The Annals of Probability
2022, Vol. 50, No. 6, 2133–2177
https://doi.org/10.1214/22-AOP1578

BROWNIAN BEES IN THE INFINITE SWARM LIMIT

BY JULIEN BERESTYCKI[1,a], ÉRIC BRUNET[2,b], JAMES NOLEN[3,c] AND SARAH PENINGTON[4,d]

[1]*Department of Statistics, University of Oxford,* [a]*julien.berestycki@stats.ox.ac.uk*
[2]*Laboratoire de Physique de l'École normale supérieure, ENS, Université PSL, CNRS, Sorbonne Université, Université Paris-Cité,* [b]*eric.brunet@ens.fr*
[3]*Department of Mathematics, Duke University,* [c]*nolen@math.duke.edu*
[4]*Department of Mathematical Sciences, University of Bath,* [d]*sp2355@bath.ac.uk*

The *Brownian bees* model is a branching particle system with spatial selection. It is a system of N particles which move as independent Brownian motions in $\mathbb{R}^d$ and independently branch at rate 1, and, crucially, at each branching event, the particle which is the furthest away from the origin is removed to keep the population size constant. In the present work we prove that, as $N \to \infty$, the behaviour of the particle system is well approximated by the solution of a free boundary problem (which is the subject of a companion paper (*Trans. Amer. Math. Soc.* **374** (2021) 6269–6329)), the *hydrodynamic limit* of the system. We then show that for this model the so-called *selection principle* holds; that is, that as $N \to \infty$, the equilibrium density of the particle system converges to the steady-state solution of the free boundary problem.

1. Introduction and main results. The *Brownian bees* model is a particular case of an N-particle branching Brownian motion (N-BBM for short) which is defined as follows. The system consists of N particles with locations in $\mathbb{R}^d$ for some dimension d. Each particle moves independently according to a Brownian motion with diffusivity $\sqrt{2}$ and branches independently into two particles at rate one. Whenever a particle in the system branches, the particle in the system which is furthest (in Euclidean distance) from the origin is immediately removed from the system so that there are exactly N particles in the system at all times. Thus, the branching events arrive according to a Poisson process with rate N. The name *Brownian bees*, suggested by Jeremy Quastel, comes from the analogy with bees swarming around a hive; throughout the paper we will refer to this process simply as N-BBM.

The particles can be labelled in a natural way which will allow us to write the N-BBM as a càdlàg $(\mathbb{R}^d)^N$-valued process. Each of the N particles carries a label from the set $\{1, \dots, N\}$. Suppose at some time τ the particle labelled k branches and the particle with label ℓ is the furthest from the origin. Then, at time τ the particle with label ℓ is removed from the system, and a new particle with label ℓ appears at the location of the particle with label k. For $k \in \{1, \dots, N\}$, let $X_k^{(N)}(t)$ denote the location of the particle with label k at time t. Then,

$$X^{(N)}(t) = \big(X_1^{(N)}(t), \dots, X_N^{(N)}(t)\big)$$

is the vector of particle locations in the N-BBM at time t. This labelling is motivated by the following equivalent description of N-BBM in terms of jumping rather than branching: at rate N the particle that is furthest from the origin jumps to the location of another particle chosen uniformly at random from all N particles. (The particular choice of ordering of the particles here plays no essential role in our results.)

Received April 2021; revised January 2022.
MSC2020 subject classifications. Primary 60J80, 35R35; secondary 60K35, 82C22.
Key words and phrases. Interacting particle systems, branching processes, free boundary problems.

We shall prove two types of result about this interacting system of particles: results about the spatial distribution of particles at a fixed time t, as the number of particles $N \to \infty$, and results about the long-term behaviour of the particle system, as time $t \to \infty$ for a fixed large number of particles N. As we show, there is a sense in which these limits commute. Showing that this so-called *selection principle* holds is a major motivation of the present work and was originally conjectured by Nathanaël Berestycki. More precisely, he predicted that, as $N \to \infty$, the particles would localise in a ball of finite radius at large times.

Our first main result is a hydrodynamic limit for the distribution of particle locations at a fixed time t, as the number of particles $N \to \infty$. This limit involves the solution of the following free boundary problem: for a probability measure μ_0 on $\mathbb{R}^d$, find $u(x,t) : \mathbb{R}^d \times (0,\infty) \to [0,\infty)$ and $R_t : (0,\infty) \to [0,\infty]$ such that

$$(1) \qquad \begin{cases} \partial_t u = \Delta u + u, & \text{for } t > 0 \text{ and } \|x\| < R_t, \\ u(x,t) = 0, & \text{for } t > 0 \text{ and } \|x\| \geq R_t, \\ u(x,t) \text{ is continuous on } \mathbb{R}^d \times (0,\infty), & \\ \int_{\mathbb{R}^d} u(x,t)\,\mathrm{d}x = 1, & \text{for } t > 0, \\ u(\cdot,t) \to \mu_0 & \text{weakly as } t \searrow 0. \end{cases}$$

In the companion paper [6] we prove that (1) has a unique solution (u, R) and that the function R_t is finite and continuous for $t > 0$.

For $t \geq 0$, we let

$$M_t^{(N)} = \max_{i \in \{1,\dots,N\}} \|X_i^{(N)}(t)\|$$

denote the maximum distance of a particle from the origin at time t. For $A \subseteq \mathbb{R}^d$ measurable, we let

$$\mu^{(N)}(A,t) = \frac{1}{N}\big|\{i \in \{1,\dots,N\} : X_i^{(N)}(t) \in A\}\big|$$

denote the proportion of particles which are in the set A at time t. In other words, $\mu^{(N)}(\mathrm{d}x,t)$ is the empirical measure of the particles at time t, that is,

$$\mu^{(N)}(\mathrm{d}x,t) = \frac{1}{N}\sum_{k=1}^{N} \delta_{X_k^{(N)}(t)}(\mathrm{d}x).$$

We can now state our hydrodynamic limit result.

THEOREM 1.1. *Suppose that μ_0 is a Borel probability measure on $\mathbb{R}^d$ and that*:

- $X_1^{(N)}(0), \dots, X_N^{(N)}(0)$ *are i.i.d. with distribution given by μ_0, and*
- (u, R) *is the solution to* (1) *with initial condition μ_0.*

Then, for any $t > 0$ and any measurable $A \subseteq \mathbb{R}^d$, almost surely,

$$\mu^{(N)}(A,t) \to \int_A u(x,t)\,\mathrm{d}x \quad \textit{and} \quad M_t^{(N)} \to R_t \quad \textit{as } N \to \infty$$

(*this holds for any coupling of the processes $(X^{(N)})_{N\in\mathbb{N}}$*).

Note that Theorem 1.1 implies that, for $t > 0$, almost surely $\mu^{(N)}(\mathrm{d}x,t) \to u(x,t)\,\mathrm{d}x$ weakly as $N \to \infty$.

Our second set of results concerns the long-term behaviour ($t \to \infty$) of the particle system for large N. We can show that, for large fixed N, the particle system converges in distribution as $t \to \infty$ to an invariant measure. For $\mathcal{X} \in (\mathbb{R}^d)^N$, we write $\mathbb{P}_{\mathcal{X}}$ to denote the probability measure under which $(X^{(N)}(t), t \geq 0)$ is an N-BBM process with $X^{(N)}(0) = \mathcal{X}$.

THEOREM 1.2. *For N sufficiently large, the process $(X^{(N)}(t), t \geq 0)$ has a unique invariant measure $\pi^{(N)}$, a probability measure on $(\mathbb{R}^d)^N$. For any $\mathcal{X} \in (\mathbb{R}^d)^N$, under $\mathbb{P}_{\mathcal{X}}$, the law of $X^{(N)}(t)$ converges in total variation norm to $\pi^{(N)}$ as $t \to \infty$,*

$$\lim_{t\to\infty} \sup_C \left|\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t) \in C\big) - \pi^{(N)}(C)\right| = 0,$$

where the supremum is over all Borel measurable sets $C \subseteq (\mathbb{R}^d)^N$.

For each $t \geq 0$, the empirical measure $\mu^{(N)}(\cdot, t)$ is a random element of $\mathcal{P}(\mathbb{R}^d)$, the set of Borel probability measures on $\mathbb{R}^d$. Theorem 1.2 implies that, as $t \to \infty$, the law of $\mu^{(N)}(\cdot, t)$ converges in total variation to the measure $\pi^{(N)} \circ H^{-1}$, where $H : (\mathbb{R}^d)^N \to \mathcal{P}(\mathbb{R}^d)$ is the map defined by $H(x_1, \dots, x_N) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and $\pi^{(N)} \circ H^{-1}$ is the pushforward of $\pi^{(N)}$ under the map H. The law of $\mu^{(N)}(\cdot, t)$ and the measure $\pi^{(N)} \circ H^{-1}$, which are both probability measures on the Polish space $\mathcal{P}(\mathbb{R}^d)$, do not depend on the particular ordering of particles used to define $X^{(N)}(t)$ as an $(\mathbb{R}^d)^N$-valued process.

We also obtain more explicit results about the long-term behaviour of the particle system. We let $U : \mathbb{R}^d \to \mathbb{R}$ denote the principal Dirichlet eigenfunction of $(-\Delta)$ in a spherical domain with radius uniquely chosen so that the eigenvalue is 1. That is, let (U, R_∞) denote the unique solution (with U continuous) to

$$\begin{cases} -\Delta U(x) = U(x), & \|x\| < R_\infty, \\ U(x) > 0, & \|x\| < R_\infty, \\ U(x) = 0, & \|x\| \geq R_\infty, \\ \displaystyle\int_{\|x\|\leq R_\infty} U(x)\,\mathrm{d}x = 1. \end{cases} \tag{2}$$

Then, (U, R_∞) (which can be written explicitly in terms of Bessel functions, see (10) in [6]) is a stationary solution to (1). In [6] we prove that any solution $(u(\cdot, t), R_t)$ of the free boundary problem (1) converges to the stationary solution (U, R_∞), as $t \to \infty$, and it turns out that this stationary solution also controls the long-term behaviour of the particle system for large N.

We shall use the following notation to denote a reasonable class of initial particle configurations. For $K > 0$ and $c \geq 0$, let:

$$\Gamma(K, c) = \left\{\mathcal{X} \in \big(\mathbb{R}^d\big)^N : \frac{1}{N}\left|\{i : \|\mathcal{X}_i\| < K\}\right| \geq c\right\}. \tag{3}$$

This is the set of particle configurations which put at least a fraction c of the particles within distance K of the origin. The following result shows that if N is large, then, at a large time t, the particles are approximately distributed according to U, and the largest particle distance from the origin is approximately R_∞.

THEOREM 1.3. *Take $K > 0$ and $c \in (0, 1]$. For $\epsilon > 0$, there exist $N_\epsilon = N_\epsilon(K, c) < \infty$ and $T_\epsilon = T_\epsilon(K, c) < \infty$ such that, for $N \geq N_\epsilon$ and $t \geq T_\epsilon$, for an initial condition $\mathcal{X} \in \Gamma(K, c)$ and $A \subseteq \mathbb{R}^d$ measurable,*

$$\mathbb{P}_{\mathcal{X}}\left(\left|\mu^{(N)}(A, t) - \int_A U(x)\,\mathrm{d}x\right| \geq \epsilon\right) < \epsilon$$
$$\text{and} \qquad \mathbb{P}_{\mathcal{X}}\big(\big|M_t^{(N)} - R_\infty\big| \geq \epsilon\big) < \epsilon.$$

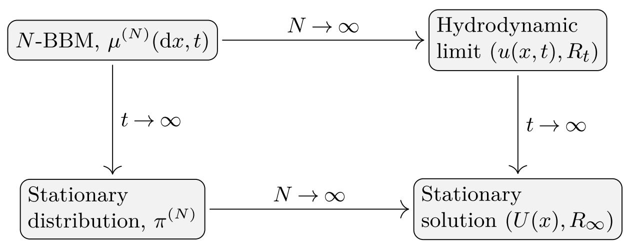
As a consequence of Theorems 1.2 and 1.3 for large N and under the invariant distribution $\pi^{(N)}$, the proportion of particles in a set A is approximately $\int_A U(x)\,\mathrm{d}x$, and the furthest particle distance from the origin is approximately R_∞.

THEOREM 1.4. *For $\epsilon > 0$ and $A \subseteq \mathbb{R}^d$ measurable,*

$$\pi^{(N)}\left(\left\{\mathcal{X} \in (\mathbb{R}^d)^N : \left|\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\mathcal{X}_i \in A\}} - \int_A U(x)\,\mathrm{d}x\right| \geq \epsilon\right\}\right) \to 0 \quad \text{as } N \to \infty \tag{4}$$

$$\text{and} \quad \pi^{(N)}\left(\left\{\mathcal{X} \in (\mathbb{R}^d)^N : \left|\max_{i \in \{1,\dots,N\}} \|\mathcal{X}_i\| - R_\infty\right| \geq \epsilon\right\}\right) \to 0 \quad \text{as } N \to \infty. \tag{5}$$

The results in this article and in the companion article [6] can be summarised in the following informal diagram.



In [6] we deal with the right-hand side of the diagram: well-posedness of the free boundary problem (1) and the long-term behaviour of its solutions. In the present article, Theorem 1.1 gives rigorous meaning to the top of the diagram, Theorem 1.2 covers the left-hand side, and Theorem 1.4 covers the bottom of the diagram.

1.1. *Related works.* The particle system we are considering is a particular case of a more general N-particle branching Brownian motion (N-BBM) with spatial selection, described as follows: The system consists of N particles moving in $\mathbb{R}^d$ with locations $(X_1^{(N)}(t), \dots, X_N^{(N)}(t))$. Each particle moves independently according to a Brownian motion with diffusivity $\sqrt{2}$ and branches independently into two particles at rate 1. Whenever a particle branches, however, the particle having least "fitness" or "score" (out of the entire ensemble) is instantly removed (killed) so that there are exactly N particles in the system at all times. The fitness of a particle is a function $\mathcal{F}(x)$ of its location $x \in \mathbb{R}^d$, and as a result, the elimination of least-fit particles tends to push the ensemble toward regions of higher fitness. Variants of this stochastic process were first studied in one spatial dimension, beginning with work of Brunet, Derrida, Mueller, and Munier [9, 10], followed by work of Bérard and Gouéré [5], on discrete-time processes, and the work of Maillard [21] on the continuous-time model involving Brownian motions. In these works the particle removed from the system is always the leftmost particle which means that they could be described by a monotone fitness function (e.g., $\mathcal{F}(r) = r$, $r \in \mathbb{R}$). The general multidimensional model, which we have just described above, was first studied by N. Berestycki and Zhao [8]; specifically, they studied the particle system with fitness functions $\mathcal{F}(x) = \|x\|$ and $\mathcal{F}(x) = \lambda \cdot x$, both of which have the effect of pushing the ensemble of particles away from the origin. The Brownian bees model that we consider in this article corresponds to the fitness function $\mathcal{F}(x) = -\|x\|$.

