and one can write

$$\int_1^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-\epsilon^2 t}}{t^{\alpha+\beta\epsilon}} = \int_0^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-\epsilon^2 t}}{t^{\alpha+\beta\epsilon}} - \int_0^1 \mathrm{d}t\,\frac{\mathrm{e}^{-\epsilon^2 t}}{t^{\alpha+\beta\epsilon}} = |\epsilon|^{2\alpha-2+2\beta\epsilon}\Gamma(1-\alpha-\beta\epsilon) - \int_0^1 \mathrm{d}t\,\frac{\mathrm{e}^{-\epsilon^2 t}}{t^{\alpha+\beta\epsilon}}. \tag{60}$$

By identification with (57), one obtains

$$A_{\alpha,\beta}(\epsilon) = -\int_0^1 \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{1}{t^{\alpha+\beta\epsilon}} \qquad \text{for } \alpha<1. \tag{61}$$

Similarly,

$$\tilde{A}_{\alpha,\beta}(\epsilon) = -\int_0^1 \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{\log t}{t^{\alpha+\beta\epsilon}} \qquad \text{for } \alpha<1. \tag{62}$$

Let $\tilde{\alpha}\in(\alpha,1)$, and let U a simply connected neighbourhood of 0 in $\mathbb{C}$ such that $\alpha+\beta\mathrm{Re}(\epsilon)<\tilde{\alpha}$ and $|\mathrm{e}^{-\epsilon^2 t}|<2$ for all $\epsilon\in U$ and $t\in[0,1]$. One can apply lemma 5 with the bounding function $g(t)=2(1+|\log t|)/t^{\tilde{\alpha}}\mathbb{1}_{\{t\in(0,1)\}}$ to show that $A_{\alpha,\beta}(\epsilon)$ and $\tilde{A}_{\alpha,\beta}(\epsilon)$ are analytic around 0.

To extend the result to $\alpha\geqslant 1$, we integrate by parts the left hand side of (57)

$$\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{1}{t^{\alpha+\beta\epsilon}} = \frac{1}{1-\alpha-\beta\epsilon}\left[-\mathrm{e}^{-\epsilon^2}+\epsilon^2\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{1}{t^{\alpha+\beta\epsilon-1}}\right]. \tag{63}$$

Then, rewriting the integrals in terms of the functions $\mathcal{A}_{\alpha,\beta}$ as in (57),

$$|\epsilon|^{2\alpha-2+2\beta\epsilon}\Gamma(1-\alpha-\beta\epsilon)+\frac{\mathbb{1}_{\{\alpha=1\}}}{\beta\epsilon}+\mathcal{A}_{\alpha,\beta}(\epsilon)$$
$$=\frac{1}{1-\alpha-\beta\epsilon}\left[-\mathrm{e}^{-\epsilon^2}+|\epsilon|^{2\alpha-2+2\beta\epsilon}\Gamma(2-\alpha-\beta\epsilon)+\frac{\mathbb{1}_{\{\alpha=2\}}}{\beta}\epsilon+\epsilon^2\mathcal{A}_{\alpha-1,\beta}(\epsilon)\right]. \quad (64)$$

With the property $x\Gamma(x)=\Gamma(x+1)$, the terms with the Γ functions cancel and one is left with

$$\mathcal{A}_{\alpha,\beta}(\epsilon)=-\frac{\mathbb{1}_{\{\alpha=1\}}}{\beta\epsilon}+\frac{1}{1-\alpha-\beta\epsilon}\left[-\mathrm{e}^{-\epsilon^2}+\frac{\mathbb{1}_{\{\alpha=2\}}}{\beta}\epsilon+\epsilon^2\mathcal{A}_{\alpha-1,\beta}(\epsilon)\right]. \quad (65)$$

For convenience let us also write the special case $\alpha=1$:

$$\mathcal{A}_{1,\beta}(\epsilon)=\frac{\mathrm{e}^{-\epsilon^2}-1}{\beta\epsilon}-\frac{\epsilon}{\beta}\mathcal{A}_{0,\beta}(\epsilon). \quad (66)$$

It is then clear from these equations that, except for $(\alpha=1,\beta=0)$ or $(\alpha=2,\beta=0)$, one has

$$\{\epsilon\mapsto\mathcal{A}_{\alpha-1,\beta}(\epsilon)\text{ analytic around }0\}\implies\{\epsilon\mapsto\mathcal{A}_{\alpha,\beta}(\epsilon)\text{ analytic around }0\}. \quad (67)$$

As $\mathcal{A}_{\alpha,\beta}$ is analytic around 0 for $\alpha<1$, this implies by induction that $\mathcal{A}_{\alpha,\beta}$ is analytic around 0 for all α if $\beta\neq0$, and for all $\alpha\notin\{1,2,3,\dots\}$ if $\beta=0$.

We proceed in the same way for $\tilde{\mathcal{A}}_{\alpha,\beta}$. Integrating by parts the integral in (58),

$$\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2t}\frac{\log t}{t^{\alpha+\beta\epsilon}}=\frac{1}{1-\alpha-\beta\epsilon}\left[\epsilon^2\int_1^\infty\mathrm{d}t\,\mathrm{e}^{-\epsilon^2t}\frac{\log t}{t^{\alpha+\beta\epsilon-1}}-\int_1^\infty\mathrm{d}t\,\mathrm{e}^{-\epsilon^2t}\frac{1}{t^{\alpha+\beta\epsilon}}\right]. \quad (68)$$

We replace all the integrals using (57) and (58) and notice, using $x\Gamma(x)=\Gamma(x+1)$ and $\Gamma(x)+x\Gamma'(x)=\Gamma'(x+1)$, that all the terms involving Γ functions cancel, *i.e.*:

$$[-2\log|\epsilon|\Gamma(1-\alpha-\beta\epsilon)+\Gamma'(1-\alpha-\beta\epsilon)]$$
$$=\frac{1}{1-\alpha-\beta\epsilon}[-2\log|\epsilon|\Gamma(2-\alpha-\beta\epsilon)+\Gamma'(2-\alpha-\beta\epsilon)-\Gamma(1-\alpha-\beta\epsilon)]. \quad (69)$$

Then, the remaining terms are

$$\frac{\mathbb{1}_{\{\alpha=1\}}}{(\beta\epsilon)^2}+\tilde{\mathcal{A}}_{\alpha,\beta}(\epsilon)=\frac{1}{1-\alpha-\beta\epsilon}\left[\frac{\mathbb{1}_{\{\alpha=2\}}}{\beta^2}+\epsilon^2\tilde{\mathcal{A}}_{\alpha-1,\beta}(\epsilon)-\frac{\mathbb{1}_{\{\alpha=1\}}}{\beta\epsilon}-\mathcal{A}_{\alpha,\beta}(\epsilon)\right] \quad (70)$$

In particular, for $\alpha=1$,

$$\tilde{\mathcal{A}}_{1,\beta}(\epsilon)=-\frac{\epsilon}{\beta}\tilde{\mathcal{A}}_{0,\beta}(\epsilon)+\frac{\mathcal{A}_{1,\beta}(\epsilon)}{\beta\epsilon}. \quad (71)$$

Notice from (66) that $\mathcal{A}_{1,\beta}(0)=0$. Hence we have again, except if $\alpha\in\{1,2,3,\dots\}$ and $\beta=0$

$$\{\epsilon\mapsto\tilde{\mathcal{A}}_{\alpha-1,\beta}(\epsilon)\text{ analytic around }0\}\implies\{\epsilon\mapsto\tilde{\mathcal{A}}_{\alpha,\beta}(\epsilon)\text{ analytic around }0\}, \quad (72)$$

and the proof is finished in the same way as for $\mathcal{A}_{\alpha,\beta}$.

□

Lemma 6 gives asymptotic expansions of $\mathrm{e}^{-\epsilon^2t}$ times exact power laws of t. The next lemma deals with the case of approximate power laws.