In the setting of one spatial dimension and with monotone fitness function $\mathcal{F}(r) = r$, $r \in \mathbb{R}$, De Masi, Ferrari, Presutti, and Soprano-Loto [13] determined the hydrodynamic limit of the particle system. For $t > 0$, define the measure

$$\mu^{(N)}(\mathrm{d}r, t) = \frac{1}{N}\sum_{k=1}^{N} \delta_{X_k^{(N)}(t)}(\mathrm{d}r).$$

De Masi et al. proved that if the initial particle locations $X_1^{(N)}(0), \dots, X_N^{(N)}(0)$ are i.i.d., with certain assumptions on the distribution of $X_1^{(N)}(0)$, then the family of empirical measures

$\mu^{(N)}(\mathrm{d}r, t)$ converges, as $N \to \infty$, to a limit which can be identified with a solution $u(r, t)$ to a free boundary problem,

$$\begin{cases} \partial_t u = \partial_r^2 u + u, & r > \gamma_t, t > 0, \\ u(r, t) = 0, & r \le \gamma_t, t > 0, \\ \int_{\gamma_t}^{\infty} u(r, t)\, \mathrm{d}r = 1, & t > 0, \end{cases} \tag{6}$$

where the free boundary at $r = \gamma_t \in \mathbb{R}$ is related to u through the integral constraint. Global existence of solutions to this free boundary problem was proved by J. Berestycki, Brunet, and Penington [7]. De Masi et al. also state that, for fixed N, the particle system (seen from the leftmost particle) converges in distribution as $t \to \infty$ to an invariant measure ν_N, but they did not prove asymptotic results about the shape of the cloud of particles under ν_N as $N \to \infty$. As discussed in Section 1.2 below, a related one-dimensional result plays a fundamental role in our work. We use some coupling ideas similar to those in the proof of the hydrodynamic limit result in [13], but we obtain a more quantitative result for our particle system (see Proposition 1.5 below) which does not require the initial particle locations to be i.i.d. random variables. This, together with results about the long-term behaviour of the free boundary problem (1) from [6], allows us to control the long-term behaviour of the Brownian bees particle system for large N.

Building on the approach of [13], Beckman [4] derived a similar hydrodynamic limit in the one-dimensional setting with symmetric fitness $\mathcal{F}(r) = -|r|$ which coincides with our case if $d = 1$. Beckman also studied the long-term behaviour of the N-BBM in one dimension, with a nonmonotone fitness function of the form $\mathcal{F}(r) = r + \psi(r)$, ψ being periodic, and proved existence of a stationary distribution in a certain moving reference frame. In earlier work, Durrett and Remenik [15] studied a related branching-selection model in which non-diffusing particles in $\mathbb{R}$ are born at random locations but do not move during their lifetimes. They showed that the hydrodynamic limit of this particle system is given by a nonlocal free boundary problem. Bérard and Gouéré [5] determined asymptotics for the speed of the N-particle branching random walk. Their work does not involve a hydrodynamic limit, but the couplings they use in their Section 3 are also closely related to those in the present paper.

Another related model is the Fleming–Viot system studied by Burdzy, Hołyst, and March [11]. In that model, particles diffuse within a bounded domain having *fixed* boundary; whenever a particle hits the boundary, it is instantly killed, and one of the internal particles simultaneously branches, preserving the total mass. As in our case, the stationary distribution for that system also converges to the principal eigenfunction of the Laplacian (as the number of particles goes to infinity). This eigenfunction is a quasi-stationary distribution for the diffusion conditioned on nonextinction; see also Collet et al. [12] and references therein, for other related works on quasi-stationary distributions.

In [2], Asselah, Ferrari, Groisman, and Jonckheere considered a slightly different Fleming–Viot particle system. In their work the N particles live on $\{0, 1, 2, \dots\}$, move independently as continuous-time subcritical Galton–Watson processes, and are killed when they hit 0 (each time a particle is killed, one of the remaining $N - 1$ particles, chosen at random, branches). Recall that, for a single sub-critical Galton–Watson process conditioned on nonextinction, there exists an infinite family of quasi-stationary distributions. (By contrast, observe that a diffusion on a bounded domain conditioned on not exiting the domain has a unique quasi-stationary distribution.) Asselah et al. showed that, for each N, the Fleming–Viot particle system has a unique invariant distribution, and that its stationary empirical distribution converges as $N \to \infty$ to the *minimal* quasi-stationary distribution of the Galton–Watson process conditioned on nonextinction (which is the quasi-stationary distribution with the minimal expected time of extinction). This has been called the *selection principle* in the literature. It is reminiscent of the fact that the solution of the Fisher-KPP equation started from a fast decreasing initial condition converges to the minimal-velocity travelling wave (see, in particular, [18] and the note [17] of Groisman and Jonkheere). This principle is conjectured to hold in quite broad generality. For instance, for the one-dimensional N-BBM studied in [13], it is conjectured that the unique invariant distribution of the system seen from the leftmost particle converges, as $N \to \infty$, to the centred minimal-velocity travelling wave solution of (6) (which is given by $\gamma_t = 2t$, $u(2t + r, t) = re^{-r}\mathbb{1}_{\{r \geq 0\}}$).

Finally, we mention the very recent work [1] of Addario-Berry, Lin, and Tendron in which a variant of the Brownian bees model with the following selection rule is considered: each time one of the N particles branches, the particle currently furthest away from the centre of mass of the cloud of particles is removed from the system. Addario-Berry et al. show that the movement of the centre of mass, appropriately rescaled, converges to a Brownian motion.

1.2. *One-dimensional results and outline of the article.* The first step in the proofs of Theorems 1.1, 1.2, 1.3, and 1.4 is to control the proportion of particles within distance r of the origin at a fixed time t, when the number of particles N is very large.

For $r > 0$, let $\mathcal{B}(r) = \{x \in \mathbb{R}^d : \|x\| < r\}$ be the open ball of radius r centred at the origin. Suppose that (u, R) solves (1) with some initial probability measure μ_0, and let $v : [0, \infty) \times (0, \infty) \to [0, 1]$ denote the mass of u within distance r of the origin at time t,

$$v(r, t) = \int_{\mathcal{B}(r)} u(x, t)\,\mathrm{d}x.$$

Then, $r \mapsto v(r, t)$ is nondecreasing and $v(r, t) = 1$ for $r \geq R_t$. Let $v_0(r) = \mu_0(\mathcal{B}(r))$; then, by Lemma 6.2 of [6], v satisfies the following parabolic obstacle problem:

$$\begin{cases} 0 \leq v(r, t) \leq 1, & \text{for } t > 0, r \geq 0, \\ \partial_t v = \partial_r^2 v - \dfrac{d-1}{r}\partial_r v + v, & \text{if } v(r, t) < 1, \\ v(0, t) = 0, & \text{for } t > 0, \\ v(r, t) \text{ is continuous on } [0, \infty) \times (0, \infty), & \\ \partial_r v(\cdot, t) \text{ is continuous on } [0, \infty), & \text{for } t > 0, \\ v(\cdot, t) \to v_0 & \text{in } L^1_{\text{loc}} \text{ as } t \searrow 0. \end{cases} \tag{7}$$

We prove in Theorem 2.1 of [6] that, for any measurable $v_0 : [0, \infty) \to [0, 1]$, (7) has a unique solution.

The following result is a hydrodynamic limit result for the distances of particles from the origin, and will be an important step in the proofs of Theorems 1.1 and 1.3. Introduce

$$F^{(N)}(r, t) := \mu^{(N)}\big(\mathcal{B}(r), t\big) \tag{8}$$

as the proportion of particles within distance r of the origin at time t. Then, Proposition 1.5 below says that, for any initial configuration of particles at a fixed time t, the proportion $F^{(N)}(r, t)$ is close to the solution $v^{(N)}(r, t)$ of (7) with initial condition v_0 determined by the initial configuration of particles. The bound does not depend on the initial particle configuration; this will be crucial when the result is used in the proof of Theorem 1.3.

PROPOSITION 1.5. *There exists* $c_1 \in (0, 1)$ *such that for* N *sufficiently large, for* $t > 0$, *and any* $\mathcal{X} \in (\mathbb{R}^d)^N$,

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{r \geq 0}\big|F^{(N)}(r, t) - v^{(N)}(r, t)\big| \geq e^{2t}N^{-c_1}\Big) \leq e^t N^{-1-c_1},$$

where $v^{(N)}$ *is the solution of* (7) *with* $v_0(r) = F^{(N)}(r, 0)$ $\forall r \geq 0$.

The next result uses Proposition 1.5 to get an upper bound on the largest particle distance from the origin which holds over a time interval of fixed length. This will then allow us to compare the particle system to a system in which particles are killed, if they are further than a deterministic distance from the origin, which will enable us to prove the d-dimensional hydrodynamic limit in Theorem 1.1.

PROPOSITION 1.6. *There exists* $c_2 \in (0, 1)$ *such that, under the assumptions of Theorem* 1.1, *for any* $0 < \eta < T$, *for* N *sufficiently large (depending on* d, μ_0, η, *and* T),

$$\mathbb{P}\big(\exists t \in [\eta, T] : M_t^{(N)} > R_t + \eta\big) \leq N^{-1-c_2}.$$

In the case where μ_0 has compact support, the proof of Proposition 1.6 can easily be extended to bound the probability that there exists $t \in (0, T]$ with $M_t^{(N)} > R_t + \eta$. However, an upper bound on $M_t^{(N)}$ in the time interval $[\eta, T]$ (for an arbitrarily small η) is enough to allow us to prove Theorem 1.1.

Using Proposition 1.5 and results about the long-term behaviour of solutions to the obstacle problem (7) from the companion paper [6], we can also prove the following result about the long-term behaviour of particle distances from the origin when N is large. For $r \geq 0$, let

$$V(r) = \int_{\mathcal{B}(r)} U(x)\,\mathrm{d}x, \tag{9}$$

where U is defined in (2).

PROPOSITION 1.7. *Take* $K > 0$ *and* $c \in (0, 1]$. *For* $\epsilon > 0$, *there exist* $N_\epsilon = N_\epsilon(K, c) < \infty$ *and* $T_\epsilon = T_\epsilon(K, c) < \infty$ *such that, for* $N \geq N_\epsilon$ *and* $t \geq T_\epsilon$, *for an initial condition* $\mathcal{X} \in \Gamma(K, c)$,

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{r\geq 0}\big|F^{(N)}(r, t) - V(r)\big| \geq \epsilon\Big) < \epsilon, \tag{10}$$

$$\mathbb{P}_{\mathcal{X}}\big(\big|M_t^{(N)} - R_\infty\big| \geq \epsilon\big) < \epsilon, \tag{11}$$

$$\textit{and} \quad \mathbb{P}_{\mathcal{X}}\Big(\sup_{s\in[0,1]} M_{t+s}^{(N)} > R_\infty + \epsilon\Big) < \epsilon. \tag{12}$$

Using (12), we can compare the particle system at large times to a system in which particles are killed if they are further than distance $R_\infty + \epsilon$ from the origin. This, together with (10), will allow us to prove Theorem 1.3.

The rest of the article is laid out as follows. In Section 2 we recall results from [6] which will be used in this article. In Section 3 we define notation which will be used throughout the proofs. Then, in Section 4 we prove Propositions 1.5 and 1.6, and in Section 5 we use Proposition 1.6 to prove Theorem 1.1. In Section 6 we prove Proposition 1.7 and use this to prove Theorem 1.3 and, finally, Theorems 1.2 and 1.4.

2. Results from [6]. In this section, we state some results from [6] which play a key role in the present work. The first one is Theorem 1.1 in [6], which says that the free boundary problem (1) has a unique solution and that, moreover, the free boundary radius R_t is continuous.

THEOREM 2.1 (Theorem 1.1 in [6]). *Let* μ_0 *be a Borel probability measure on* $\mathbb{R}^d$. *Then, there exists a unique classical solution to problem* (1). *Furthermore*:

- $t \mapsto R_t$ *is continuous (and finite) for* $t > 0$.

- *As* $t \searrow 0$, $R_t \to R_0 := \inf\{r > 0 : \mu_0(\mathcal{B}(r)) = 1\} \in [0, \infty]$.
- *For* $t > 0$ *and* $\|x\| < R_t$, $u(x,t) > 0$.

For a Borel probability measure μ_0 on $\mathbb{R}^d$, let (u, R) denote the solution of (1). Let $(B_t)_{t\ge 0}$ denote a d-dimensional Brownian motion with diffusivity $\sqrt{2}$, and, for $x \in \mathbb{R}^d$, write $\mathbb{P}_x$ for the probability measure under which $B_0 = x$. For $t > 0$, define a family of measures on $\mathbb{R}^d$ according to

$$\rho_t(x, A) = \mathbb{P}_x\big(B_t \in A, \|B_s\| < R_s \forall s \in (0,t)\big)$$

for all Borel sets $A \subseteq \mathbb{R}^d$. Then, $\rho_t(x, \mathrm{d}z)$ is absolutely continuous with respect to the Lebesgue measure, so it has a density. Abusing notation, we denote this density by $\rho_t(x,z)$. Then, by Proposition 6.1 in [6],

$$u(z,t) = e^t \int_{\mathbb{R}^d} \mu_0(\mathrm{d}x)\rho_t(x,z), \quad z \in \mathbb{R}^d, t > 0. \tag{13}$$

Define the cumulative distribution of the norm process $\|B_t\|$, conditional on $\|B_0\| = y$, as

$$w(y,r,t) := \mathbb{P}\big(\|B_t\| < r \,|\, \|B_0\| = y\big). \tag{14}$$

Then, the function $r \mapsto g(y,r,t) := \partial_r w(y,r,t)$ is the density of $\|B_t\|$, conditional on $\|B_0\| = y$; in other words, g is the transition density of the d-dimensional Bessel process with diffusivity $\sqrt{2}$. The function

$$G(y,r,t) := -\partial_y w(y,r,t) \tag{15}$$

is the fundamental solution of the equation

$$\partial_t G = \partial_r^2 G - \frac{d-1}{r}\partial_r G, \qquad G(y,0,t) = 0, \qquad G(y,r,0) = \delta(r-y). \tag{16}$$

(See Section 3 of [6] for more details on the properties of G.) For $t, r, y > 0$, the fundamental solution G and transition density g are smooth functions of their arguments and are related by

$$\partial_r G = -\partial_y g. \tag{17}$$

For a given initial condition $v_0^\ell \in L^\infty(0,\infty)$, we let

$$v^\ell(r,t) = e^t \int_0^\infty \mathrm{d}y G(y,r,t) v_0^\ell(y). \tag{18}$$

This v^ℓ is a solution to the linear problem

$$\begin{cases} \partial_t v^\ell = \partial_r^2 v^\ell - \dfrac{d-1}{r}\partial_r v^\ell + v^\ell, & \text{for } t > 0, r \ge 0, \\ v^\ell(0,t) = 0, & \text{for } t > 0, \\ v^\ell(\cdot,t) \to v_0^\ell & \text{in } L^1_{\mathrm{loc}} \text{ as } t \searrow 0, \end{cases} \tag{19}$$

and it is the unique solution to (19) which is bounded on $[0,\infty) \times [0,T]$ for each $T > 0$. In the particular case $v_0^\ell(r) = \mathbb{1}_{\{y \le r\}}$, we have $v^\ell(r,t) = e^t w(y,r,t)$.