Lemma 7. *Let $f(\epsilon,t)$ be a family of functions such that, for a certain neighbourhood U of 0 in $\mathbb{C}$,*

- *$\epsilon \mapsto f(\epsilon,t)$ is analytic in U for all $t>0$,*
- *There exists a $C>0$ and two real constants α and β such that, for all $\epsilon \in U\cap\mathbb{R}$,*

$$|f(\epsilon,t)| \le \frac{C}{t^{\alpha+\beta\epsilon}} \quad \text{for } t>1, \qquad |f(\epsilon,t)| \le C \quad \text{for } t \le 1. \tag{73}$$

Then there exists a polynomial $\mathcal{P}$ such that, for ϵ real, non-zero, and $|\epsilon|$ small enough,

$$\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t} f(\epsilon,t) = \mathcal{P}(\epsilon) + \begin{cases} \mathcal{O}\left(|\epsilon|^{2\alpha-2}\right) & \text{if } \alpha \notin \{1,2,3,\dots\}, \\ \mathcal{O}\left(|\epsilon|^{2\alpha-2}\log|\epsilon|\right) & \text{if } \alpha \in \{1,2,3,\dots\}, \end{cases} \qquad \text{as } \epsilon \to 0. \tag{74}$$

Proof. We first consider $\alpha \leqslant 1$. In that case, the polynomial $\mathcal{P}(\epsilon)$ plays no role as it is asymptotically smaller than the $\mathcal{O}$ term. Thus, we simply need to bound the integral for $\epsilon \in U\cap\mathbb{R}$:

$$\left|\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t} f(\epsilon,t)\right| \leqslant C + C\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{1}{t^{\alpha+\beta\epsilon}}. \tag{75}$$

The remaining integral is given by (57) except for the case $\alpha=1$ and $\beta=0$:

$$\int_1^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}\frac{1}{t^{\alpha+\beta\epsilon}} = \begin{cases} |\epsilon|^{2\alpha-2+2\beta\epsilon}\Gamma(1-\alpha-\beta\epsilon) + \mathcal{A}_{\alpha,\beta}(\epsilon) = \mathcal{O}(|\epsilon|^{2\alpha-2}) & \text{if } \alpha<1, \\ |\epsilon|^{2\beta\epsilon}\Gamma(-\beta\epsilon) + \frac{1}{\beta\epsilon} + \mathcal{A}_{\alpha,\beta}(\epsilon) = \mathcal{O}(\log|\epsilon|) & \text{if } \alpha=1, \end{cases} \tag{76}$$

where we used $|\epsilon|^{2\beta\epsilon} = 1+\mathcal{O}(\epsilon\log|\epsilon|)$ and $\Gamma(-\beta\epsilon) = \frac{\Gamma(1-\beta\epsilon)}{-\beta\epsilon} = -\frac{1}{\beta\epsilon}+\mathcal{O}(1)$. One checks independently that the case $\alpha=1$ and $\beta=0$ gives also $\mathcal{O}(\log|\epsilon|)$.

For $\alpha>1$, we proceed by induction. Pick $\tilde\alpha \in (1,\alpha)$, and make the neighbourhood U of 0 small enough that $\alpha+\beta\epsilon>\tilde\alpha$ for $\epsilon \in U\cap\mathbb{R}$. Then

$$|f(\epsilon,t)| \leqslant \frac{C}{t^{\alpha+\beta\epsilon}} \leqslant \frac{C}{t^{\tilde\alpha}} \qquad \text{for all } t>1 \text{ and } \epsilon \in U\cap\mathbb{R}. \tag{77}$$

Integrating by parts,

$$\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t} f(\epsilon,t) = F(\epsilon,0) - \epsilon^2\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t}F(\epsilon,t) \qquad \text{with } F(\epsilon,t) = \int_t^\infty \mathrm{d}t'\,f(\epsilon,t'). \tag{78}$$

Using lemma 5, the function $\epsilon \mapsto F(\epsilon,t)$ is analytic in U for all $t \geqslant 0$. Furthermore, for some $\tilde C$,

$$|F(\epsilon,t)| \leqslant \frac{\tilde C}{t^{\alpha-1+\beta\epsilon}} \quad \text{for } t>1, \qquad F(\epsilon,t)| \leqslant \tilde C \quad \text{for } t \leqslant 1. \tag{79}$$

Then, assuming that the lemma holds for $\alpha-1$, we can apply it to the integral with F in (78); after Taylor-expanding the analytic function $F(\epsilon,0)$, we see that the result holds for f.

□

5. Expansions in ϵ

In section 3, we have shown the first part of proposition 2 (called lemma 2), which states that for $\epsilon \in (-1, \gamma - 1)$,

$$\int_0^\infty \mathrm{d}t\,\varphi(\epsilon,t)\mathrm{e}^{-\epsilon^2 t+(1+\epsilon)(\mu_t-2t)} = \int \mathrm{d}x\,h_0(x)\mathrm{e}^{(1+\epsilon)x} - \mathbb{1}_{\{\epsilon>0\}}\Phi(2+2\epsilon). \quad (80)$$

In this section, we use the results of section 4 to make some small ϵ expansions and prove the second part of proposition 2 (called lemma 3) and our main result, theorem 2.

To prove lemma 3, it remains to show (55):

$$\int_0^\infty \mathrm{d}t\left[\varphi(\epsilon,t)-\hat{\varphi}(\epsilon)\right]\mathrm{e}^{-\epsilon^2 t+(1+\epsilon)(\mu_t-2t)} = \mathcal{P}(\epsilon)+\mathcal{O}(\epsilon^3), \quad (81)$$

for some polynomial $\mathcal{P}(\epsilon)$, under the hypotheses that $\gamma > 1$ and (12) holds, *i.e.* that there exists $C > 0$, $t_0 \geqslant 0$ and a (real) neighbourhood U of 0 such that

$$\left|\varphi(\epsilon,t)-\hat{\varphi}(\epsilon)\right| \leqslant \frac{C}{t} \qquad \text{for } t > t_0 \text{ and } \epsilon \in U. \quad (82)$$

Proof of lemma 3. Recall from (2) that $F(h) \leqslant Ch^{1+p}$ for some $p > 0$ and some constant C. Recall the definitions (15) of φ and $\hat{\varphi}$:

$$\varphi(\epsilon,t) := \int \mathrm{d}z\,F[h(\mu_t+z,t)]\mathrm{e}^{(1+\epsilon)z}, \qquad \hat{\varphi}(\epsilon) := \int \mathrm{d}z\,F[\omega(z)]\mathrm{e}^{(1+\epsilon)z}. \quad (83)$$

For each $t > 0$, these functions of ϵ are analytic in the region $V = \{\epsilon \in \mathbb{C};\, -1 < \operatorname{Re}\epsilon < p\}$. Indeed, lemma 1 with $r = 1$ and (2) give that $0 \leqslant F[h(\mu_t+z,t)] \leqslant C_t\mathrm{e}^{-(1+p)z}$ for some function of time C_t. Using that bound for $z > 0$ and the bound $0 \leqslant F[h(\mu_t+z,t)] \leqslant 1$ for $z < 0$, we get from lemma 5 that $\epsilon \mapsto \varphi(\epsilon,t)$ is analytic in the domain $\{\epsilon \in \mathbb{C};\, -1+a < \operatorname{Re}\epsilon < p-a\}$ for any $a > 0$, and is therefore analytic on the domain V defined above. The same argument works in the same way for $\hat{\varphi}(\epsilon)$.

Then, as $\gamma > 1$, Bramson's result implies that $\mu_t = 2t - \frac{3}{2}\log t + a + o(1)$. With (82), we see that there is a constant $C' > 0$ such that

$$\left|\varphi(\epsilon,t)-\hat{\varphi}(\epsilon)\right|\mathrm{e}^{(1+\epsilon)(\mu_t-2t)} \leqslant \frac{C'}{t^{\frac{5}{2}+\frac{3}{2}\epsilon}} \qquad \text{for } t > t_0 \text{ and } \epsilon \in U. \quad (84)$$

Pick $\beta \in (1,\gamma)$, and make the neighbourhood U smaller if needed so that $\epsilon \in U \implies 0.5 < 1+\epsilon < \beta$. Then, recalling the definition (21) of g, since $F(h) < h$, and since $\mathrm{e}^{(1+\epsilon)z} \leqslant \mathrm{e}^{0.5z} + \mathrm{e}^{\beta z}$ for all z,

$$\varphi(\epsilon,t)\mathrm{e}^{(1+\epsilon)\mu_t} \leqslant g(0.5,t) + g(\beta,t) \qquad \text{for } t > 0 \text{ and } \epsilon \in U, \quad (85)$$

which remains bounded for $t \in [0,t_0]$ according to lemma 1. Similarly, $\hat{\varphi}(\epsilon)$ is bounded for $\epsilon \in U$ and we see that the left hand side of (84) is uniformly bounded by some constant for $t \leqslant t_0$ and $\epsilon \in U$. Then, (81) is a direct application of lemma 7. This concludes the proof of lemma 3, and of proposition 2.

□

We now turn to the proof of theorem 2.

Proof of theorem 2. We assume that the hypotheses of that theorem hold; they imply in particular that lemma 3, the second form of the magical relation, holds:

$$\hat{\varphi}(\epsilon)\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\epsilon^2 t+(1+\epsilon)(\mu_t-2t)} = \int \mathrm{d}x\,h_0(x)\mathrm{e}^{(1+\epsilon)x} - \mathbb{1}_{\{\epsilon>0\}}\Phi(2+2\epsilon) + \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3). \quad (86)$$

Furthermore, the hypothesis that $\int \mathrm{d}x\, h_0(x)\mathrm{e}^{rx} < \infty$ for some $r > 1$ (*i.e.* $\gamma > 1$) implies (see lemma 5) that $\int \mathrm{d}x\, h_0(x)\mathrm{e}^{(1+\epsilon)x}$ is an analytic function of ϵ around 0, so that it can be absorbed into the $\mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3)$ term. Finally, we write (86) as

$$\hat{\varphi}(\epsilon)I(\epsilon) = -\mathbb{1}_{\{\epsilon>0\}}\Phi(2+2\epsilon) + \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3) \qquad \text{with } I(\epsilon) = \int_1^\infty \mathrm{d}t\, \mathrm{e}^{-\epsilon^2 t + (1+\epsilon)(\mu_t - 2t)}. \tag{87}$$

Notice that we defined $I(\epsilon)$ as an integral from 1 to ∞, not 0 to ∞. We are allowed to do this because the remaining integral from 0 to 1 is an analytic function of ϵ around 0; multiplied by $\hat{\varphi}(\epsilon)$ (another analytic function), it can be absorbed into $\mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3)$ term.