For $t > 0$ and $m \in \mathbb{R}$, we define the operators G_t and C_m by letting

$$G_t f(r) = \int_0^\infty \mathrm{d}y G(y,r,t) f(y) \quad \text{and} \quad C_m f(r) = \min\big(f(r), m\big). \tag{20}$$

In particular, $v^\ell = e^t G_t v_0^\ell$. By Lemma 3.1 in [6], we have that $|\int_0^\infty G(y,r,t)h(y)\,\mathrm{d}y| \le \|h\|_{L^\infty}$, and so, for $f, g \in L^\infty[0,\infty)$ and $t > 0$,

$$\|G_t f - G_t g\|_{L^\infty} \le \|f - g\|_{L^\infty}. \tag{21}$$

Suppose $v_0 : [0, \infty) \to [0, 1]$ is nondecreasing, and let v denote the solution of the obstacle problem (7) with initial condition v_0. For $\delta > 0$ and $k \in \mathbb{N}_0$, we let

$$v^{k,\delta,-} = \left(e^{\delta} G_{\delta} C_{e^{-\delta}}\right)^k v_0, \qquad v^{k,\delta,+} = \left(C_1 e^{\delta} G_{\delta}\right)^k v_0. \tag{22}$$

Then, by Lemmas 4.3 and 4.4 in [6], we have the following result.

LEMMA 2.2 (Lemmas 4.3 and 4.4 in [6]). *For any $\delta > 0$ and $k \in \mathbb{N}_0$,*

$$v^{k,\delta,-}(r) \le v(r, k\delta) \le v^{k,\delta,+}(r) \quad \forall r \ge 0 \quad \text{and} \quad \left\| v^{k,\delta,+} - v^{k,\delta,-} \right\|_{L^\infty} \le \left(e^{k\delta} + 1\right)\left(e^{\delta} - 1\right).$$

We shall also use the following result which was proved as part of Theorem 2.1 in [6]. The result says that the solution of (7) is continuous with respect to the initial condition in the following sense.

LEMMA 2.3 (From Theorem 2.1 in [6]). *Let v and $\tilde{v}$ be the solutions to (7) corresponding to the initial conditions v_0 and $\tilde{v}_0$. Then, for $t > 0$,*

$$\left\| v(\cdot, t) - \tilde{v}(\cdot, t) \right\|_{L^\infty} \le e^t \| v_0 - \tilde{v}_0 \|_{L^\infty}.$$

Recall the definition of V in (9). The following result, which follows directly from Theorem 2.2 in [6], gives us control over how quickly the solution $v(\cdot, t)$ of the obstacle problem (7) converges to V.

PROPOSITION 2.4 (From Theorem 2.2 in [6]). *For $c \in (0, 1]$, $K > 0$, and $\epsilon > 0$, there exists $t_\epsilon = t_\epsilon(c, K) \in (0, \infty)$ such that the following holds. Suppose $v_0 : [0, \infty) \to [0, 1]$ is nondecreasing with $v_0(K) \ge c$, and let v solve the obstacle problem (7) with initial condition v_0. For $t > 0$, let $R_t = \inf\{r \ge 0 : v(r, t) = 1\}$. Then, for $t \ge t_\epsilon$,*

$$\left| v(r, t) - V(r) \right| < \epsilon \quad \forall r \ge 0 \quad \text{and} \quad |R_t - R_\infty| < \epsilon.$$

The final result from [6], which we need in this article, is Proposition 5.10, which says that for large K and small c, if an initial condition v_0 has mass at least c within distance K of the origin, then the solution v of (7) has mass at least $2c$ within distance $K - 1$ of the origin during a fixed time interval $[t_0, 2t_0]$.

PROPOSITION 2.5 (Proposition 5.10 in [6]). *There exist $t_0 > 1$ and $c_0 \in (0, 1/2)$ such that, for all $c \in (0, c_0]$, all $K \ge 2$, and all $t_1 \in [t_0, 2t_0]$, for $v_0 : [0, \infty) \to [0, 1]$ measurable, the condition*

$$v_0(r) \ge c \mathbb{1}_{\{r \ge K\}} \quad \forall r \ge 0$$

implies that

$$v(r, t_1) \ge 2c \mathbb{1}_{\{r \ge K-1\}} \quad \forall r \ge 0,$$

and

$$v(r, nt_1) \ge \min\left(2c_0, 2^n c\right) \mathbb{1}_{\{r \ge \max(K-n, 1)\}}, \quad \forall r \ge 0, n \in \mathbb{N},$$

where $v(r, t)$ denotes the solution of (7) with initial condition v_0.

3. Notation. From now on, we let $(B_t)_{t\geq 0}$ denote a d-dimensional Brownian motion with diffusivity $\sqrt{2}$, and for $x \in \mathbb{R}^d$, we write $\mathbb{P}_x$ for the probability measure under which $B_0 = x$ and write $\mathbb{E}_x$ for the corresponding expectation.

The locations (positions) of a collection of m particles in $\mathbb{R}^d$ are written as a vector $\mathcal{X} \in (\mathbb{R}^d)^m$. The size of the vector (i.e., the number of particles in the collection) is written $|\mathcal{X}|$ (i.e., $|\mathcal{X}| = m$ for $\mathcal{X} \in (\mathbb{R}^d)^m$). The individual locations in $\mathcal{X}$ are written $\mathcal{X}_k$ for $k \in \{1, \dots, |\mathcal{X}|\}$,

$$\mathcal{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m) \quad \text{with } m = |\mathcal{X}|.$$

We extend some set notation to vectors. Specifically, we write $\mathcal{X} \subseteq \mathcal{Y}$ to mean that all the particles in $\mathcal{X}$ are also in $\mathcal{Y}$,

$$\mathcal{X} \subseteq \mathcal{Y} \quad \Leftrightarrow \quad \exists j : \{1, \dots, |\mathcal{X}|\} \to \{1, \dots, |\mathcal{Y}|\} \text{ injective such that } \mathcal{X}_k = \mathcal{Y}_{j(k)}\ \forall k.$$

We write $|A \cap \mathcal{X}|$ for the number of particles in $\mathcal{X}$ which lie in some set $A \subseteq \mathbb{R}^d$,

$$|A \cap \mathcal{X}| = \big|\{k \in \{1, \dots, |\mathcal{X}|\} : \mathcal{X}_k \in A\}\big|.$$

If $\mathcal{X}$ and $\mathcal{Y}$ are two vectors of particles with locations in $\mathbb{R}^d$, we write $\mathcal{X} \preceq \mathcal{Y}$ to mean that the vector $\mathcal{X}$ contains more particles than the vector $\mathcal{Y}$ in any ball centred on the origin,

$$\mathcal{X} \preceq \mathcal{Y} \quad \Leftrightarrow \quad \big|\mathcal{X} \cap \mathcal{B}(r)\big| \geq \big|\mathcal{Y} \cap \mathcal{B}(r)\big| \quad \text{for all } r > 0, \tag{23}$$

where we recall that $\mathcal{B}(r)$ is the centred open ball of radius r,

$$\mathcal{B}(r) = \{x \in \mathbb{R}^d : \|x\| < r\}.$$

Notice that $\mathcal{X} \preceq \mathcal{Y}$ implies that $|\mathcal{X}| \geq |\mathcal{Y}|$.

The order in which the particle locations are written within a vector $\mathcal{X}$ is irrelevant for the operations $\subseteq$, $\cap$ and, $\preceq$ described above.

It will be useful to compare the N-BBM to the standard d-dimensional binary branching Brownian motion (BBM) without selection in which particles move independently in $\mathbb{R}^d$, according to Brownian motions with diffusivity $\sqrt{2}$, and branch into two particles at rate 1. The BBM may be labelled using the Ulam–Harris scheme (see [20] and references therein). Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathbb{N}^n$, and for $t \geq 0$, let $\mathcal{N}_t^+ \subset \mathcal{U}$ denote the set of Ulam–Harris labels of the particles in the BBM at time t. (If the BBM has m particles at time 0, then the particles initially have labels $1, \dots, m$ and for each $t \geq 0$, $\mathcal{N}_t^+ \subset \bigcup_{n=1}^{\infty} \{1, \dots, m\} \times \{1, 2\}^{n-1}$; e.g., the particles labelled $(7, 1)$ and $(7, 2)$ are the two children of the seventh original particle; the particle $(7, 2, 1)$ is a grandchild of the seventh original particle and is the first child of $(7, 2)$.) For $u \in \mathcal{N}_t^+$, let $X_u^+(t)$ denote the location of particle u at time t, and for $s \in [0, t)$, let $X_u^+(s)$ denote the location of its ancestor in the BBM at time s. We write $X^+(t) = (X_u^+(t))_{u \in \mathcal{N}_t^+}$ for the vector of particle locations in the BBM at time t, with particles ordered lexicographically by their Ulam–Harris labels.

In the proofs we shall often use the following standard coupling between the BBM and the N-BBM: consider a standard d-dimensional binary BBM, as described above. In addition to their spatial location, let each particle carry a colour attribute, either red or blue. When a blue particle branches, the two offspring particles are coloured blue, and simultaneously, the blue particle furthest from the origin turns red. When a red particle branches, the two offspring particles are coloured red. The system begins with N blue particles at time 0. The set of blue particles is a realisation of the N-BBM, while the entire collection of particles (blue and red) is a realisation of standard BBM. Specifically, for $t \geq 0$, let $\mathcal{N}_t^{(N)} \subseteq \mathcal{N}_t^+$ denote the set of labels of blue particles at time t. Then, $\mathcal{N}_t^{(N)}$ is always a set of size N, and there exists an enumeration $(u_k)_{k=1}^N$ of $\mathcal{N}_t^{(N)}$ such that $X^{(N)}(t) = (X_{u_1}^+(t), \dots, X_{u_N}^+(t))$. Recall that we let

$M_t^{(N)} = \max_{k\in\{1,\dots,N\}} \|X_k^{(N)}(t)\|$, the maximum distance of a particle in the N-BBM from the origin at time t. Notice that, for $t \geq 0$, almost surely

$$\mathcal{N}_t^{(N)} = \{u \in \mathcal{N}_t^+ : \|X_u^+(s)\| \leq M_s^{(N)} \ \forall s \in [0,t]\}. \tag{24}$$

We usually write $\mathcal{X}$ for the initial configuration of the N-BBM and of the BBM. In some cases where we compare an N-BBM and a BBM with different initial conditions, we write $\mathcal{X}^+$ for the initial condition of the BBM. Expectations and laws started from an initial condition $\mathcal{X}$ are written $\mathbb{E}_\mathcal{X}$ and $\mathbb{P}_\mathcal{X}$, respectively. We also write $(\mathcal{F}_t)_{t\geq 0}$ for the natural filtration of $(X^{(N)}(t), t \geq 0)$, that is, $\mathcal{F}_t = \sigma((X^{(N)}(s), s \leq t))$.

4. One-dimensional hydrodynamic limit results. In this section we prove the hydrodynamic limit results about the distances of particles from the origin, Propositions 1.5 and 1.6.

For $t \geq 0$ and $r > 0$, recall the definition (8) of the cumulative distribution function

$$F^{(N)}(r,t) = \mu^{(N)}(\mathcal{B}(r),t) = \frac{1}{N}\big|X^{(N)}(t) \cap \mathcal{B}(r)\big| = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\|X_i^{(N)}(t)\|<r\}}, \tag{25}$$

and introduce

$$F^+(r,t) = \frac{1}{N}\big|X^+(t) \cap \mathcal{B}(r)\big| = \frac{1}{N}\sum_{u\in\mathcal{N}_t^+} \mathbb{1}_{\{\|X_u^+(t)\|<r\}}.$$

4.1. *Proof of Proposition* 1.5.

4.1.1. *Upper bound for the proof of Proposition* 1.5. Recall from (20) that, for $m \in \mathbb{R}$ and $f : [0,\infty) \to \mathbb{R}$, we let $C_m f(r) = \min(f(r), m)$. The following proposition will play a crucial role in the proof of Proposition 1.5; it says that the random function F^+, corresponding to the BBM, stochastically dominates the random function $F^{(N)}$ corresponding to the N-BBM, if both processes start from the same particle configuration.

PROPOSITION 4.1. *Suppose $\mathcal{X} = (\mathcal{X}_1, \dots, \mathcal{X}_N) \in (\mathbb{R}^d)^N$. There exists a coupling of the N-BBM $X^{(N)}(t)$ started from $\mathcal{X}$ and of the BBM $X^+(t)$ also started from $\mathcal{X}$ such that*

$$F^{(N)}(\cdot,t) \leq C_1 F^+(\cdot,t) \quad \forall t \geq 0.$$

(*Equivalently, $X^+(t) \preceq X^{(N)}(t)$ for all $t \geq 0$.*) *In particular, if $f : [0,\infty) \to \mathbb{R}\cup\{+\infty\}$ is measurable, then, for all $t \geq 0$,*

$$\mathbb{P}_\mathcal{X}\Big(\sup_{r\geq 0}\big(F^{(N)}(r,t) - f(r)\big) \geq 0\Big) \leq \mathbb{P}_\mathcal{X}\Big(\sup_{r\geq 0}\big(C_1F^+(r,t) - f(r)\big) \geq 0\Big).$$

PROOF. This is a direct property of the standard coupling between the N-BBM $X^{(N)}(t)$ and the BBM $X^+(t)$, as described in Section 3. Observe that, with $X^{(N)}(0) = \mathcal{X} = X^+(0)$, under the coupling described in Section 3, for $t \geq 0$ we have $X^{(N)}(t) \subseteq X^+(t)$. It follows that $F^{(N)}(r,t) \leq F^+(r,t)$ for all $r > 0$ and $t \geq 0$. Therefore, since $F^{(N)}(r,t) \leq 1$ also holds for all $r > 0$ and $t \geq 0$, we have

$$F^{(N)}(r,t) \leq C_1 F^+(r,t), \quad r > 0, t \geq 0$$

which completes the proof. □

Recall the definition of the operator G_t from (20), and introduce

$$v^\ell(r,t) = e^t G_t v_0^\ell(r) \quad \text{with } v_0^\ell(r) = F^+(r,0), \tag{26}$$

the solution of (19) with initial condition determined by the initial configuration $X^+(0)$ of the BBM.

LEMMA 4.2. *For N large enough ($N \geq 48$ is sufficient), for all $\mathcal{X} \in (\mathbb{R}^d)^m$ with $m \leq N$ and all $t > 0$,*

$$\sup_{r>0} \mathbb{P}_{\mathcal{X}}\big(\big|F^+(r,t) - v^\ell(r,t)\big| \geq N^{-1/5}\big) \leq 13e^{4t}N^{-6/5},$$

and

$$\mathbb{P}_{\mathcal{X}}\big(N^{-1}\big(\big|X^+(t)\big| - e^t|\mathcal{X}|\big) \geq N^{-1/5}\big) \leq 13e^{4t}N^{-6/5}.$$

PROOF. Recall that $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion with diffusivity $\sqrt{2}$. We claim that, for $r > 0$ and $t \geq 0$,

$$v^\ell(r,t) = \frac{1}{N}\sum_{i=1}^{|\mathcal{X}|} e^t \mathbb{P}_{\mathcal{X}_i}\big(\|B_t\| < r\big). \tag{27}$$

Indeed, by the definition of G_t in (20),

$$\begin{aligned}
G_t F^+(r,0) &= \frac{1}{N}\int_0^\infty G(y,r,t)\sum_{i=1}^{|\mathcal{X}|} \mathbb{1}_{\{\|\mathcal{X}_i\|<y\}}\,\mathrm{d}y \\
&= \frac{1}{N}\sum_{i=1}^{|\mathcal{X}|}\int_{\|\mathcal{X}_i\|}^\infty G(y,r,t)\,\mathrm{d}y \\
&= -\frac{1}{N}\sum_{i=1}^{|\mathcal{X}|}\int_{\|\mathcal{X}_i\|}^\infty \int_0^r \partial_y g\big(y,r',t\big)\,\mathrm{d}r'\,\mathrm{d}y \\
&= \frac{1}{N}\sum_{i=1}^{|\mathcal{X}|}\int_0^r g\big(\|\mathcal{X}_i\|,r',t\big)\,\mathrm{d}r' \\
&= \frac{1}{N}\sum_{i=1}^{|\mathcal{X}|}\mathbb{P}_{\mathcal{X}_i}\big(\|B_t\| < r\big),
\end{aligned}$$

where the third line follows by (16) and (17) and the last two lines follow since $g(y,\cdot,t)$ is the density at time t of a d-dimensional Bessel process started at y. This proves the claim (27).

In the BBM X^+ started from $\mathcal{X}$, for $i \in \{1,\dots,|\mathcal{X}|\}$, denote by $\mathcal{N}_t^{+,i}$ the labels of the family of particles at time t descended from the ith particle in $\mathcal{X}$, that is, recalling that particles are labelled according to the Ulam–Harris scheme, let

$$\mathcal{N}_t^{+,i} = \big\{u \in \mathcal{N}_t^+ : u = (i,u_2,\dots)\big\}.$$

Then, letting $X^{+,i}(t) = (X_u^+(t))_{u\in\mathcal{N}_t^{+,i}}$, the processes $X^{+,i}$ form a family of independent BBMs, and for each i the process $X^{+,i}$ is started from a single particle at location $\mathcal{X}_i$. Fix a time $t > 0$, and write $n_i = |\mathcal{N}_t^{+,i}|$ for the number of particles descended from $\mathcal{X}_i$ at time t and

$$n_i(r) = \big|\big\{u \in \mathcal{N}_t^{+,i} : X_u^+(t) \in \mathcal{B}(r)\big\}\big|$$

for the number of particles at time t descended from $\mathcal{X}_i$ which lie within distance r of the origin. Then, $F^+(r,t) = \frac{1}{N}\sum_{i=1}^{|\mathcal{X}|} n_i(r)$ and by the many-to-one lemma (see [19]) and (27),

$$\frac{1}{N}\sum_{i=1}^{|\mathcal{X}|}\mathbb{E}_{\mathcal{X}}\big[n_i(r)\big] = \frac{1}{N}\sum_{i=1}^{|\mathcal{X}|} e^t\mathbb{P}_{\mathcal{X}_i}\big(\|B_t\| < r\big) = v^\ell(r,t).$$

Therefore,

$$\begin{aligned}
&\mathbb{E}_{\mathcal{X}}\big[\big(F^+(r,t)-v^\ell(r,t)\big)^4\big]\\
&\quad=\frac{1}{N^4}\,\mathbb{E}_{\mathcal{X}}\Bigg[\Bigg(\sum_{i=1}^{|\mathcal{X}|}\big(n_i(r)-\mathbb{E}_{\mathcal{X}}[n_i(r)]\big)\Bigg)^4\Bigg]\\
&\quad=\frac{1}{N^4}\Bigg(\sum_{i=1}^{|\mathcal{X}|}\mathbb{E}_{\mathcal{X}}\big[\big(n_i(r)-\mathbb{E}_{\mathcal{X}}[n_i(r)]\big)^4\big]+6\sum_{\substack{i,j=1\\ i<j}}^{|\mathcal{X}|}\operatorname{Var}_{\mathcal{X}}\big(n_i(r)\big)\operatorname{Var}_{\mathcal{X}}\big(n_j(r)\big)\Bigg).
\end{aligned}$$

Note that

$$\mathbb{E}_{\mathcal{X}}\big[\big(n_i(r)-\mathbb{E}_{\mathcal{X}}[n_i(r)]\big)^4\big]\le\mathbb{E}_{\mathcal{X}}\big[n_i(r)^4+\mathbb{E}_{\mathcal{X}}[n_i(r)]^4\big]\le 2\,\mathbb{E}_{\mathcal{X}}\big[n_i(r)^4\big]\le 2\,\mathbb{E}_{\mathcal{X}}\big[n_i^4\big] \tag{28}$$

and that

$$\operatorname{Var}_{\mathcal{X}}\big(n_i(r)\big)\le\mathbb{E}_{\mathcal{X}}\big[n_i(r)^2\big]\le\mathbb{E}_{\mathcal{X}}\big[n_i^2\big]. \tag{29}$$

Furthermore, since n_i has geometric distribution with parameter e^{-t},

$$\mathbb{E}_{\mathcal{X}}\big[n_i^4\big]\le 24e^{4t}\quad\text{and}\quad\mathbb{E}_{\mathcal{X}}\big[n_i^2\big]\le 2e^{2t}. \tag{30}$$

We now have that, since $|\mathcal{X}|\le N$,

$$\mathbb{E}_{\mathcal{X}}\big[\big(F^+(r,t)-v^\ell(r,t)\big)^4\big]\le\frac{1}{N^4}\big(N48e^{4t}+3N^2 4e^{4t}\big)\le\big(48N^{-3}+12N^{-2}\big)e^{4t}.$$

By the same argument, since $|X^+(t)|=\sum_{i=1}^{|\mathcal{X}|}n_i$ and $\sum_{i=1}^{|\mathcal{X}|}\mathbb{E}_{\mathcal{X}}[n_i]=e^t|\mathcal{X}|$, we also have

$$\mathbb{E}_{\mathcal{X}}\big[\big(N^{-1}\big(|X^+(t)|-e^t|\mathcal{X}|\big)\big)^4\big]\le\big(48N^{-3}+12N^{-2}\big)e^{4t}.$$

By Markov's inequality the stated results follow as soon as

$$N^{4/5}\big(48N^{-3}+12N^{-2}\big)\le 13N^{-6/5}$$

which is equivalent to $N\ge 48$. $\square$

Next, we upgrade the estimate in Lemma 4.2 to be uniform in r, at the expense of slightly slower decay in N. Recall the definition of v^ℓ in (26).

LEMMA 4.3. *For all $N\ge 1$, all $\mathcal{X}\in(\mathbb{R}^d)^m$ with $m\le N$, and all $t>0$ such that $e^t|\mathcal{X}|N^{-1}\ge 1-N^{-1/10}$,*

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{r\ge 0}\big|C_1F^+(r,t)-C_1v^\ell(r,t)\big|\ge 3N^{-1/10}\Big)\le 13e^{4t}N^{-11/10}.$$

PROOF. Observe that there is nothing to prove if $N^{1/10}<3$; as in that case the probability is actually 0. Thus, we assume henceforth in this proof that $N^{1/10}\ge 3$. Fix $t>0$ such that $e^t|\mathcal{X}|N^{-1}\ge 1-N^{-1/10}$. For $k\in\mathbb{N}_0$ with $k\le\lfloor N^{1/10}\rfloor$, let

$$r_k=\inf\bigg\{y\ge 0: v^\ell(y,t)\ge\frac{k}{\lfloor N^{1/10}\rfloor}\bigg\}.$$

Recall the formula for $v^\ell(\cdot,t)$ in (27) in the proof of Lemma 4.2. Note that $F^+(\cdot,t)$ and $v^\ell(\cdot,t)$ are nondecreasing, that $v^\ell(\cdot,t)$ is continuous, and that (r_k) for $k=0,1,\dots,\lfloor N^{1/10}\rfloor$ is an increasing sequence with $r_0=0$. As $r\to\infty$, $v^\ell(r,t)\to e^tN^{-1}|\mathcal{X}|\ge 1-N^{-1/10}$, so $r_k<\infty$ for $k\le\lfloor N^{1/10}\rfloor-2$.

Suppose that, for every $k \in \{1, \dots, \lfloor N^{1/10} \rfloor - 2\}$,

$$|F^+(r_k, t) - v^\ell(r_k, t)| \le N^{-1/5}. \tag{31}$$

Then, for $r \ge 0$, if $r \in [r_k, r_{k+1}]$ for some $k \in \{0, \dots, \lfloor N^{1/10} \rfloor - 3\}$, we have

$$\frac{k}{\lfloor N^{1/10} \rfloor} - N^{-1/5} \le F^+(r_k, t) \le F^+(r, t) \le F^+(r_{k+1}, t) \le N^{-1/5} + \frac{k+1}{\lfloor N^{1/10} \rfloor}$$

and

$$v^\ell(r, t) \in \left[\frac{k}{\lfloor N^{1/10} \rfloor}, \frac{k+1}{\lfloor N^{1/10} \rfloor}\right].$$

Since $\frac{\lfloor N^{1/10} \rfloor - 2}{\lfloor N^{1/10} \rfloor} + N^{-1/5} \le 1$ for all $N \ge 1$, we see that, for all $r \le r_{\lfloor N^{1/10} \rfloor - 2}$, we have $C_1 F^+(r, t) = F^+(r, t)$ and $C_1 v^\ell(r, t) = v^\ell(r, t)$. Thus, assuming (31) holds,

$$\begin{aligned} |C_1 F^+(r, t) - C_1 v^\ell(r, t)| &\le N^{-1/5} + (\lfloor N^{1/10} \rfloor)^{-1} \\ &< 3N^{-1/10} \end{aligned}$$

for all $r \le r_{\lfloor N^{1/10} \rfloor - 2}$ (since $N^{1/10} \ge 3$ implies $(\lfloor N^{1/10} \rfloor)^{-1} < 2N^{-1/10}$).

If instead $r \ge r_k$ where $k = \lfloor N^{1/10} \rfloor - 2$, then

$$v^\ell(r, t) \ge \frac{k}{\lfloor N^{1/10} \rfloor} = 1 - 2(\lfloor N^{1/10} \rfloor)^{-1},$$

and by (31),

$$F^+(r, t) \ge F^+(r_k, t) \ge 1 - N^{-1/5} - 2(\lfloor N^{1/10} \rfloor)^{-1}.$$

Hence,

$$|C_1 F^+(r, t) - C_1 v^\ell(r, t)| < 3N^{-1/10}$$

(since $N^{1/10} \ge 3$ implies $N^{-1/10} + 2N^{1/10}/\lfloor N^{1/10} \rfloor < 3$). Thus, if $N^{1/10} \ge 3$ and (31) holds for each $k \in \{1, \dots, \lfloor N^{1/10} \rfloor - 2\}$, then

$$\sup_{r \ge 0} |C_1 F^+(r, t) - C_1 v^\ell(r, t)| < 3N^{-1/10}.$$

Now, by a union bound and Lemma 4.2, for $N^{1/10} \ge 3$,

$$\begin{aligned} &\mathbb{P}_{\mathcal{X}}\Big(\sup_{r \ge 0} |C_1 F^+(r, t) - C_1 v^\ell(r, t)| \ge 3N^{-1/10}\Big) \\ &\quad \le \mathbb{P}_{\mathcal{X}}\big(\exists k \in \{1, \dots, \lfloor N^{1/10} \rfloor - 2\} : |F^+(r_k, t) - v^\ell(r_k, t)| \ge N^{-1/5}\big) \\ &\quad \le N^{1/10} \cdot 13 e^{4t} N^{-6/5} \\ &\quad = 13 e^{4t} N^{-11/10} \end{aligned}$$

which completes the proof. □

COROLLARY 4.4. *For all $N \ge 1$, for $\mathcal{X} \in (\mathbb{R}^d)^N$, and $t > 0$,*

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{r \ge 0}\big(F^{(N)}(r, t) - C_1 e^t G_t F^{(N)}(r, 0)\big) \ge 3N^{-1/10}\Big) \le 13 e^{4t} N^{-11/10}.$$

PROOF. This follows immediately from Lemma 4.3, Proposition 4.1, and the definition of v^ℓ in (26). $\square$

As in (22), for $\delta > 0$ and $k \in \mathbb{N}_0$, let

$$v^{k,\delta,+} = (C_1 e^\delta G_\delta)^k F^{(N)}(\cdot, 0).$$

We now apply Corollary 4.4 repeatedly on successive timesteps to prove the following result.