The proof can be split into three steps; in step 1, we compute a small ϵ expansion of $I(\epsilon)$. That expansion, valid for any sign of ϵ, is highly irregular and involves logarithmic terms and powers of both ϵ and $|\epsilon|$. We show incidently how this expansion for $\epsilon < 0$ allows to recover the numerical coefficients in (7). More importantly, this leads with (87) to a first expression of $\Phi(2+2\epsilon)$ which still involves $\hat{\varphi}(\epsilon)$. In step 2, we compute a small ϵ expansion of $\hat{\varphi}(\epsilon)$. Finally, step 3 is some elementary but tedious algebra needed to obtain the final form of $\Phi(2+\epsilon)$.

Step 1 Using the lemmas proved in section 4, we now compute a small ϵ expansion of $I(\epsilon)$. We have assumed that the position μ_t has a large t expansion given by (7); actually, let us simply write

$$\mu_t = 2t - \frac{3}{2}\log t + a + \frac{b}{\sqrt{t}} + c\frac{\log t}{t} + r(t) \qquad \text{with } r(t) = \mathcal{O}\left(\frac{1}{t}\right), \tag{88}$$

and we will recover the values of b and c as given in (7). We have

$$\begin{aligned} \mathrm{e}^{(1+\epsilon)(\mu_t-2t)} &= \frac{\mathrm{e}^{(1+\epsilon)a}}{t^{\frac{3}{2}+\frac{3}{2}\epsilon}}\mathrm{e}^{\frac{b(1+\epsilon)}{\sqrt{t}} + \frac{c(1+\epsilon)\log t}{t} + (1+\epsilon)r(t)} \\ &= \mathrm{e}^{(1+\epsilon)a}\left[\frac{1}{t^{\frac{3}{2}+\frac{3}{2}\epsilon}} + \frac{b(1+\epsilon)}{t^{2+\frac{3\epsilon}{2}}} + \frac{c(1+\epsilon)\log t}{t^{\frac{5}{2}+\frac{3}{2}\epsilon}}\right] + R(t,\epsilon) \end{aligned} \tag{89}$$

with

$$R(t,\epsilon) = \frac{\mathrm{e}^{(1+\epsilon)a}}{t^{\frac{3}{2}+\frac{3}{2}\epsilon}}\left[\mathrm{e}^{\frac{b(1+\epsilon)}{\sqrt{t}} + \frac{c(1+\epsilon)\log t}{t} + (1+\epsilon)r(t)} - \left(1 + \frac{b(1+\epsilon)}{\sqrt{t}} + \frac{c(1+\epsilon)\log t}{t}\right)\right]. \tag{90}$$

For $|u| < 1$, we have the bound $|\mathrm{e}^u - (1+u-v)| \leqslant |\mathrm{e}^u - (1+u)| + |v| \leqslant u^2 + |v|$. Applying this to $u = \frac{b(1+\epsilon)}{\sqrt{t}} + \frac{c(1+\epsilon)\log t}{t} + (1+\epsilon)r(t)$ and $v = (1+\epsilon)r(t)$, we see easily that there exists a $C > 0$ and a real neighbourhood U of 0 such that

$$|R(t,\epsilon)| \leqslant \frac{C}{t^{\frac{5}{2}+\frac{3}{2}\epsilon}} \qquad \text{for all } t > 1 \text{ and all } \epsilon \in U. \tag{91}$$

As $\epsilon \mapsto R(t,\epsilon)$ is analytic around 0 for all $t > 0$, a direct application of lemma 6, and in particular of (59), and of lemma 7 gives from (87), (89) and (91):

$$\begin{aligned} I(\epsilon) = \mathrm{e}^{(1+\epsilon)a}\Bigg[&|\epsilon|^{1+3\epsilon}\Gamma\left(-\frac{1}{2} - \frac{3}{2}\epsilon\right) + b(1+\epsilon)|\epsilon|^{2+3\epsilon}\Gamma\left(-1 - \frac{3}{2}\epsilon\right) \\ &+ c(1+\epsilon)|\epsilon|^{3+3\epsilon}\left(-2\log|\epsilon|\,\Gamma\left(-\frac{3}{2} - \frac{3}{2}\epsilon\right) + \Gamma'\left(-\frac{3}{2} - \frac{3}{2}\epsilon\right)\right)\Bigg] + \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3). \end{aligned} \tag{92}$$

The three analytic functions from lemma 6 have been absorbed into the $\mathcal{P}(\epsilon)+\mathcal{O}(\epsilon^3)$ term of lemma 7. We expand all the Gamma functions; the expansion of the second one is irregular:

$$\Gamma\left(-1-\frac{3}{2}\epsilon\right)=\frac{\Gamma(-\frac{3}{2}\epsilon)}{-1-\frac{3}{2}\epsilon}=\frac{\Gamma(1-\frac{3}{2}\epsilon)}{-\frac{3}{2}\epsilon(-1-\frac{3}{2}\epsilon)}=\frac{2}{3\epsilon}\frac{1+\gamma_E\frac{3}{2}\epsilon+\mathcal{O}(\epsilon^2)}{1+\frac{3}{2}\epsilon}$$
$$=\frac{2}{3\epsilon}+\gamma_E-1+\mathcal{O}(\epsilon), \tag{93}$$

where $\gamma_E=-\Gamma'(1)\simeq 0.577$ is Euler's gamma constant. We obtain

$$I(\epsilon)=\mathrm{e}^{(1+\epsilon)a}|\epsilon|^{3\epsilon}\left[\Gamma\left(-\frac{1}{2}\right)|\epsilon|-\frac{3}{2}\Gamma'\left(-\frac{1}{2}\right)\epsilon|\epsilon|+b\left(\frac{2}{3}\epsilon+\left(\gamma_E-\frac{1}{3}\right)\epsilon^2\right)\right.$$
$$\left.-2c\Gamma\left(-\frac{3}{2}\right)|\epsilon|^3\log|\epsilon|\right]+\mathcal{P}(\epsilon)+\mathcal{O}(\epsilon^3),$$
$$=\mathrm{e}^{(1+\epsilon)a}|\epsilon|^{3\epsilon}\left[\left(\Gamma\left(-\frac{1}{2}\right)|\epsilon|+\frac{2}{3}b\epsilon\right)+\left(b\left(\gamma_E-\frac{1}{3}\right)\epsilon^2-\frac{3}{2}\Gamma'\left(-\frac{1}{2}\right)\epsilon|\epsilon|\right)\right.$$
$$\left.-2c\Gamma\left(-\frac{3}{2}\right)|\epsilon|^3\log|\epsilon|\right]+\mathcal{P}(\epsilon)+\mathcal{O}(\epsilon^3). \tag{94}$$

Notice how the expansion mixes terms such as ϵ and $|\epsilon|$. For reference, we recall that

$$\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi},\quad \Gamma'\left(-\frac{1}{2}\right)=-2\sqrt{\pi}\,(2-\gamma_E-2\log 2)\,,\quad \Gamma\left(-\frac{3}{2}\right)=\frac{4}{3}\sqrt{\pi}. \tag{95}$$

Before going further, we show how to recover the values of b and c. Notice in (87) that, for $\epsilon<0$ (and since $\hat{\varphi}(\epsilon)$ is analytic around 0), we must have $I(\epsilon)=\mathcal{P}(\epsilon)+\mathcal{O}(\epsilon^3)$. In particular, there must remain no $\log|\epsilon|$ term in the expansion (94) for $\epsilon<0$. There is a $\log|\epsilon|$ term explicitly written in (94), and others in the expansion of the prefactor $|\epsilon|^{3\epsilon}=1+3\epsilon\log|\epsilon|+\frac{9}{2}\epsilon^2\log^2|\epsilon|+\cdots$. Developing, we obtain a term $\left(\Gamma(-\frac{1}{2})|\epsilon|+\frac{2}{3}b\epsilon\right)3\epsilon\log|\epsilon|$; that term must cancel for $\epsilon<0$, hence, with (95)

$$b=\frac{3}{2}\Gamma\left(-\frac{1}{2}\right)=-3\sqrt{\pi}. \tag{96}$$

Then, c must be chosen in order to prevent a term $\epsilon^3\log|\epsilon|$ from appearing when $\epsilon<0$. This leads to

$$3\left(b\left(\gamma_E-\frac{1}{3}\right)+\frac{3}{2}\Gamma'\left(-\frac{1}{2}\right)\right)+2c\Gamma\left(-\frac{3}{2}\right)=0. \tag{97}$$

With (95) and (96), this leads to

$$c=\frac{9}{8}(5-6\log 2). \tag{98}$$

Using the values (96) and (98) of b and c in (88) gives back the expression (7) of the position μ_t of the front. Let us make two remarks:

- If we try to add in (88) extra terms of the form $C(\log t)^n/t^\alpha$, we would obtain non-cancellable singularities (terms containing $\log|\epsilon|$ or non integral powers of $|\epsilon|$) in the expansion of $I(\epsilon)$. We conclude that if μ_t can be written as an expansion in terms of the form $C(\log t)^n/t^\alpha$, then the only terms that may appear are those written in (88).

- We used the hypothesis that $\int \mathrm{d}x\, h_0(x)\mathrm{e}^{rx} < \infty$ for some $r > 1$ (*i.e.* $\gamma > 1$), only once, to get rid of the $\int \mathrm{d}x\, h_0(x)\mathrm{e}^{(1+\epsilon)x}$ term in (86). If we relax this hypothesis and simply assume $\int \mathrm{d}x\, h_0(x)x\mathrm{e}^{x} < \infty$ (this is needed to reach (86)), we would have at this point that, for $\epsilon < 0$, $\hat{\varphi}(\epsilon)I(\epsilon) = \int \mathrm{d}x\, h_0(x)\mathrm{e}^{(1+\epsilon)x} + \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3)$. We have just shown that, if the position μ_t of the front is given by (7), then $I(\epsilon) = \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3)$ for $\epsilon < 0$. We thus see that

$$\mu_t \text{ given by (7)} \implies \int \mathrm{d}x\, h_0(x)\mathrm{e}^{(1+\epsilon)x} = \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3) \text{ for } \epsilon < 0 \\ \iff \int \mathrm{d}x\, h_0(x)x^3\mathrm{e}^{x} < \infty. \tag{99}$$

  (We omit the proof of the last equivalence.) Conversely, if $\int \mathrm{d}x\, h_0(x)x^3\mathrm{e}^{x} = \infty$, then the asymptotic expansion for small negative ϵ of $\int \mathrm{d}x\, h_0(x)\mathrm{e}^{(1+\epsilon)x}$ will feature some singular terms larger than ϵ^3, and the expression of μ_t needs to be modified in such a way that $\hat{\varphi}(\epsilon)I(\epsilon)$ matches those singular terms.

We return to the expression (94) of $I(\epsilon)$ without making any assumption on the sign of ϵ, and we make the substitution

$$|\epsilon| = -\epsilon + 2\epsilon\mathbb{1}_{\{\epsilon>0\}}, \qquad |\epsilon|^3 = -\epsilon^3 + 2\epsilon^3\mathbb{1}_{\{\epsilon>0\}}. \tag{100}$$

We have tuned b and c so that one obtains $I(\epsilon) = \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3)$ for $\epsilon < 0$. For ϵ of either sign, we have three extra terms multiplied by $\mathbb{1}_{\{\epsilon>0\}}$, corresponding to the three terms with $|\epsilon|$ or $|\epsilon|^3$ in (94):

$$I(\epsilon) = \mathbb{1}_{\{\epsilon>0\}}\mathrm{e}^{(1+\epsilon)a}\epsilon^{3\epsilon}\left[2\Gamma\left(-\frac{1}{2}\right)\epsilon - 3\Gamma'\left(-\frac{1}{2}\right)\epsilon^2 - 4c\Gamma\left(-\frac{3}{2}\right)\epsilon^3\log\epsilon\right] + \mathcal{P}(\epsilon) + \mathcal{O}(\epsilon^3). \tag{101}$$

Comparing with (87), we see that we must have (only for $\epsilon > 0$, of course):

$$\Phi(2+2\epsilon) = \hat{\varphi}(\epsilon)\mathrm{e}^{(1+\epsilon)a}\epsilon^{3\epsilon}\left[-2\Gamma\left(-\frac{1}{2}\right)\epsilon + 3\Gamma'\left(-\frac{1}{2}\right)\epsilon^2 + 4c\Gamma\left(-\frac{3}{2}\right)\epsilon^3\log\epsilon\right] + \mathcal{O}(\epsilon^3). \tag{102}$$

This expression will be, after some transformations, our main result (8).

Step 2 We now make a small ϵ expansion of $\hat{\varphi}(\epsilon)$. From the definition (15) of $\hat{\varphi}(\epsilon)$ and the equation (4) followed by ω, one has

$$\hat{\varphi}(\epsilon) = \int \mathrm{d}z\, F[\omega(z)]\mathrm{e}^{(1+\epsilon)z} = \int \mathrm{d}z\left[\omega''(z) + 2\omega'(z) + \omega(z)\right]\mathrm{e}^{(1+\epsilon)z}. \tag{103}$$

This function $\hat{\varphi}(\epsilon)$ is analytic around $\epsilon = 0$, but we need to assume $-1 < \epsilon < 0$ to split the integral into three terms and integrate by parts. (Recall that $\omega(z) \sim \tilde{\alpha}ze^{-z}$ as $z \to \infty$.)

$$\begin{aligned}\hat{\varphi}(\epsilon) &= \int \mathrm{d}z\,\omega''(z)\mathrm{e}^{(1+\epsilon)z} + 2\int \mathrm{d}z\,\omega'(z)\mathrm{e}^{(1+\epsilon)z} + \int \mathrm{d}z\,\omega(z)\mathrm{e}^{(1+\epsilon)z},\\ &= \left[(1+\epsilon)^2 - 2(1+\epsilon) + 1\right]\int \mathrm{d}z\,\omega(z)\mathrm{e}^{(1+\epsilon)z} = \epsilon^2\int \mathrm{d}z\,\omega(z)\mathrm{e}^{(1+\epsilon)z},\\ &= \epsilon^2\mathrm{e}^{-(1+\epsilon)\alpha}\int \mathrm{d}z\,\omega(z-a)\mathrm{e}^{(1+\epsilon)z} \qquad \text{for } -1 < \epsilon < 0.\end{aligned} \tag{104}$$

Recall (5): for any $q \in (0,p)$,

$$\omega(z-a) = (\alpha z + \beta)\mathrm{e}^{-z} + \mathcal{O}(\mathrm{e}^{-(1+q)z}) \quad \text{as } z \to \infty, \tag{105}$$

Then $\int \mathrm{d}z\,\omega(z-\alpha)\mathrm{e}^{(1+\epsilon)z} = \alpha/\epsilon^2 - \beta/\epsilon + \mathcal{O}(1)$ and

$$\hat{\varphi}(\epsilon)\mathrm{e}^{(1+\epsilon)a} = \alpha - \beta\epsilon + \mathcal{O}(\epsilon^2). \tag{106}$$

Even though the intermediate steps are only valid for $\epsilon < 0$, the final result is also valid for $\epsilon > 0$ (small enough) by analyticity.

Step 3 We now gather the different terms and make the final simplifications. We use (106) in (102) and we replace c and the Gamma functions by their values (95) and (98) to obtain

$$\begin{aligned}\Phi(2+2\epsilon) &= (\alpha-\beta\epsilon)\epsilon^{3\epsilon}\Big[4\sqrt{\pi}\epsilon - 6\sqrt{\pi}(2-\gamma_E-2\log 2)\epsilon^2 + 6(5-6\log 2)\sqrt{\pi}\epsilon^3\log\epsilon\Big] \\ &\quad + \mathcal{O}(\epsilon^3) \\ &= \sqrt{\pi}(\alpha-\beta\epsilon)\epsilon^{3\epsilon}\Big[4\epsilon - 6(2-\gamma_E-2\log 2)\epsilon^2 + 6(5-6\log 2)\epsilon^3\log\epsilon\Big] + \mathcal{O}(\epsilon^3).\end{aligned} \tag{107}$$

It remains to develop with the term $\epsilon^{3\epsilon} = 1 + 3\epsilon\log\epsilon + \frac{9}{2}\epsilon^2\log^2\epsilon + \cdots$; only the coefficient of $\epsilon^3\log\epsilon$ requires to combine two terms: $3\times(-6)(2-\gamma_E-2\log 2) + 6(5-6\log 2) = 6(3\gamma_E - 1)$. We obtain.

$$\begin{aligned}\Phi(2+2\epsilon) = \sqrt{\pi}(\alpha-\beta\epsilon)\,&\big[4\epsilon + 12\epsilon^2\log\epsilon - 6(2-\gamma_E-2\log 2)\epsilon^2 \\ &+18\epsilon^3\log^2\epsilon + 6(3\gamma_E-1)\epsilon^3\log\epsilon\big] + \mathcal{O}(\epsilon^3).\end{aligned} \tag{108}$$

The last step is to replace ϵ by $\epsilon/2$

$$\begin{aligned}\Phi(2+\epsilon) &= \sqrt{\pi}\Big(\alpha - \frac{\beta}{2}\epsilon\Big)\Big[2\epsilon + 3\epsilon^2\log\frac{\epsilon}{2} - 3\Big(1-\frac{\gamma_E}{2}-\log 2\Big)\epsilon^2 + \frac{9}{4}\epsilon^3\log^2\frac{\epsilon}{2} \\ &\quad + \frac{3}{4}(3\gamma_E-1)\epsilon^3\log\frac{\epsilon}{2}\Big] + \mathcal{O}(\epsilon^3) \\ &= \sqrt{\pi}\Big(\alpha-\frac{\beta}{2}\epsilon\Big)\Big[2\epsilon + 3\epsilon^2\log\epsilon - 3\Big(1-\frac{\gamma_E}{2}\Big)\epsilon^2 + \frac{9}{4}\epsilon^3\log^2\epsilon \\ &\quad + \frac{3}{4}(3\gamma_E - 6\log 2 - 1)\epsilon^3\log\epsilon\Big] + \mathcal{O}(\epsilon^3),\end{aligned} \tag{109}$$

which is (11). This completes the proof of theorem 2.