PROPOSITION 4.5. *For all $N \geq 1$, $\delta > 0$, $K \in \mathbb{N}$ and $\mathcal{X} \in (\mathbb{R}^d)^N$,*

$$\mathbb{P}_\mathcal{X}\Big(\sup_{r \geq 0}\big(F^{(N)}(r, K\delta) - v^{K,\delta,+}(r)\big) \geq 3Ke^{K\delta}N^{-1/10}\Big) \leq 13Ke^{4\delta}N^{-11/10}.$$

PROOF. For $k \in \{1, \dots, K\}$ and $r \geq 0$, we can write

$$\begin{aligned} F^{(N)}(r, k\delta) - v^{k,\delta,+}(r) = {} & F^{(N)}(r, k\delta) - C_1 e^\delta G_\delta F^{(N)}\big(r, (k-1)\delta\big) \\ & + C_1 e^\delta G_\delta F^{(N)}\big(r, (k-1)\delta\big) \\ & - C_1 e^\delta G_\delta v^{k-1,\delta,+}(r). \end{aligned} \tag{32}$$

To control (32), we define the events

$$E_k := \Big\{\sup_{r \geq 0}\big(F^{(N)}(r, k\delta) - C_1 e^\delta G_\delta F^{(N)}\big(r, (k-1)\delta\big)\big) < 3N^{-1/10}\Big\}$$

and

$$E_* = \bigcap_{k=1}^K E_k.$$

Then, on the event E_*, for each $k \in \{1, \dots, K\}$, we have by (32) that, for $r \geq 0$,

$$F^{(N)}(r, k\delta) - v^{k,\delta,+}(r) < 3N^{-1/10} + C_1 e^\delta G_\delta F^{(N)}\big(r, (k-1)\delta\big) - C_1 e^\delta G_\delta v^{k-1,\delta,+}(r).$$

Note that since $G \geq 0$ and $\int_0^\infty G(y, r, t)\,\mathrm{d}y \leq 1$ by (14) and (15), we have that $G_\delta f - G_\delta g \leq \max(0, \sup_{r \geq 0}(f(r) - g(r))$ for any $f, g : [0, \infty) \to \mathbb{R}$. Moreover, $C_1 f(r) - C_1 g(r) \leq \max(0, f(r) - g(r))$ for any $f, g : [0, \infty) \to \mathbb{R}$. Therefore, for $r \geq 0$,

$$C_1 e^\delta G_\delta f(r) - C_1 e^\delta G_\delta g(r) \leq e^\delta \max\Big(0, \sup_{r \geq 0}\big(f(r) - g(r)\big)\Big).$$

It follows that

$$F^{(N)}(r, k\delta) - v^{k,\delta,+}(r) < 3N^{-1/10} + e^\delta \max\Big(0, \sup_{y \geq 0}\big(F^{(N)}\big(y, (k-1)\delta\big) - v^{k-1,\delta,+}(y)\big)\Big)$$

also holds on the event E_*. By iterating this argument, it follows that, for $k \in \{1, \dots, K\}$,

$$\sup_{r \geq 0}\big(F^{(N)}(r, k\delta) - v^{k,\delta,+}(r)\big) < 3ke^{k\delta}N^{-1/10}$$

holds on E_*.

To estimate $\mathbb{P}_\mathcal{X}(E_*^c)$, we use a union bound and Corollary 4.4 with $t = \delta$. Specifically,

$$\mathbb{P}_\mathcal{X}(E_*^c) = \mathbb{P}_\mathcal{X}\left(\bigcup_{k=1}^K E_k^c\right) \leq \sum_{k=1}^K \mathbb{P}_\mathcal{X}(E_k^c).$$

By the Markov property, for $k \in \{1, \dots, K\}$,

$$\mathbb{P}_\mathcal{X}(E_k^c) = \mathbb{E}_\mathcal{X}\big[\mathbb{P}_\mathcal{X}\big(E_k^c | \mathcal{F}_{(k-1)\delta}\big)\big] = \mathbb{E}_\mathcal{X}\big[H\big(X^{(N)}((k-1)\delta)\big)\big],$$

where $H : (\mathbb{R}^d)^N \to \mathbb{R}$ is defined by

$$H(\mathcal{X}') = \mathbb{P}_{\mathcal{X}'}\Big(\sup_{r\geq 0}\big(F^{(N)}(r,\delta) - C_1 e^{\delta} G_\delta F^{(N)}(r,0)\big) \geq 3N^{-1/10}\Big).$$

Therefore, by Corollary 4.4,

$$\mathbb{P}_{\mathcal{X}}\big(E_*^c\big) \leq \sum_{k=1}^{K} 13e^{4\delta}N^{-11/10} = 13Ke^{4\delta}N^{-11/10}.$$

The result follows. □

4.1.2. *Lower bound for the proof of Proposition 1.5.* We begin by proving that under a suitable coupling, the random function $F^{(N)}$ for the N-BBM stochastically dominates the random function F^+ for the BBM with an initial condition consisting of less than N particles. This result is very similar to the lower bound in Theorem 5.1 of [14].

Recall from our definition of the $\preceq$ notation in (23) in Section 3 that

$$X^{(N)}(t) \preceq X^+(t) \quad \Leftrightarrow \quad F^{(N)}(\cdot,t) \geq F^+(\cdot,t).$$

Moreover, the relation $\mathcal{X} \preceq \mathcal{X}^+$ is not affected by the ordering of the points in the vectors $\mathcal{X}$ and $\mathcal{X}^+$.

PROPOSITION 4.6. *Suppose $\mathcal{X} = (\mathcal{X}_1,\dots,\mathcal{X}_N) \in (\mathbb{R}^d)^N$ and $\mathcal{X}^+ = (\mathcal{X}_1^+,\dots,\mathcal{X}_m^+) \in (\mathbb{R}^d)^m$ with $m \leq N$ and such that $\mathcal{X} \preceq \mathcal{X}^+$. There exists a coupling of the N-BBM $X^{(N)}(t)$ started from $\mathcal{X}$ and of the BBM $X^+(t)$ started from $\mathcal{X}^+$ such that, for $t \geq 0$,*

$$X^{(N)}(t) \preceq X^+(t) \quad \text{if } \big|X^+(t)\big| \leq N.$$

In particular, under that coupling, $F^{(N)}(\cdot,t) \geq F^+(\cdot,t)$ if $|X^+(t)| \leq N$. Then, if $f : [0,\infty) \to \mathbb{R}$ is measurable, for $t \geq 0$,

$$\mathbb{P}_{\mathcal{X}}\Big(\inf_{r\geq 0}\big(F^{(N)}(r,t) - f(r)\big) > 0\Big) \geq \mathbb{P}_{\mathcal{X}^+}\Big(\inf_{r\geq 0}\big(F^+(r,t) - f(r)\big) > 0, \big|X^+(t)\big| \leq N\Big). \tag{33}$$

PROOF. The coupling of the processes $X^{(N)}$ and X^+ is similar in spirit to the lower bound in Section 5.4 of [14].

Let τ_ℓ^+ for $\ell \in \mathbb{N}$ denote the successive branch times of the BBM process,

$$\tau_\ell^+ = \inf\{t \geq 0 : \big|X^+(t)\big| = m + \ell\}.$$

By induction on $|\mathcal{X}^+|$, we claim that it is sufficient to find a coupling of the processes $X^{(N)}$ and X^+ on the same probability space with $X^{(N)}(0) = \mathcal{X}$ and $X^+(0) = \mathcal{X}^+$ such that

$$X^{(N)}(t) \preceq X^+(t), \quad \forall t \in \begin{cases} [0,\tau_1^+], & \text{if } m < N, \\ [0,\tau_1^+), & \text{if } m = N \end{cases} \tag{34}$$

holds almost surely. Indeed, assume that (34) holds. If $m = N$, then the proposition is proved. If $m < N$, then $X^{(N)}(\tau_1^+) \preceq X^+(\tau_1^+)$, and the construction of the coupling can be repeated up to time τ_2^+ using $X^{(N)}(\tau_1^+)$ and $X^+(\tau_1^+)$ as the new initial particle configurations, by the strong Markov property. By induction the property $X^{(N)}(t) \preceq X^+(t)$ holds for $t \in [0,\tau_{N-m+1}^+)$, where τ_{N-m+1}^+ is the first time at which there are $N+1$ particles in the BBM X^+, and the proposition is proved.

We now show that (34) holds. Set $\tau_0 = 0$, and let $(\tau_i)_{i=1}^\infty$ be the arrival times in a Poisson process with rate N so that $(\tau_{i+1} - \tau_i)_{i\geq 0}$ is a family of independent $\mathrm{Exp}(N)$ random variables. These τ_i for $i \geq 1$ will define the branch times for the N-BBM process $X^{(N)}$. The coupling will ensure that $\tau_1^+ = \tau_p$ for some $p \in \mathbb{N}$, where we recall that τ_1^+ is the time of the first branching event in the BBM X^+.

We now construct the motion of the particles for $t \in (0, \tau_1)$. Given $x, x^+ \in \mathbb{R}^d$ with $\|x\| \leq \|x^+\|$, we say that (B, B^+) are a pair of spherically-ordered Brownian motions starting from (x, x^+) if B and B^+ are Brownian motions in $\mathbb{R}^d$ (with diffusivity $\sqrt{2}$), starting from $B_0 = x$ and $B_0^+ = x^+$, and such that, with probability one, $\|B_t\| \leq \|B_t^+\|$ holds for all $t \geq 0$. There are multiple ways to construct such a pair. For example, B and B^+ might evolve as independent Brownian motions in $\mathbb{R}^d$ up to the first time T at which $\|B_T\| = \|B_T^+\|$; after that time they are coupled in such a way that $\|B_t\| = \|B_t^+\|$ for all $t \geq T$, for example, by taking $B_t - B_T = \Theta(B_t^+ - B_T^+)$ with $\Theta : \mathbb{R}^d \to \mathbb{R}^d$ being an orthogonal transformation such that $\Theta B_T^+ = B_T$. Alternatively, one could use the skew-product decomposition of Brownian motion (see, e.g., the proof of Proposition 2.10 in [8]), driving the radial components of B and B^+ by the same Bessel process.

Since the condition $\mathcal{X} \preceq \mathcal{X}^+$ is invariant under permutation of the indices of points in $\mathcal{X}$ and $\mathcal{X}^+$, it suffices to assume that the vectors $\mathcal{X}$ and $\mathcal{X}^+$ are ordered in such a way that

$$\|\mathcal{X}_k\| \leq \|\mathcal{X}_k^+\|, \quad \forall k \in \{1, \dots, m\},$$

where we recall that $m = |\mathcal{X}^+| \leq N$. (Order the vectors $\mathcal{X}$ and $\mathcal{X}^+$, e.g., so that the points with the lowest indices are the points closest to the origin.)

Then, for $t \in (0, \tau_1)$, we define $(X_k^{(N)}(t), X_k^+(t))$ for $k \in \{1, \dots, m\}$ to be m independent pairs of spherically-ordered Brownian motions starting from $(\mathcal{X}_k, \mathcal{X}_k^+)$, and for $k \in \{m+1, \dots, N\}$, the $X_k^{(N)}(t)$ are defined to be independent Brownian motions starting from $\mathcal{X}_k$. Hence, for $t \in (0, \tau_1)$, we have $\|X_k^{(N)}(t)\| \leq \|X_k^+(t)\|$ for $k \in \{1, \dots, m\}$ which, in turn, implies that

$$X^{(N)}(t) \preceq X^+(t) \quad \forall t < \tau_1.$$

We now describe the first branching event at time τ_1.

Defining $X^{(N)}(\tau_1-) = (X_1^{(N)}(\tau_1-), \dots, X_N^{(N)}(\tau_1-)) = \lim_{t \nearrow \tau_1} X^{(N)}(t)$, and similarly defining $X^+(\tau_1-) = \lim_{t \nearrow \tau_1} X^+(t)$, we have

$$\|X_k^{(N)}(\tau_1-)\| \leq \|X_k^+(\tau_1-)\|, \quad \forall k \in \{1, \dots, m\}.$$

Let j_1, a random variable uniformly distributed on $\{1, \dots, N\}$, be the index of the branching particle in the N-BBM at time τ_1. Let k_1 denote the index of the particle in $X^{(N)}(\tau_1-)$ with maximal distance from the origin. If $j_1 > m$, then at time τ_1, the N-BBM branches, but the BBM does not. The particle in $X^{(N)}$ with index j_1 is duplicated; the particle in $X^{(N)}$ with index k_1 is eliminated. More precisely, let

$$X_k^{(N)}(\tau_1) = X_k^{(N)}(\tau_1-) \quad \text{for } k \neq k_1,$$
$$X_{k_1}^{(N)}(\tau_1) = X_{j_1}^{(N)}(\tau_1-), \quad \text{and} \quad X^+(\tau_1) = X^+(\tau_1-).$$

Then, for $k \in \{1, \dots, m\}$,

$$\|X_k^{(N)}(\tau_1)\| \leq \|X_k^{(N)}(\tau_1-)\| \leq \|X_k^+(\tau_1-)\| = \|X_k^+(\tau_1)\|,$$

and, in particular, $X^{(N)}(\tau_1) \preceq X^+(\tau_1)$. The construction is then repeated to extend the definition from time τ_1 to τ_2: take new pairs of spherically-ordered Brownian motions to determine the motion of particles up to time τ_2, pick the branching particle j_2 in the N-BBM uniformly at random, and so on until time τ_{i^+}, where $i^+ = \inf\{i \geq 1 : j_i \leq m\}$. This time $\tau_{i^+} = \tau_1^+$ is a branching time for both the N-BBM and the BBM. Observe that $\tau_1^+ \sim \text{Exp}(m)$.

In the construction of the coupling, so far we have that, for $t < \tau_1^+$,

$$\|X_k^{(N)}(t)\| \le \|X_k^+(t)\| \quad \forall k \in \{1, \dots, m\},$$

and so, in particular, $X^{(N)}(t) \preceq X^+(t)$. At time $\tau_1^+ - = \tau_{i^+} -$, the particles with index $j_{i^+} \in \{1, \dots, m\}$ from both $X^{(N)}$ and X^+ branch, and the particle in $X^{(N)}$ of maximal distance from the origin is removed. More precisely, if k_{i^+} is the index of the particle in $X^{(N)}(\tau_1^+ -)$ with maximal distance from the origin, we let

$$X_k^{(N)}(\tau_1^+) = X_k^{(N)}(\tau_1^+ -) \quad \text{for } k \ne k_{i^+}, \qquad X_{k_{i^+}}^{(N)}(\tau_1^+) = X_{j_{i^+}}^{(N)}(\tau_1^+ -),$$
$$\text{and} \quad X^+(\tau_1^+) = (X_1^+(\tau_1^+ -), \dots, X_{j_{i^+}}^+(\tau_1^+ -), X_{j_{i^+}}^+(\tau_1^+ -), \dots, X_m^+(\tau_1^+ -)).$$

Suppose $m < N$. Then, $\|X_k^{(N)}(\tau_1^+)\| \le \|X_k^+(\tau_1^+ -)\|$ $\forall k \in \{1, \dots, m\}$, with $k \ne k_{i^+}$, and also $\|X_{k_{i^+}}^{(N)}(\tau_1^+)\| \le \|X_{j_{i^+}}^+(\tau_1^+ -)\|$. Moreover, if $k_{i^+} \le m$, then $\|X_{m+1}^{(N)}(\tau_1^+)\| \le \|X_{k_{i^+}}^{(N)}(\tau_1^+ -)\| \le \|X_{k_{i^+}}^+(\tau_1^+ -)\|$. It follows that $X^{(N)}(\tau_1^+) \preceq X^+(\tau_1^+)$ if $m < N$. This concludes the coupling construction to achieve (34), and the proof of proposition is now complete. □

We now use Proposition 4.6 to prove a lower bound on $F^{(N)}(\cdot, \delta)$.

LEMMA 4.7. *For N large enough ($N \ge (2e)^{10}$ is sufficient), for all $\mathcal{X} \in (\mathbb{R}^d)^N$, and for all $\delta \in (0, 1)$,*

$$\mathbb{P}_{\mathcal{X}}\Big(\inf_{r \ge 0}\big(F^{(N)}(r, \delta) - e^\delta G_\delta C_{e^{-\delta} - N^{-1/5}} F^{(N)}(r, 0)\big) \le -4N^{-1/10}\Big) \le 26 e^{4\delta} N^{-11/10}.$$

PROOF. Take $\delta \in (0, 1)$ and $\mathcal{X} \in (\mathbb{R}^d)^N$. For $r \ge 0$, let $f_{\mathcal{X}}(r) = N^{-1}|\mathcal{X} \cap \mathcal{B}(r)|$; note that if $X^{(N)}(0) = \mathcal{X}$, then $F^{(N)}(r, 0) = f_{\mathcal{X}}(r)$. Let $\mathcal{X}^+ \subset \mathcal{X}$ consist of the $\lfloor N(e^{-\delta} - N^{-1/5})\rfloor$ particles in $\mathcal{X}$ which are closest to the origin. Let $(X^+(t), t \ge 0)$ be the BBM started from $X^+(0) = \mathcal{X}^+$. Observe that $F^+(r, 0) = C_{\lfloor N(e^{-\delta} - N^{-1/5})\rfloor / N} f_{\mathcal{X}}(r)$ so that

$$C_{e^{-\delta} - N^{-1/5}} f_{\mathcal{X}}(r) - N^{-1} \le F^+(r, 0) \le C_{e^{-\delta} - N^{-1/5}} f_{\mathcal{X}}(r) \quad \forall r \ge 0. \tag{35}$$

Let R_1 be the event

$$R_1 = \{|X^+(\delta)| \le N\}.$$

Since $|\mathcal{X}^+| \le N(e^{-\delta} - N^{-1/5})$, one has $N^{-1} e^\delta |\mathcal{X}^+| - 1 \le -N^{-1/5} e^\delta \le -N^{-1/5}$, and Lemma 4.2 implies that, for $N \ge 48$,

$$\mathbb{P}_{\mathcal{X}^+}(R_1^c) \le \mathbb{P}_{\mathcal{X}^+}\big(N^{-1}(|X^+(\delta)| - e^\delta |\mathcal{X}^+|) \ge N^{-1/5}\big) \le 13 e^{4\delta} N^{-6/5}.$$

Let R_2 be the event

$$R_2 = \Big\{\sup_{r \ge 0} |C_1 F^+(r, \delta) - C_1 v^\ell(r, \delta)| < 3N^{-1/10}\Big\},$$

where, as in (26) in Section 4.1.1, we let $v^\ell(r, \delta) = e^\delta G_\delta F^+(r, 0)$. Since $|\mathcal{X}^+| > N(e^{-\delta} - N^{-1/5}) - 1$, one has $e^\delta N^{-1}|\mathcal{X}^+| \ge 1 - e^\delta N^{-1/5} - e^\delta N^{-1} \ge 1 - N^{-1/10}$ if N is sufficiently large that $e(N^{-1/5} + N^{-1}) \le N^{-1/10}$, for instance, if $N \ge (2e)^{10}$. Then, by Lemma 4.3 we know that, for all $N \ge (2e)^{10}$,

$$\mathbb{P}_{\mathcal{X}^+}(R_2^c) \le 13 e^{4\delta} N^{-11/10}.$$

Since $e^\delta N^{-1}|\mathcal{X}^+| < 1$, we have $v^\ell(\cdot, \delta) < 1$ (by (27) in the proof of Lemma 4.2). Therefore, by (35) we have that, for $r \ge 0$,

$$C_1 v^\ell(r, \delta) = v^\ell(r, \delta) = e^\delta G_\delta F^+(r, 0) \ge e^\delta G_\delta C_{e^{-\delta} - N^{-1/5}} f_{\mathcal{X}}(r) - e^\delta N^{-1}.$$

On the event R_1 we also have $F^+(\cdot,\delta) \le N^{-1}|X^+(\delta)| \le 1$, and hence $C_1 F^+(\cdot,\delta) = F^+(\cdot,\delta)$. This shows that on the event $R_1 \cap R_2$, we have both $|X^+(\delta)| \le N$ and

$$\begin{aligned}\inf_{r\ge 0}\big(F^+(r,\delta) - e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} f_{\mathcal{X}}(r)\big) &\ge \inf_{r\ge 0}\big(C_1 F^+(r,\delta) - C_1 v^\ell(r,\delta)\big) - e^\delta N^{-1}\\ &> -3N^{-1/10} - e^\delta N^{-1}.\end{aligned}$$

Note that since $\mathcal{X}^+ \subset \mathcal{X}$, we have $\mathcal{X} \preceq \mathcal{X}^+$. Therefore, by Proposition 4.6 and taking N sufficiently large that $eN^{-1} \le N^{-1/10}$ (which again holds when $N \ge (2e)^{10}$),

$$\begin{aligned}&\mathbb{P}_{\mathcal{X}}\Big(\inf_{r\ge 0}\big(F^{(N)}(r,\delta) - e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} f_{\mathcal{X}}(r)\big) > -4N^{-1/10}\Big)\\ &\quad\ge \mathbb{P}_{\mathcal{X}^+}\Big(\inf_{r\ge 0}\big(F^+(r,\delta) - e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} f_{\mathcal{X}}(r)\big) > -4N^{-1/10}, \big|X^+(\delta)\big| \le N\Big)\\ &\quad\ge \mathbb{P}_{\mathcal{X}^+}(R_1 \cap R_2)\\ &\quad\ge 1 - \mathbb{P}_{\mathcal{X}^+}\big(R_1^c\big) - \mathbb{P}_{\mathcal{X}^+}\big(R_2^c\big)\\ &\quad\ge 1 - 13e^{4\delta}N^{-6/5} - 13e^{4\delta}N^{-11/10},\end{aligned}$$

which completes the proof, since $f_{\mathcal{X}}(\cdot) = F^{(N)}(\cdot,0)$ if $X^{(N)}(0) = \mathcal{X}$. □

As in (22), for $k \in \mathbb{N}_0$ and $\delta > 0$, let

$$v^{k,\delta,-} = \big(e^\delta G_\delta C_{e^{-\delta}}\big)^k F^{(N)}(\cdot,0).$$

By applying Lemma 4.7 repeatedly, we can prove the following lower bound, which, together with the upper bound in Proposition 4.5, will allow us to prove Proposition 1.5.

PROPOSITION 4.8. *For N large enough ($N \ge (2e)^{10}$ is sufficient), for all $\mathcal{X} \in (\mathbb{R}^d)^N$, $\delta \in (0,1)$, and $K \in \mathbb{N}$,*

$$\mathbb{P}_{\mathcal{X}}\Big(\inf_{r\ge 0}\big(F^{(N)}(r,K\delta) - v^{K,\delta,-}(r)\big) \le -5Ke^{K\delta}N^{-1/10}\Big) \le 26Ke^{4\delta}N^{-11/10}.$$

PROOF. For $k \in \{1,\dots,K\}$ and $r \ge 0$, we can write

$$\begin{aligned}&F^{(N)}(r,k\delta) - v^{k,\delta,-}(r)\\ &\quad= F^{(N)}(r,k\delta) - e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} F^{(N)}\big(r,(k-1)\delta\big)\\ &\qquad+ e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} F^{(N)}\big(r,(k-1)\delta\big) - e^\delta G_\delta C_{e^{-\delta}} F^{(N)}\big(r,(k-1)\delta\big)\\ &\qquad+ e^\delta G_\delta C_{e^{-\delta}} F^{(N)}\big(r,(k-1)\delta\big) - e^\delta G_\delta C_{e^{-\delta}} v^{k-1,\delta,-}(r).\end{aligned} \tag{36}$$

For the second line on the right-hand side of (36), note that $\|C_{e^{-\delta}-N^{-1/5}} f - C_{e^{-\delta}} f\|_{L^\infty} \le N^{-1/5}$ for any $f : [0,\infty) \to \mathbb{R}$, and so, using (21),

$$\big\|e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} F^{(N)}\big(\cdot,(k-1)\delta\big) - e^\delta G_\delta C_{e^{-\delta}} F^{(N)}\big(\cdot,(k-1)\delta\big)\big\|_{L^\infty} \le e^\delta N^{-1/5}. \tag{37}$$

For the third line on the right-hand side of (36), observe that $C_m f(r) - C_m g(r) \ge \min(0, f(r) - g(r))$ for any $m \in (0,1)$ and any $f, g : [0,\infty) \to \mathbb{R}$. Then, since $\min(0, f(r) - g(r)) \le 0$, we have that, for any $\delta > 0$, $m \in (0,1)$, and $r \ge 0$,

$$G_\delta C_m f(r) - G_\delta C_m g(r) \ge G_\delta \min\big(0, f(r) - g(r)\big) \ge \inf_{y\ge 0} \min\big(0, f(y) - g(y)\big),$$

where we used from (14) and (15) that $G \geq 0$ and $\int_0^\infty G(y, r, t)\,dy \leq 1$. It follows that

$$\begin{aligned}(38)\qquad &\inf_{r\geq 0}\big(G_\delta C_{e^{-\delta}} F^{(N)}(r,(k-1)\delta) - G_\delta C_{e^{-\delta}} v^{k-1,\delta,-}(r)\big)\\ &\quad \geq \min\Big(0, \inf_{y\geq 0}\big(F^{(N)}(y,(k-1)\delta) - v^{k-1,\delta,-}(y)\big)\Big).\end{aligned}$$

To control the first line of the right-hand side of (36) for $k \in \mathbb{N}$, define the event

$$E_k := \Big\{\inf_{r\geq 0}\big(F^{(N)}(r,k\delta) - e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} F^{(N)}(r,(k-1)\delta)\big) > -4N^{-1/10}\Big\},$$

and let

$$E_* = \bigcap_{k=1}^{K} E_k.$$

Then, on the event E_*, using (37) and (38), we have for each $k \in \{1, \dots, K\}$, for all $r \geq 0$,

$$\begin{aligned}F^{(N)}(r,k\delta) - v^{k,\delta,-}(r) &> -4N^{-1/10} - e^\delta N^{-1/5}\\ &\quad + e^\delta \min\Big(0, \inf_{y\geq 0}\big(F^{(N)}(y,(k-1)\delta) - v^{k-1,\delta,-}(y)\big)\Big).\end{aligned}$$

By iterating this bound, it follows that if $N \geq (2e)^{10}$ (and so, in particular, $eN^{-1/5} < N^{-1/10}$), then on the event E_*, for $k \in \{1, \dots, K\}$,

$$(39)\qquad \inf_{r\geq 0}\big(F^{(N)}(r,k\delta) - v^{k,\delta,-}(r)\big) > -\big(4N^{-1/10} + e^\delta N^{-1/5}\big)ke^{k\delta} > -5N^{-1/10}Ke^{K\delta}.$$

To estimate $\mathbb{P}_{\mathcal{X}}(E_*^c)$, we use a union bound and Lemma 4.7. Specifically,

$$\mathbb{P}_{\mathcal{X}}(E_*^c) = \mathbb{P}_{\mathcal{X}}\left(\bigcup_{k=1}^{K} E_k^c\right) \leq \sum_{k=1}^{K} \mathbb{P}_{\mathcal{X}}(E_k^c).$$

By the Markov property, for $k \in \{1, \dots, K\}$,

$$\mathbb{P}_{\mathcal{X}}(E_k^c) = \mathbb{E}_{\mathcal{X}}\big[\mathbb{P}_{\mathcal{X}}(E_k^c \mid \mathcal{F}_{(k-1)\delta})\big] = \mathbb{E}_{\mathcal{X}}\big[H\big(X^{(N)}((k-1)\delta)\big)\big],$$

where $H : (\mathbb{R}^d)^N \to \mathbb{R}$ is defined by

$$H(\mathcal{X}') = \mathbb{P}_{\mathcal{X}'}\Big(\inf_{r\geq 0}\big(F^{(N)}(r,\delta) - e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} F^{(N)}(r,0)\big) \leq -4N^{-1/10}\Big).$$

Therefore, by Lemma 4.7 for $N \geq (2e)^{10}$,

$$\mathbb{P}_{\mathcal{X}}(E_*^c) \leq \sum_{k=1}^{K} 26e^{4\delta} N^{-11/10} = 26Ke^{4\delta}N^{-11/10}.$$