□

It now remains to prove proposition 1 to obtain theorem 1.

6. Proof of proposition 1

We start by recalling the main results of [Gra19]:

Theorem 3 (Cole Graham 2019 [Gra19]). *Let $h(x,t)$ be the solution to the Fisher–KPP equation (1) with $F(h) = h^2$ and with initial condition $h_0(x)$. Assume that $0 \leqslant h_0 \leqslant 1$ and that h_0 is a compact perturbation of the step function. There exist α_0 and α_1 in $\mathbb{R}$ depending on the* *initial data h_0 such that the following holds. For any $\gamma > 0$, there exists $C_\gamma > 0$ also depending on h_0 such that for all $x \in \mathbb{R}$ and all $t \geqslant 3$*

$$\left|h(\sigma_t + x, t) - U_{\text{app}}(x,t)\right| \leqslant \frac{C_\gamma(1+|x|)\mathrm{e}^{-x}}{t^{\frac{3}{2}-\gamma}}, \tag{110}$$

where

$$\sigma_t = 2t - \frac{3}{2}\log t + \alpha_0 - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6\log 2)\frac{\log t}{t} + \frac{\alpha_1}{t} \tag{111}$$

and

$$U_{\text{app}}(x,t) = \phi(x) + \frac{1}{t}\psi(x) + \mathcal{O}(t^{\gamma-3/2}) \quad \text{locally uniformly in } x. \tag{112}$$

Here, $\phi(x)$ is the critical travelling wave translated in such a way that [Gra19, equation (1.2)]

$$\phi(x) = A_0 x\mathrm{e}^{-x} + \mathcal{O}(\mathrm{e}^{-(1+q)x}) \quad \text{as } x \to \infty, \tag{113}$$

and $\psi(x)$ satisfies [Gra19, lemma 5 with $\psi(x) = A_0\mathrm{e}^{-x}V_1^-(x)$ as written in the Proof of theorem 3 p 1985]

$$\psi(x) \text{ is bounded,} \qquad \psi(x) \sim -\frac{A_0}{4}x^3\mathrm{e}^{-x} \quad \text{as } x \to \infty. \tag{114}$$

Furthermore, there exist smooth functions V_1^+, V_2^+ and V_3^+ of $x/\sqrt{t}$ such that, for t large enough and $\gamma \in (0,3)$,

$$\begin{cases} \left|U_{\text{app}}(x,t) - \phi(x) - \frac{1}{t}\psi(x)\right| \leqslant C_\gamma \frac{\min(1,\mathrm{e}^{-x})}{t^{\frac{3}{2}-\frac{2}{3}\gamma}} & \text{for } x \leqslant t^{\gamma/6}, \\ \Big|U_{\text{app}}(x,t) - A_0 e^{-x}\Big(x\mathrm{e}^{-x^2/(4t)} + V_1^+(x/\sqrt{t}) & \\ \qquad + \frac{\log t}{\sqrt{t}}V_2^+(x/\sqrt{t}) + \frac{1}{\sqrt{t}}V_3^+(x/\sqrt{t})\Big)\Big| \leqslant C_\gamma \frac{\mathrm{e}^{-x}}{t^{\frac{3}{2}-\frac{1}{2}\gamma}} & \text{for } x > t^{\gamma/6}. \end{cases} \tag{115}$$

The V_i^+ satisfy $V_i^+(0) = V_i^+(\infty) = 0$, and so there are bounded.

Remark. • We introduced in (4) the critical travelling wave $\omega(x)$, fixing the translational invariance by imposing $\omega(0) = \frac{1}{2}$. That critical wave decays as $\omega(x) = (\tilde{\alpha}x + \tilde{\beta})\mathrm{e}^{-x} + \mathcal{O}(\mathrm{e}^{-(1+q)x})$ for large x, see text above (5). The critical wave $\phi(x)$ in [Gra19] is obtained by taking $\phi(x) = \omega(x - \tilde{\beta}/\tilde{\alpha})$, so that there remains no Cste $\times\, \mathrm{e}^{-x}$ term in its large x expansion.

• (115) is not explicitly written in [Gra19], but it can be pieced together from the proofs: in the Proof of theorem 3 p 1985, one reads $U_{\text{app}}(x,t) = A_0\mathrm{e}^{-x}V_{\text{app}}(x,t)$ and in the proof of theorem 9, p 1986, one reads

$$V_{\text{app}}(x,t) = \mathbb{1}_{\{x<t^\epsilon\}}V^-(x,t) + \mathbb{1}_{\{x\geqslant t^\epsilon\}}V^+(x,t) + K(t)\theta(xt^{-\epsilon})\varphi(x,t). \tag{116}$$

At the end of proof (p 1995), the author takes $\epsilon = \gamma/6$; the functions φ and θ are bounded, $K(t) = \mathcal{O}(t^{3\epsilon-3/2})$, and θ is supported on $(0,2)$, see p 1986. Then, we have so far, for some C,

$$\begin{cases} \left|U_{\text{app}}(x,t) - A_0\mathrm{e}^{-x}V^-(x,t)\right| \leqslant C\frac{\mathrm{e}^{-x}\mathbb{1}_{\{x>0\}}}{t^{\frac{3}{2}-\frac{1}{2}\gamma}} & \text{for } x \leqslant t^{\gamma/6}, \\ \left|U_{\text{app}}(x,t) - A_0\mathrm{e}^{-x}V^+(x,t)\right| \leqslant C\frac{\mathrm{e}^{-x}}{t^{\frac{3}{2}-\frac{1}{2}\gamma}} & \text{for } x > t^{\gamma/6}. \end{cases} \tag{117}$$

(C is some positive constant independent of x and t which can change at each occurrence.)

We start with the first line; the function V^- is given at the top of p 1986:

$$V^-(x,t) = V_0^-(x+\zeta_t) + \frac{1}{t}V_1^-(x+\zeta_t), \tag{118}$$

with

$$V_0^-(x) = A_0^{-1}\mathrm{e}^x\phi(x), \qquad V_1^-(x) = A_0^{-1}\mathrm{e}^x\psi(x), \qquad \zeta(t) = \mathcal{O}(t^{4\epsilon-\frac{3}{2}}) = \mathcal{O}(t^{\frac{2}{3}\gamma-\frac{3}{2}}). \tag{119}$$

(See respectively p 1972, proof of theorem 3 p 1985, and bottom of p 1985.)

From p 1972 and lemma 5 p 1973, we have $(V_0^-)'(x) \sim 1$ and $(V_1^-)'(x) \sim -\frac{3}{4}x^2$ as $x \to \infty$, and $(V_0^-)'(x) = \mathcal{O}(\mathrm{e}^x)$ and $(V_1^-)'(x) = \mathcal{O}(\mathrm{e}^x)$ as $x \to -\infty$. Thus, $|(V_0^-)'(x)|$ and $|(V_1^-)'(x)/t|$ are both bounded by $C\min(\mathrm{e}^x, 1)$ for all $t > 1$ and all $x < \sqrt{t}$. This implies that

$$\left|V^-(x,t) - V_0^-(x) - \frac{1}{t}V_1^-(x)\right| \leqslant C\min(\mathrm{e}^x,1)\zeta_t \leqslant C\frac{\min(\mathrm{e}^x,1)}{t^{\frac{3}{2}-\frac{2}{3}\gamma}} \quad \text{for } t>1 \text{ and } x<\sqrt{t}. \tag{120}$$

Multiplying by $A_0\mathrm{e}^{-x}$ and using (119),

$$\left|A_0\mathrm{e}^{-x}V^-(x,t) - \phi(x) - \frac{1}{t}\psi(x)\right| \leqslant C\frac{\min(1,\mathrm{e}^{-x})}{t^{\frac{3}{2}-\frac{2}{3}\gamma}} \quad \text{for } t>1 \text{ and } x<\sqrt{t}. \tag{121}$$

Combining with the first line of (117) under the assumption $\gamma < 3$, we obtain the first line of (115), as the bounding term in (117) is small compared to the bounding term in (121).

We now turn to the second line of (117). The function V^+, only defined for $x > 0$, is given in (3.4) p 1973 in terms of $\tau = \log t$ and $\eta = x/\sqrt{t}$:

$$V^+(x,t) = \mathrm{e}^{\tau/2}V_0^+(\eta) + V_1^+(\eta) + \tau\mathrm{e}^{-\tau/2}V_2^+(\eta) + \mathrm{e}^{-\tau/2}V_3^+(\eta), \tag{122}$$

with

$$V_0^+(\eta) = \eta\mathrm{e}^{-\eta^2/4}, \qquad \text{and so} \qquad \mathrm{e}^{\tau/2}V_0^+(\eta) = x\mathrm{e}^{-x^2/(4t)}. \tag{123}$$

(Top of p 1974: $V_0^+(\eta) = q_0\phi_0(\eta)$ for some real q_0; middle of p 1974: $q_0 = 1$; bottom of p 1973: $\phi_0(\eta) = \eta\mathrm{e}^{-\eta^2/4}$.) Using (122) and (123) in the second line of (117) gives the second line of (115). The V_i^+ are smooth (they are solutions on some differential equations written pp 1974, 1975), and satisfy $V_i^+(0) = V_i^+(\infty) = 0$, see line after (3.4) p 1973.

We wrote (115) with the accuracy provided by the proofs of [Gra19], but we actually need a less precise version, only up to order $1/t$:

Corollary 1. *With the notations and hypotheses of theorem 3, for any $\gamma \in (0, 1/2]$, if t is large enough,*

$$\begin{cases} \left|h(\sigma_t + x, t) - \phi(x)\right| \leqslant C_\gamma \dfrac{(1+|x|^3)\mathrm{e}^{-x}}{t} & \text{for } x \leqslant t^{\gamma/6}, \\ \left|h(\sigma_t + x, t) - \phi(x)\right| \leqslant C_\gamma x\mathrm{e}^{-x} & \text{for } x > t^{\gamma/6}. \end{cases} \tag{124}$$

Proof. Recall from (114) that ψ is bounded and $\psi(x) \sim Cx^3\mathrm{e}^{-x}$ as $x \to \infty$. This implies that $|\psi(x)| \leqslant C\min\left(1, (1+|x|^3)\mathrm{e}^{-x}\right)$ for some constant C. Then, the first line of (115) implies that

$$\left|U_{\mathrm{app}}(x,t) - \phi(x)\right| \leqslant C\frac{\min\left(1,(1+|x|^3)\mathrm{e}^{-x}\right)}{t} \qquad \text{for } x \leqslant t^{\gamma/6} \tag{125}$$

for some other constant C. With (110), this implies the first line of (124). (Recall $\gamma \leqslant \frac{1}{2}$.)

In the second line of (115), the quantities V_i^+ are bounded. As $x > t^{\gamma/6} \geqslant 1$, we have

$$\left|U_{\mathrm{app}}(x,t)\right| \leqslant Cx\mathrm{e}^{-x} \qquad \text{for } x > t^{\gamma/6}. \tag{126}$$

As we also have $\phi(x) \sim A_0 x e^{-x}$, we obtain $\left|U_{\text{app}}(x,t) - \phi(x)\right| \leqslant Cxe^{-x} \text{for} x > t^{\gamma/6}$. which gives, with (110), the second line of (124).

□

Unfortunately, theorem *3* and, consequently, corollary 1 are very imprecise for $x < 0$. We will need the following result to complement corollary 1:

Lemma 8. *With the notations and hypotheses of theorem 3, there exists C and t_0 depending on the initial condition h_0 such that, for $t \geqslant t_0$,*

$$\left|h(\sigma_t + x, t) - \phi(x)\right| \leqslant \frac{C}{t} \qquad \text{for } x \leqslant 0. \tag{127}$$

Proof. Choose $\alpha \in (0, \frac{1}{2})$ and let $x_0 = \phi^{-1}(\frac{1}{2} + \alpha)$. It suffices to prove $|h(\sigma_t + x, t) - \phi(x)| \leqslant C/t$ for $x \leqslant x_0$: if $x_0 \geqslant 0$, then (127) follows; if $x_0 < 0$, then (124) provides the required bound for $x \in [x_0, 0]$.

Let

$$\delta(x,t) = h(\sigma_t + x, t) - \phi(x). \tag{128}$$

By substitution, one obtains

$$\begin{aligned} \partial_t \delta = \frac{\mathrm{d}}{\mathrm{d}t} h(\sigma_t + x, t) &= \partial_x^2(\phi + \delta) + \dot{\sigma}_t \partial_x(\phi + \delta) + (\phi + \delta) - (\phi + \delta)^2, \\ &= \phi'' + \dot{\sigma}_t \phi' + \phi - \phi^2 + \partial_x^2 \delta + \dot{\sigma}_t \partial_x \delta + \delta - 2\delta\phi - \delta^2, \\ &= (\dot{\sigma}_t - 2)\phi' + \partial_x^2 \delta + \dot{\sigma}_t \partial_x \delta + (1 - 2\phi - \delta)\delta, \end{aligned} \tag{129}$$

where we used in the last step that $\phi'' + 2\phi' + \phi - \phi^2 = 0$.

As $\delta(x,t)$ converges uniformly to 0 [Bra83], there is a time $t_0 > 0$ such that $|\delta(x,t)| \leqslant \alpha$ for all x and all $t \geqslant t_0$. Recalling that $\phi(x_0) = \frac{1}{2} + \alpha$ and ϕ is a decreasing function, we have then

$$1 - 2\phi(x) + |\delta(x,t)| \leqslant 1 - 2\phi(x_0) + \alpha = -\alpha \qquad \text{for } x \leqslant x_0 \text{ and } t \geqslant t_0. \tag{130}$$

From respectively (124) and $|\delta(x,t_0)| \leqslant \alpha$, one can find $C > 0$ such that

$$|\delta(x_0,t)| \leqslant \frac{C}{t} \quad \text{for } t \geqslant t_0, \qquad |\delta(x,t_0)| \leqslant \frac{C}{t_0} \quad \text{for } x \leqslant x_0. \tag{131}$$

As $\phi' < 0$ is bounded and $0 < 2 - \dot{\sigma}_t \sim \frac{3}{2t}$ for t large enough, one can increase t_0 and C such that, furthermore,

$$0 \leqslant \phi'(x)(\dot{\sigma}_t - 2) \leqslant \alpha\frac{C}{t} - \frac{C}{t^2} \quad \text{for } t \geqslant t_0 \text{ and } x \leqslant x_0. \tag{132}$$

(The reason for the negligible C/t^2 term will soon become apparent.) Let $\hat{\delta}$ be the solution to

$$\partial_t \hat{\delta} = \alpha\frac{C}{t} - \frac{C}{t^2} + \partial_x^2 \hat{\delta} + \dot{\sigma}_t \partial_x \hat{\delta} - \alpha\delta \quad \text{for } x < x_0, t > t_0, \qquad \hat{\delta}(x_0,t) = \frac{C}{t}, \qquad \hat{\delta}(x,t_0) = \frac{C}{t_0}. \tag{133}$$

We consider (129) for $x < x_0$ and $t > t_0$, taking as 'initial' condition $\delta(x,t_0)$ and as boundary condition $\delta(x_0,t)$. Using the comparison principle between δ and $\hat{\delta}$, and then between $-\delta$ and $\hat{\delta}$, one obtains with (130)–(132) that $|\delta(x,t)| \leqslant \hat{\delta}(x,t)$ for all $x \leqslant x_0$ and $t \geqslant t_0$. But the solution to (133) is $\hat{\delta}(x,t) = \frac{C}{t}$, hence $|\delta(x,t)| \leqslant \frac{C}{t}$ for $t \geqslant t_0$ and $x \leqslant x_0$.

□

We can now prove proposition 1.

Proof of proposition 1. The fact that (7) holds is already proved in [Gra19, corollary 4], as an easy corollary of theorem 3, which states:

$$\mu_t = \sigma_t + \phi^{-1}\left(\frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{t}\right), \tag{134}$$

so that a in (7) is given by $a = \alpha_0 + \phi^{-1}(\frac{1}{2})$. It remains to prove that (12) with $F(h) = h^2$ holds:

$$\left|\int \mathrm{d}x\,\mathrm{e}^{rx} h(\mu_t + x, t)^2 - \int \mathrm{d}x\,\mathrm{e}^{rx}\omega(x)^2\right| \leqslant \frac{C}{t} \qquad \text{for } t > t_0 \text{ and } r \in U, \tag{135}$$

where $C > 0$ and $t_0 > 0$ are some constants, and where U is some real neighbourhood of 1. We choose to take $U = [0,01, 1.99]$.