The result follows by (39). $\square$

4.1.3. *Combining the upper and lower bounds for the proof of Proposition* 1.5. We can now complete the proof of Proposition 1.5. Let $v^{(N)}$ denote the solution of (7) with initial condition $v_0(r) = F^{(N)}(r,0)$ for $r \geq 0$. By Lemma 2.2 we have that, for $\delta > 0$, $k \in \mathbb{N}_0$, and $r \geq 0$,

$$v^{k,\delta,-}(r) = \big(e^\delta G_\delta C_{e^{-\delta}}\big)^k F^{(N)}(r,0) \leq v^{(N)}(r,k\delta) \leq \big(C_1 e^\delta G_\delta\big)^k F^{(N)}(r,0) = v^{k,\delta,+}(r)$$

and

$$\big\|v^{k,\delta,+} - v^{k,\delta,-}\big\|_{L^\infty} \leq \big(e^{k\delta}+1\big)\big(e^\delta - 1\big).$$

For $N \geq (2e)^{10}$, $\mathcal{X} \in (\mathbb{R}^d)^N$, $\delta \in (0,1)$, and $K \in \mathbb{N}$, Proposition 4.5 yields that

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{r\geq 0}\big(F^{(N)}(r,K\delta) - v^{(N)}(r,K\delta)\big) \geq 3Ke^{K\delta}N^{-1/10} + \big(e^{K\delta}+1\big)\big(e^{\delta}-1\big)\Big) \leq 13Ke^{4\delta}N^{-11/10},$$

and by Proposition 4.8,

$$\mathbb{P}_{\mathcal{X}}\Big(\inf_{r\geq 0}\big(F^{(N)}(r,K\delta) - v^{(N)}(r,K\delta)\big) \leq -5Ke^{K\delta}N^{-1/10} - \big(e^{K\delta}+1\big)\big(e^{\delta}-1\big)\Big) \leq 26Ke^{4\delta}N^{-11/10}.$$

It follows that, for $N \geq (2e)^{10}$, for $\mathcal{X} \in (\mathbb{R}^d)^N$, $\delta \in (0,1)$, and $K \in \mathbb{N}$,

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{r\geq 0}\big|F^{(N)}(r,K\delta) - v^{(N)}(r,K\delta)\big| \geq 5Ke^{K\delta}N^{-1/10} + \big(e^{K\delta}+1\big)\big(e^{\delta}-1\big)\Big) \leq 39Ke^{4\delta}N^{-11/10}.$$

Take $t > 0$, and let $K = \lceil N^{1/20}t\rceil$ and $\delta = t/K$. Then, $\delta \leq N^{-1/20} \leq 1$, and using that $e^x - 1 \leq 2x$ if $x \in [0,1]$, we see that, for all $N \geq 1$,

$$\begin{aligned}5Ke^{K\delta}N^{-1/10} + \big(e^{K\delta}+1\big)\big(e^{\delta}-1\big) &\leq 5\big(N^{1/20}t+1\big)e^tN^{-1/10} + \big(e^t+1\big)\big(e^{N^{-1/20}}-1\big)\\ &\leq 5(t+1)e^tN^{-1/20} + 4e^tN^{-1/20}\\ &\leq 9e^{2t}N^{-1/20}.\end{aligned}$$

Furthermore, by taking N large enough that $39e^{4N^{-1/20}} \leq 40$, we have $39Ke^{4\delta}N^{-11/10} \leq 40(t+1)N^{-21/20}$. Therefore, for N sufficiently large (for instance, $N \geq 158^{20}$ ensures that $39e^{4N^{-1/20}} \leq 40$), for any $\mathcal{X} \in (\mathbb{R}^d)^N$, and $t > 0$,

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{r\geq 0}\big|F^{(N)}(r,t) - v^{(N)}(r,t)\big| \geq 9e^{2t}N^{-1/20}\Big) \leq 40e^tN^{-21/20}.$$

This completes the proof of Proposition 1.5.

4.2. *Proof of Proposition* 1.6. We begin with the following lemma, which is a consequence of Proposition 1.5 and a concentration estimate for $F^{(N)}(\cdot,0)$, in the case where $X_1^{(N)}(0),\dots,X_N^{(N)}(0)$ are i.i.d. with some fixed distribution μ_0. This lemma will be used later to argue that at a fixed time t, $F^{(N)}(R_t,t)$ is close to 1, where (u,R) is the solution of the free boundary problem (1) with initial condition μ_0.

LEMMA 4.9. *There exist constants $c_3 \in (0,1)$ and $N_0 \in \mathbb{N}$ such that the following holds. Let μ_0 be a given probability distribution on $\mathbb{R}^d$, and suppose $X_1^{(N)}(0),\dots,X_N^{(N)}(0)$ are i.i.d. with distribution μ_0. Let v denote the solution of* (7) *with initial condition $v_0(r) = \mu_0(\mathcal{B}(r))$. Then, for $N \geq N_0$ and $t \geq 0$,*

$$\mathbb{P}\big(\big\|F^{(N)}(\cdot,t) - v(\cdot,t)\big\|_{L^\infty} \geq e^{2t}N^{-c_3}\big) \leq e^tN^{-1-c_3}.$$

The constant N_0 does not depend on μ_0. The difference between Lemma 4.9 and Proposition 1.5 is that the initial condition for $v(\cdot,t)$ in Lemma 4.9 is given by $v_0(r) = \mu_0(\mathcal{B}(r))$, whereas the initial condition for $v^{(N)}(\cdot,t)$ in Proposition 1.5 is given by $v_0(r) = F^{(N)}(r,0)$.

PROOF. Recall from Proposition 1.5 that there is a constant $c_1 > 0$ such that, for N large enough (not depending on the initial particle configuration), for $t \geq 0$,

$$\mathbb{P}\big(\|F^{(N)}(\cdot,t) - v^{(N)}(\cdot,t)\|_{L^\infty} \geq e^{2t}N^{-c_1}\big) \leq e^t N^{-1-c_1},$$

where $v^{(N)}$ is the solution of (7) with initial condition $v_0^{(N)}(r) = F^{(N)}(r,0)$. By Lemma 2.3,

$$\|v^{(N)}(\cdot,t) - v(\cdot,t)\|_{L^\infty} \leq e^t \|F^{(N)}(\cdot,0) - \mu_0(\mathcal{B}(\cdot))\|_{L^\infty}. \tag{40}$$

The function $v_0(r) = \mu_0(\mathcal{B}(r))$ is the cumulative distribution function for each of the real-valued random variables $\|X_i^{(N)}(0)\|$, $i = 1, \dots, N$, which are independent. Therefore, it follows immediately from Corollary 1 and Comment 2(iii) of [22] (which is a sharp, quantitative version of the Glivenko–Cantelli theorem) that

$$\mathbb{P}\big(\|F^{(N)}(\cdot,0) - \mu_0(\mathcal{B}(\cdot))\|_{L^\infty} > \epsilon\big) \leq 2e^{-2N\epsilon^2} \tag{41}$$

holds for all $\epsilon > 0$ and $N \geq 1$.

For $\epsilon > 0$ to be chosen, let E be the event

$$E = \big\{\|F^{(N)}(\cdot,0) - \mu_0(\mathcal{B}(\cdot))\|_{L^\infty} \leq \epsilon\big\}.$$

Then, for $c_3 > 0$ to be determined, N large enough for Proposition 1.5 to hold, and $t \geq 0$,

$$\begin{aligned}
&\mathbb{P}\big(\|F^{(N)}(\cdot,t) - v(\cdot,t)\|_{L^\infty} \geq e^{2t}N^{-c_3}\big) \\
&\quad\leq \mathbb{P}\big(\|F^{(N)}(\cdot,t) - v^{(N)}(\cdot,t)\|_{L^\infty} + \|v^{(N)}(\cdot,t) - v(\cdot,t)\|_{L^\infty} \geq e^{2t}N^{-c_3}\big) \\
&\quad\leq \mathbb{P}\big(\|F^{(N)}(\cdot,t) - v^{(N)}(\cdot,t)\|_{L^\infty} + \|v^{(N)}(\cdot,t) - v(\cdot,t)\|_{L^\infty} \geq e^{2t}N^{-c_3}, E\big) + \mathbb{P}(E^c) \\
&\quad\leq \mathbb{P}\big(\|F^{(N)}(\cdot,t) - v^{(N)}(\cdot,t)\|_{L^\infty} + e^t\epsilon \geq e^{2t}N^{-c_3}, E\big) + 2e^{-2N\epsilon^2}
\end{aligned}$$

by (40) and (41). Choose $c_3 \in (0, \min(c_1, 1/2))$ and $\epsilon = \frac{1}{2}N^{-c_3}$. Then, for N large enough, one has $e^{2t}N^{-c_3} - e^t\epsilon \geq e^{2t}N^{-c_1}$ for all $t \geq 0$ (it is sufficient for the inequality to hold at $t = 0$ which is the case when $N^{c_1 - c_3} \geq 2$). Therefore,

$$\mathbb{P}\big(\|F^{(N)}(\cdot,t) - v(\cdot,t)\|_{L^\infty} \geq e^{2t}N^{-c_3}\big) \leq e^t N^{-1-c_1} + 2e^{-\frac{1}{2}N^{1-2c_3}}.$$

Since $1 - 2c_3 > 0$, we may take N larger if necessary so that $e^t N^{-1-c_1} + 2e^{-\frac{1}{2}N^{1-2c_3}} \leq e^t N^{-1-c_3}$ for all $t \geq 0$ (it is sufficient for the inequality to hold at $t = 0$), and the proof is complete. □

Now consider an N-BBM $X^{(N)}$ started from an initial condition $\mathcal{X} \in (\mathbb{R}^d)^N$. Recall the coupling in Section 3 between $X^{(N)}$ and X^+, where $X^+(t) = (X_u^+(t))_{u \in \mathcal{N}_t^+}$ is the vector of particle locations at time t in a standard BBM started from the same initial condition $\mathcal{X}$ such that under the coupling, for all $t \geq 0$, $X^{(N)}(t) \subseteq X^+(t)$. Recall also from (24) that under the coupling, for $t \geq 0$, almost surely

$$\begin{aligned}
&\{X_k^{(N)}(t)\}_{k=1}^N = \{X_u^+(t)\}_{u \in \mathcal{N}_t^{(N)}}, \\
&\qquad \text{where } \mathcal{N}_t^{(N)} = \{u \in \mathcal{N}_t^+ : \|X_u^+(s)\| \leq M_s^{(N)}\ \forall s \in [0,t]\},
\end{aligned} \tag{42}$$

and $M_s^{(N)} = \max_{k \in \{1,\dots,N\}} \|X_k^{(N)}(s)\|$ is the maximum distance of a particle in the N-BBM from the origin at time s.

For Lemmas 4.10 to 4.12, we also introduce two quantities. For any particle with label $u \in \mathcal{N}_t^+$ in the BBM at time t, let $B_u(s)$ be the displacement of that particle at time $s \in [0,t]$ from its location at time 0,

$$B_u(s) = X_u^+(s) - X_u^+(0).$$

For $\epsilon > 0$ and $r > 0$, let $Z_\epsilon(r)$ denote the number of particles in the BBM at time ϵ which started (at time 0) from a particle in $\mathcal{B}(r)$,

$$Z_\epsilon(r) = \left|\left\{u \in \mathcal{N}_\epsilon^+ : X_u^+(0) \in \mathcal{B}(r)\right\}\right|.$$

LEMMA 4.10. *For $N \geq 1$ and $\epsilon, r > 0$, if the following two conditions hold*:

- $Z_\epsilon(r) > N$,
- $\max_{u \in \mathcal{N}_{2\epsilon}^+} \sup_{s \in [0,2\epsilon]} \|B_u(s)\| \leq \frac{1}{3}\epsilon^{1/3}$,

then

$$M_s^{(N)} \leq r + \epsilon^{1/3} \quad \forall s \in [\epsilon, 2\epsilon].$$

PROOF. Assume the hypotheses hold. Then, we must have

$$\exists s^* \in [0, \epsilon] : \quad M_{s^*}^{(N)} \leq r + \frac{1}{3}\epsilon^{1/3}.$$

Indeed, if we had $M_s^{(N)} > r + \frac{1}{3}\epsilon^{1/3}$ $\forall s \in [0, \epsilon]$, then by (42) and using that $\|B_u(s)\| = \|X_u^+(s) - X_u^+(0)\| \leq \frac{1}{3}\epsilon^{1/3}$,

$$\begin{aligned}\mathcal{N}_\epsilon^{(N)} &\supseteq \left\{u \in \mathcal{N}_\epsilon^+ : \|X_u^+(s)\| \leq r + \frac{1}{3}\epsilon^{1/3}\ \forall s \in [0, \epsilon]\right\} \\ &\supseteq \left\{u \in \mathcal{N}_\epsilon^+ : \|X_u^+(0)\| < r\right\},\end{aligned}$$

which is impossible, as the set on the right-hand side has size $Z_\epsilon(r)$, and we assumed that $Z_\epsilon(r) > N$.