In (135), make the change of variable $x \to x + \sigma_t - \mu_t$ in the first integral, and the change $x \to x - \phi^{-1}(\frac{1}{2})$ in the second. Recalling that $\omega(x - \phi^{-1}(\frac{1}{2})) = \phi(x)$ and factorising by $\mathrm{e}^{r(\sigma_t - \mu_t)}$, we obtain that (135) is equivalent to

$$\mathrm{e}^{r(\sigma_t-\mu_t)}\left|\int \mathrm{d}x\,\mathrm{e}^{rx} h(\sigma_t + x, t)^2 - \mathrm{e}^{r(\mu_t - \sigma_t - \phi^{-1}(\frac{1}{2}))}\int \mathrm{d}x\,\mathrm{e}^{rx}\phi(x)^2\right| \leqslant \frac{C}{t} \quad \text{for } t > t_0 \text{ and } r \in U. \tag{136}$$

The prefactor $\mathrm{e}^{r(\sigma_t - \mu_t)}$ is bounded for $r \in U$ and $t > 1$, and can be dropped. As $\int \mathrm{d}x\,\mathrm{e}^{rx}\phi(x)^2$ is bounded for $r \in U$, and since (134) holds, one has for some C and t_0:

$$\left|\int \mathrm{d}x\,\mathrm{e}^{rx}\phi(x)^2 - \mathrm{e}^{r(\mu_t - \sigma_t - \phi^{-1}(\frac{1}{2}))}\int \mathrm{d}x\,\mathrm{e}^{rx}\phi(x)^2\right| \leqslant \frac{C}{t} \quad \text{for } t > t_0 \text{ and } r \in U, \tag{137}$$

and then (135) is equivalent to

$$\left|\int \mathrm{d}x\,\mathrm{e}^{rx} h(\sigma_t + x, t)^2 - \int \mathrm{d}x\,\mathrm{e}^{rx}\phi(x)^2\right| \leqslant \frac{C}{t} \qquad \text{for } t > t_0 \text{ and } r \in U. \tag{138}$$

We now show that (138) holds.

First notice that there exists $C > 0$ such that, for all x and all t large enough,

$$h(\sigma_t + x, t) \leqslant C\phi(x). \tag{139}$$

Indeed, from (124), $|h(\sigma_t + x, t) - \phi(x)| \leqslant 2C_\gamma x\mathrm{e}^{-x}$ for $x \geqslant 1$ and t large enough (we used $x^2 \leqslant t$ in the first line, since $\gamma \leqslant 1/2$). Since $\phi(x) \sim A_0 x\mathrm{e}^{-x}$ for large x, this implies that $h(\sigma_t + x, t) \leqslant C\phi(x)$ for some C is $x \geqslant 1$ and t large enough. Making C larger if needed so that $C\phi(1) \geqslant 1$ ensures that the relation also holds for $x \leqslant 1$ since ϕ is a decreasing function and $h \leqslant 1$.

Then, for another constant C, for all t large enough and all $r \in U$,

$$\begin{aligned}\left|\int \mathrm{d}x\,\mathrm{e}^{rx}\Big[h(\sigma + x, t)^2 - \phi(x)^2\Big]\right|, &\leqslant \int \mathrm{d}x\,\mathrm{e}^{rx}\Big|h(\sigma + x, t) - \phi(x)\Big| \times \Big(h(\sigma + x, t) + \phi(x)\Big) \\ &\leqslant C\int \mathrm{d}x\,\mathrm{e}^{rx}\phi(x)\Big|h(\sigma + x, t) - \phi(x)\Big|.\end{aligned} \tag{140}$$

We cut the integral in three ranges: $x < 0$, $0 < x < t^{\gamma/6}$ and $x > t^{\gamma/6}$. In the first range, we use $r \geqslant 0.01$, $\phi \leqslant 1$ and (127). In the two other ranges, we use $r \leqslant 1.99$ and (124):

$$\left| \int \mathrm{d}x\, \mathrm{e}^{rx} \left[h(\sigma + x, t)^2 - \phi(x)^2 \right] \right| \leqslant C \int_{-\infty}^{0} \mathrm{d}x\, \mathrm{e}^{0.01x} \frac{1}{t} + C \int_{0}^{t^{\gamma/6}} \mathrm{d}x\, \mathrm{e}^{1.99x} \phi(x) \frac{(1+x^3)\mathrm{e}^{-x}}{t}$$
$$+ C \int_{t^{\gamma/6}}^{\infty} \mathrm{d}x\, \mathrm{e}^{1.99x} \phi(x) x \mathrm{e}^{-x},$$
$$\leqslant \frac{C}{t} + \frac{C}{t} + C t^{\gamma/3} \mathrm{e}^{-0.01 t^{\gamma/6}} \leqslant \frac{C}{t}, \tag{141}$$

where we used $\mathrm{e}^{1.99x}\phi(x)x\mathrm{e}^{-x} \leqslant Cx^2\mathrm{e}^{-0.01x}$. This concludes the proof. □

7. Conclusion

In this paper, we study the quantity $\Phi(c)$ appearing in (8), which describes the behaviour of the solution to the Fisher–KPP equation (1) at time t and position ct for $c > 2$. We first showed that (8) holds for values of $c > 2$ satisfying (19), and we computed a small $\epsilon = c - 2$ expansion of the quantity $\Phi(c)$ appearing in (8), up to the order $\mathcal{O}(\epsilon^3)$, see (11). The expansion depends on the initial condition and the nonlinear term in (1) through two numbers α and β which characterise the shifted travelling wave reached by the front, see (3) and (5). Although, $\Phi'(2)$ exists, $\Phi''(2)$ does not. The expansion (11) is surprisingly irregular, with several logarithmic corrections.

Our method to reach this result relies on so-called magical relation between the position μ_t of the front, the initial condition h_0, and the quantity $\Phi(c)$, see proposition 2. This approach relates in some way the large t expansion (7) of the position μ_t of the front and the small ϵ expansion of $\Phi(2+\epsilon)$.

As explained in the proofs of the present paper and in [BBD18], the magical relation also allows to predict non-rigorously the coefficients of the large t expansion of the position of the front for all initial conditions. It would be interesting to turn this approach into a proof.

We believe that our result is universal; however, the proofs in this paper rely on knowing the large t expansion of the position of the front, and on some other technical condition (12) which has only been proved for the Fisher–KPP equation (1) with the $F(h) = h^2$ nonlinearity, and an initial condition which is a compact perturbation of the step function. Therefore, our result is only proved in that situation.

All the results in this paper could be easily extended to the front studied in [BBD17, BBD18, BBP19], where the nonlinearity in the Fisher–KPP equation is replaced by a moving boundary: $\partial_t h = \partial_x^2 h + h$ if $x > \mu_t$ and $h(x,t) = 1$ if $x \leqslant \mu_t$ with h differentiable at $x = \mu_t$. Then, as can be shown rigorously, the magical relation (16) still holds with $\varphi(\epsilon, t) = \hat{\varphi}(\epsilon) = 1/(1+\epsilon)$, and we believe that (11) also holds; the only result missing to prove it with our method is that the large t expansion of μ_t is also given by (7) for that model. (The technical condition (12) is not needed in that case.)

A front which satisfies $h(2t - \frac{3}{2}\log t + z, t) \to \omega(z-a)$ with $\omega(z) \simeq (\tilde{\alpha} z + \tilde{\beta})\mathrm{e}^{-z}$ is sometimes called a 'pulled front'. Bramson's conditions (2) for the nonlinearity $F(h)$ imply that the front h is pulled, but it is known that one can still have a pulled front in some situations where (2), and in particular the condition $F(h) > 0$, is not satisfied [Saa03]. It would be interesting to check whether our results hold for any pulled front, even when (2) is not satisfied. (Note: we use $F(h) > 0$ several times to state that h is smaller than the solution of the linearised equation, so our proofs will need some non-trivial changes. The fact that $F(h)$ is differentiable is only used in the proof of lemma 2, and we never use $F'(h) \geqslant 0$ or $F'(h) = \mathcal{O}(h^p)$; we do use however $F(h) = \mathcal{O}(h^{1+p})$ several times.)

The magical relation could also be used to compute $\Phi(c)$ for large c. As is clear from inspecting (16), this would require studying the early times of the evolution of the front. This point was already noticed in [DMS16].

Beyond the results themselves, the method used to reach them are, in our opinion, quite unexpected and interesting. We feel that there remains many aspects of the Fisher–KPP equation that could be better understood, and the magical relation might be a useful tool to that purpose.

Data availability statement

No new data were created or analysed in this study.

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Appendix

We prove lemma 4:

Lemma 4. *Let $\beta \in (0,\gamma)$. For $t > 0$, the quantities $h(x,t)$, $|\partial_x h(x,t)|$, $|\partial_x^2 h(x,t)|$ and $|\partial_t h(x,t)|$ are bounded by $A(t)\max(1,\mathrm{e}^{-\beta x})$ for some locally bounded function A.*

Proof. We already know that the result holds for $h(x,t)$ from lemma 1 and $0 < h(x,t) < 1$. The Fisher–KPP equation (1) will then provide the result for $\partial_t h$ once it is proved for $\partial_x^2 h$. We now focus on $\partial_x h$ and $\partial_x^2 h$. Following Uchiyama [Uch78, section 4], we use the following representations:

$$h(x,t) = \int \mathrm{d}y\, p(x-y,t)h_0(y) + \int_0^t \mathrm{d}s \int \mathrm{d}y\, p(x-y,t-s)f[h(y,s)], \tag{142}$$

$$\partial_x h(x,t) = \int \mathrm{d}y\, \partial_x p(x-y,t)h_0(y) + \int_0^t \mathrm{d}s \int \mathrm{d}y\, \partial_x p(x-y,t-s)f[h(y,s)], \tag{143}$$

$$\partial_x^2 h(x,t) = \int \mathrm{d}y\, \partial_x^2 p(x-y,t)h_0(y) + \int_0^t \mathrm{d}s \int \mathrm{d}y\, \partial_x p(x-y,t-s)f'[h(y,s)]\partial_x h(y,s), \tag{144}$$

where $f(h) = h - F(h)$ and

$$p(x,t) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{x^2}{4t}}. \tag{145}$$

By using $0 \leqslant h_0 \leqslant 1$ and $0 \leqslant f[h] \leqslant h \leqslant 1$ in (143), Uchiyama shows that

$$|\partial_x h(x,t)| \leqslant \int \mathrm{d}y\, |\partial_x p(x-y,t)| + \int_0^t \mathrm{d}s \int \mathrm{d}y\, |\partial_x p(x-y,t-s)| = \frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{t}} + 2\sqrt{t}\right), \tag{146}$$

Let $\zeta := \max|f'|$; by using (146) (and $0 \leqslant h_0 \leqslant 1$) in (144), he also obtains, in the same way,

$$|\partial_x^2 h(x,t)| \leqslant \frac{1}{2t} + \zeta \times (1+t). \tag{147}$$

We now need to show that $|\partial_x h(x,t)|$ and $|\partial_x^2 h(x,t)|$ are bounded by $A(t)\mathrm{e}^{-\beta x}$ for some locally bounded function A and for $\beta \in (0,\gamma)$.

We bound the right-hand-sides of (143) and (144), starting with the terms involving h_0. Choose $p>1$ such that $p\beta<\gamma$, and let q be the Hölder conjugate of p, *i.e.* such that $1/p+1/q=1$. By Hölder's inequality applied to $h_0(y)^{1/p}\mathrm{e}^{\beta y}\times\partial_x p(x-y,t)h_0(y)^{1/q}\mathrm{e}^{-\beta y}$, we obtain

$$\left|\int \mathrm{d}y\,\partial_x p(x-y,t)h_0(y)\right| \leqslant \left[\int \mathrm{d}y\,h_0(y)\mathrm{e}^{\beta p y}\right]^{\frac{1}{p}}\left[\int \mathrm{d}y\,|\partial_x p(x-y,t)|^q h_0(y)\mathrm{e}^{-\beta q y}\right]^{\frac{1}{q}}. \tag{148}$$

The first integral in the right-hand-side is $g(\beta p,0)$, which is finite since we took $\beta p<\gamma$. We focus on the second integral, which we first bound using $h_0(y)\leqslant 1$; then

$$\begin{aligned}\int \mathrm{d}y\,|\partial_x p(x-y,t)|^q\mathrm{e}^{-\beta q y} &= \mathrm{e}^{-\beta q x}\int \mathrm{d}y\,|\partial_x p(y,t)|^q\mathrm{e}^{\beta q y},\\ &= \mathrm{e}^{-\beta q x}\sqrt{t}\int \mathrm{d}y\,|\partial_x p(y\sqrt{t},t)|^q\mathrm{e}^{\beta q y\sqrt{t}},\\ &= \mathrm{e}^{-\beta q x}t^{-q+\frac{1}{2}}\int \mathrm{d}y\,|\partial_x p(y,1)|^q\mathrm{e}^{\beta q y\sqrt{t}}.\end{aligned} \tag{149}$$

Indeed, as $p(y\sqrt{t},t)=p(y,1)/\sqrt{t}$, we have that $\partial_x p(y\sqrt{t},t)=\partial_x p(y,1)/t$. The remaining integral on the right-hand-side converges because of the Gaussian bounds in $\partial_x p(y,1)$ and gives some continuous function of t defined for all $t\geqslant 0$. Then, in (148),

$$\left|\int \mathrm{d}y\,\partial_x p(x-y,t)h_0(y)\right| \leqslant \frac{B_1(t)}{t^{1-\frac{1}{2q}}}\mathrm{e}^{-\beta x}, \tag{150}$$

for some function B_1 continuous on $[0,\infty)$. It is crucial for what follows that $B_1(0)$ is finite, so that the divergence of $B_1(t)/t^{1-1/(2q)}$ as $t\searrow 0$ is integrable.

Note: we are about to introduce functions B_2, B_3, etc. As for B_1, all these functions are implicitly defined and continuous on $[0,\infty)$.

The same method for the second derivative, using this time $\partial_x^2 p(y\sqrt{t},t)=\partial_x^2 p(y,1)/t^{3/2}$, gives

$$\left|\int \mathrm{d}y\,\partial_x^2 p(x-y,t)h_0(y)\right| \leqslant \frac{B_2(t)}{t^{\frac{3}{2}-\frac{1}{2q}}}\mathrm{e}^{-\beta x}. \tag{151}$$

We now turn to the second term in the right hand side of (143). We first write, from lemma 1,

$$0\leqslant f[h(y,s)]\leqslant h(y,s)\leqslant C\frac{\mathrm{e}^{(1+\beta^2)s}}{\sqrt{s}}\mathrm{e}^{-\beta y}\leqslant C\frac{\mathrm{e}^{(1+\beta^2)t}}{\sqrt{s}}\mathrm{e}^{-\beta y}, \tag{152}$$

for $0<s\leqslant t$, with C a constant. Then, as in (149),

$$\begin{aligned}\int \mathrm{d}y\,|\partial_x p(x-y,t-s)|\mathrm{e}^{-\beta y} &= \mathrm{e}^{-\beta x}\int \mathrm{d}y\,|\partial_x p(y,t-s)|\mathrm{e}^{\beta y} = \frac{\mathrm{e}^{-\beta x}}{\sqrt{t-s}}\int \mathrm{d}y\,|\partial_x p(y,1)|\mathrm{e}^{\beta y\sqrt{t-s}}\\ &\leqslant \frac{\mathrm{e}^{-\beta x}}{\sqrt{t-s}}\int \mathrm{d}y\,|\partial_x p(y,1)|\mathrm{e}^{\beta\max(y,0)\sqrt{t}} = \frac{B_3(t)}{\sqrt{t-s}}\mathrm{e}^{-\beta x},\end{aligned} \tag{153}$$

for $0\leqslant s<t$. This leads with (152) to

$$\left|\int \mathrm{d}y\,\partial_x p(x-y,t-s)f[h(y,s)]\right| \leqslant \frac{B_4(t)}{\sqrt{s(t-s)}}\mathrm{e}^{-\beta x}, \tag{154}$$

for $0 < s < t$. Then,

$$\left|\int_0^t \mathrm{d}s \int \mathrm{d}y\, \partial_x p(x-y,t-s) f[h(y,s)]\right| \leqslant \pi B_4(t) \mathrm{e}^{-\beta x}. \tag{155}$$

With (150), into (143), we obtain

$$\left|\partial_x h(x,t)\right| \leqslant \frac{B_5(t)}{t^{1-\frac{1}{2q}}} \mathrm{e}^{-\beta x}. \tag{156}$$

We now turn to the second term in the right hand side of (144). We bound $f'[h(y,s)]$ by $\zeta := \max|f'|$; then using (156) with (153), we obtain

$$\left|\int \mathrm{d}y\, \partial_x p(x-y,t-s) f'[h(y,s)] \partial_x h(y,s)\right| \leqslant \zeta \frac{B_3(t) B_5(s)}{\sqrt{t-s}\, s^{1-\frac{1}{2q}}} \mathrm{e}^{-\beta x}, \tag{157}$$

and, finally, since the integral on s is finite,

$$\left|\int_0^t \mathrm{d}s \int \mathrm{d}y\, \partial_x p(x-y,t-s) f'[h(y,s)) \partial_x h(y,s)\right| \leqslant B_6(t) \mathrm{e}^{-\beta x}. \tag{158}$$

With (151), into (144):

$$\left|\partial_x^2 h(x,t)\right| \leqslant \frac{B_7(t)}{t^{\frac{3}{2}-\frac{1}{2q}}} \mathrm{e}^{-\beta x}, \tag{159}$$

and the proof is complete. □

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