Then, for any $s \in [s^*, 2\epsilon]$, for any particle with label $u \in \mathcal{N}_s^{(N)}$, its ancestor at time s^* must have been within distance $M_{s^*}^{(N)}$ of the origin, that is, $\|X_u^+(s^*)\| \leq M_{s^*}^{(N)}$. Hence,

$$M_s^{(N)} \leq M_{s^*}^{(N)} + \max_{u \in \mathcal{N}_s^+} \|B_u(s) - B_u(s^*)\| \leq \left(r + \frac{1}{3}\epsilon^{1/3}\right) + 2 \cdot \frac{1}{3}\epsilon^{1/3}$$

which completes the proof. □

Recall from the definition of Γ in (3) that, for $r > 0$ and $\delta \in [0, 1)$, $\mathcal{X} \in \Gamma(r, 1-\delta)$ means that at least a fraction $1-\delta$ of the N particles of the vector $\mathcal{X}$ are in $\mathcal{B}(r)$, and, in particular, for $t \geq 0$,

$$X^{(N)}(t) \in \Gamma(r, 1-\delta) \quad \Leftrightarrow \quad F^{(N)}(r, t) \geq 1 - \delta. \tag{43}$$

LEMMA 4.11. *There exists a constant $c_4 > 0$ such that*, *for any $N \in \mathbb{N}$*, *$r > 0$*, *and $\epsilon \in (0, 1)$*, *if $\mathcal{X} \in \Gamma(r, 1 - \frac{1}{4}\epsilon)$*, *then*

$$\mathbb{P}_\mathcal{X}\left(Z_\epsilon(r) \leq N\right) \leq e^{-c_4 \epsilon N}.$$

PROOF. Note that since each particle in the BBM branches independently at rate 1 and since we assumed that the number $|\mathcal{X} \cap \mathcal{B}(r)|$ of initial particles in $\mathcal{B}(r)$ is at least $N(1 - \frac{1}{4}\epsilon)$, we have

$$Z_\epsilon(r) \geq N\left(1 - \frac{1}{4}\epsilon\right) + \xi,$$

where ξ is a Poisson random variable with mean $N(1-\frac{1}{4}\epsilon)\epsilon$. Hence, for any $c>0$, by Markov's inequality,

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}\big(Z_\epsilon(r)\le N\big) &\le \mathbb{P}\Big(\xi\le\frac{1}{4}\epsilon N\Big)\\ &= \mathbb{P}\big(e^{-c\xi}\ge e^{-\frac{1}{4}c\epsilon N}\big)\\ &\le e^{\frac{1}{4}c\epsilon N}\,\mathbb{E}\big[e^{-c\xi}\big]\\ &= e^{\frac{1}{4}c\epsilon N+N(1-\frac{1}{4}\epsilon)\epsilon(e^{-c}-1)}\\ &\le e^{N(1-\frac{1}{4}\epsilon)\epsilon(\frac{1}{2}c+e^{-c}-1)},\end{aligned}$$

where we used $\frac{1}{4}\epsilon\le\frac{1}{2}\epsilon(1-\frac{1}{4}\epsilon)$ in the last line. Fixing $c>0$ sufficiently small that $\frac{1}{2}c+e^{-c}-1=-c'<0$, it follows that

$$\mathbb{P}_{\mathcal{X}}\big(Z_\epsilon(r)\le N\big)\le e^{-c'\epsilon N(1-\frac{1}{4}\epsilon)}\le e^{-\frac{1}{2}c'\epsilon N}. \qquad \square$$

LEMMA 4.12. *Fix $b\in(0,1/2)$. Then, there exists $N_0=N_0(d,b)$ such that, for all $N\ge N_0$ and $r>0$, if $\mathcal{X}\in\Gamma(r,1-\frac{1}{4}\epsilon)$ where $\epsilon=N^{-b}$, then*

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{s\in[\epsilon,2\epsilon]}M_s^{(N)}>r+\epsilon^{1/3}\Big)\le e^{-\epsilon^{-1/4}}.$$

PROOF. Take $\mathcal{X}\in\Gamma(r,1-\frac{1}{4}\epsilon)$. By Lemma 4.10, observe that

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}\Big(\sup_{s\in[\epsilon,2\epsilon]}M_s^{(N)}>r+\epsilon^{1/3}\Big) &\le \mathbb{P}_{\mathcal{X}}\big(Z_\epsilon(r)\le N\big)\\ &\quad+\mathbb{P}_{\mathcal{X}}\Big(\exists u\in\mathcal{N}_{2\epsilon}^{+}:\sup_{s\in[0,2\epsilon]}\|B_u(s)\|>\frac{1}{3}\epsilon^{1/3}\Big).\end{aligned}\tag{44}$$

By Lemma 4.11 the first term on the right-hand side is bounded by $e^{-c_4\epsilon N}$. We focus on the second term. By the many-to-one lemma, recalling that we let $(B_s)_{s\ge0}$ denote a d-dimensional Brownian motion with diffusivity $\sqrt{2}$,

$$\mathbb{P}_{\mathcal{X}}\Big(\exists u\in\mathcal{N}_{2\epsilon}^{+}:\sup_{s\in[0,2\epsilon]}\|B_u(s)\|>\frac{1}{3}\epsilon^{1/3}\Big)\le Ne^{2\epsilon}\,\mathbb{P}_0\Big(\sup_{s\in[0,2\epsilon]}\|B_s\|>\frac{1}{3}\epsilon^{1/3}\Big).$$

Letting $\xi_{1,s},\dots,\xi_{d,s}$ denote the d coordinates of B_s, which are themselves independent one-dimensional Brownian motions,

$$\begin{aligned}\mathbb{P}_0\Big(\sup_{s\in[0,2\epsilon]}\|B_s\|>\frac{1}{3}\epsilon^{1/3}\Big) &= \mathbb{P}_0\Big(\sup_{s\in[0,2\epsilon]}(\xi_{1,s}^2+\dots+\xi_{d,s}^2)>\frac{1}{9}\epsilon^{2/3}\Big)\\ &\le \mathbb{P}_0\Big(\sup_{s\in[0,2\epsilon]}\xi_{1,s}^2>\frac{1}{9d}\epsilon^{2/3}\text{ or }\dots\text{ or }\sup_{s\in[0,2\epsilon]}\xi_{d,s}^2>\frac{1}{9d}\epsilon^{2/3}\Big)\\ &\le d\,\mathbb{P}_0\Big(\sup_{s\in[0,2\epsilon]}|\xi_{1,s}|>\frac{1}{3\sqrt{d}}\epsilon^{1/3}\Big)\\ &\le 4d\,\mathbb{P}_0\Big(\xi_{1,2\epsilon}>\frac{1}{3\sqrt{d}}\epsilon^{1/3}\Big)\\ &\le 4d\exp\Big(-\frac{\epsilon^{-1/3}}{72d}\Big),\end{aligned}\tag{45}$$

where the fourth line follows by the reflection principle and the last line by a Gaussian tail bound.

By (44) we now have that

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{s\in[\epsilon,2\epsilon]} M_s^{(N)} > r+\epsilon^{1/3}\Big) \le e^{-c_4\epsilon N} + Ne^{2\epsilon}\cdot 4d\exp\Big(-\frac{\epsilon^{-1/3}}{72d}\Big) \le e^{-\epsilon^{-1/4}},$$

where the second inequality holds for N sufficiently large (depending only on d, b, and the constant c_4), since $\epsilon = N^{-b}$ and $b\in(0,1/2)$. □

We can now complete the proof of Proposition 1.6. Recall that we assume that the N-BBM is started from N i.i.d. particles with distribution given by some μ_0 and that (u,R) denotes the solution to the free boundary problem (1) with initial condition μ_0. As in Lemma 4.9, let v denote the solution of (7) with initial condition $v_0(r)=\mu_0(\mathcal{B}(r))$, and recall from Section 1.2 that $v(r,t)=\int_{\mathcal{B}(r)} u(x,t)\,\mathrm{d}x$ and so, in particular, $v(R_t,t)=1$ for $t>0$.

Take $c_3\in(0,1)$ and $N_0\in\mathbb{N}$ as in Lemma 4.9; take $N\ge N_0$, and let $\epsilon = N^{-c_3/2}$. Then, for $T>0$, by a union bound,

$$\begin{aligned}
&\mathbb{P}\big(\exists t\in[2\epsilon,T]: M_t^{(N)} > R_{\epsilon(\lfloor t/\epsilon\rfloor -1)}+\epsilon^{1/3}\big)\\
&\quad\le \sum_{k=1}^{\lfloor T/\epsilon\rfloor -1}\mathbb{P}\Big(\sup_{s\in[\epsilon,2\epsilon]} M_{\epsilon k+s}^{(N)} > R_{\epsilon k}+\epsilon^{1/3}\Big)\\
&\quad\le \sum_{k=1}^{\lfloor T/\epsilon\rfloor -1}\Big(\mathbb{P}\big(E_{\epsilon k}^c\big)+\mathbb{P}\Big(E_{\epsilon k};\ \sup_{s\in[\epsilon,2\epsilon]} M_{\epsilon k+s}^{(N)} > R_{\epsilon k}+\epsilon^{1/3}\Big)\Big).
\end{aligned}$$

The above is, of course, valid for any choice of events E_t for each $t>0$, but in order to use Lemma 4.12, we let $E_t=\{X^{(N)}(t)\in\Gamma(R_t,1-\frac{1}{4}\epsilon)\}$. Then, for N sufficiently large that $\frac14\epsilon > e^{2T}N^{-c_3}$, recalling the meaning of Γ in (43), and since $v(R_t,t)=1$, for $t\in(0,T]$,

$$\begin{aligned}
\mathbb{P}\big(E_t^c\big) &= \mathbb{P}\Big(X^{(N)}(t)\notin\Gamma\Big(R_t,1-\frac14\epsilon\Big)\Big) = \mathbb{P}\Big(F^{(N)}(R_t,t)<1-\frac14\epsilon\Big)\\
&\le \mathbb{P}\Big(\sup_{r\ge 0}\big|F^{(N)}(r,t)-v(r,t)\big|\ge e^{2T}N^{-c_3}\Big)\\
&\le e^T N^{-1-c_3}
\end{aligned}$$

by Lemma 4.9. For N sufficiently large (depending on d), for $k\in\mathbb{N}$, by Lemma 4.12 (with $b=c_3/2$) and the Markov property at time ϵk, we have

$$\mathbb{P}\Big(E_{\epsilon k};\ \sup_{s\in[\epsilon,2\epsilon]} M_{\epsilon k+s}^{(N)} > R_{\epsilon k}+\epsilon^{1/3}\Big)\le e^{-\epsilon^{-1/4}}.$$

Therefore,

$$\mathbb{P}\big(\exists t\in[2\epsilon,T]: M_t^{(N)} > R_{\epsilon(\lfloor t/\epsilon\rfloor -1)}+\epsilon^{1/3}\big)\le \lfloor T/\epsilon\rfloor\big(e^T N^{-1-c_3}+e^{-\epsilon^{-1/4}}\big)\le N^{-1-\frac13 c_3}$$

for N sufficiently large (depending on d and T). For $\eta\in(0,T)$, since $(R_t)_{t\in[\eta,T]}$ is continuous (by Theorem 2.1), for N sufficiently large,

$$R_{\epsilon(\lfloor t/\epsilon\rfloor -1)}+\epsilon^{1/3}\le R_t+\eta\quad \forall t\in[\eta,T]. \tag{46}$$

Therefore, for N sufficiently large (depending on d, μ_0, η and T),

$$\mathbb{P}\big(\exists t\in[\eta,T]: M_t^{(N)} > R_t+\eta\big)\le N^{-1-\frac13 c_3}$$

which completes the proof of Proposition 1.6.

5. Proof of Theorem 1.1: Hydrodynamic limit result for u. First, we notice that it is sufficient to prove that there exists $c_5 > 0$ such that, for any $t > 0$, $A \subseteq \mathbb{R}^d$ measurable and $\delta > 0$, for N sufficiently large (depending on d, μ_0, A, δ and t),

$$\mathbb{P}\Big(\mu^{(N)}(A,t) - \int_A u(x,t)\,\mathrm{d}x \geq \delta\Big) \leq N^{-1-c_5}. \tag{47}$$

Indeed, since $\mu^{(N)}(A,t) + \mu^{(N)}(\mathbb{R}^d \setminus A, t) = 1$ and $\int_A u(x,t)\,\mathrm{d}x + \int_{\mathbb{R}^d\setminus A} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} u(x,t)\,\mathrm{d}x = 1$,

$$\mathbb{P}\Big(\mu^{(N)}(A,t) - \int_A u(x,t)\,\mathrm{d}x \leq -\delta\Big) = \mathbb{P}\Big(\mu^{(N)}(\mathbb{R}^d \setminus A, t) - \int_{\mathbb{R}^d\setminus A} u(x,t)\,\mathrm{d}x \geq \delta\Big). \tag{48}$$

Hence, it follows from (47) that, for $t > 0$, $A \subseteq \mathbb{R}^d$ measurable, and $\delta > 0$, for N sufficiently large,

$$\mathbb{P}\Big(\Big|\mu^{(N)}(A,t) - \int_A u(x,t)\,\mathrm{d}x\Big| \geq \delta\Big) \leq 2N^{-1-c_5},$$

and so by Borel-Cantelli, $\mu^{(N)}(A,t) \to \int_A u(x,t)\,\mathrm{d}x$ almost surely as $N \to \infty$. Moreover, for $t > 0$ and $\delta > 0$, let $\delta' = 1 - \int_{\mathcal{B}(R_t-\delta)} u(x,t)\,\mathrm{d}x > 0$ by Theorem 2.1. Then,

$$\begin{aligned}\mathbb{P}\big(M_t^{(N)} < R_t - \delta\big) &= \mathbb{P}\big(\mu^{(N)}(\mathcal{B}(R_t-\delta),t) = 1\big)\\ &= \mathbb{P}\Big(\mu^{(N)}(\mathcal{B}(R_t-\delta),t) - \int_{\mathcal{B}(R_t-\delta)} u(x,t)\,\mathrm{d}x \geq \delta'\Big) \leq N^{-1-c_5}\end{aligned}$$

for N sufficiently large by (47). Also, by Proposition 1.6 with $\eta = \min(\delta, t)$, for N sufficiently large,

$$\mathbb{P}\big(M_t^{(N)} > R_t + \delta\big) \leq N^{-1-c_2}.$$

Therefore, by Borel–Cantelli, $M_t^{(N)} \to R_t$ almost surely as $N \to \infty$.

It now remains to prove (47). Let $(X^+(t), t \geq 0)$ be a BBM with the same initial particle distribution as the N-BBM, that is, such that $X^+(0) = (X_i^+(0))_{i=1}^N$, where $(X_i^+(0))_{i=1}^N$ are i.i.d. with distribution given by μ_0. Recall the coupling described in Section 3 between the N-BBM $X^{(N)}$ and the BBM X^+ such that, under the coupling, for all $t \geq 0$,

$$X^{(N)}(t) \subseteq X^+(t).$$

Take $t > 0$ and $\eta \in (0,t)$. We let $\mathcal{C}_{\eta,t}$ denote the set of locations of particles in the BBM (without killing) at time t whose ancestors at times $s \in [\eta, t]$ were always within distance $R_s + \eta$ of the origin,

$$\mathcal{C}_{\eta,t} = \big\{X_u^+(t) : u \in \mathcal{N}_t^+,\ \|X_u^+(s)\| \leq R_s + \eta\ \forall s \in [\eta,t]\big\}.$$

Notice that if $M_s^{(N)} \leq R_s + \eta\ \forall s \in [\eta, t]$, then, by (24), almost surely

$$X^{(N)}(t) \subseteq \mathcal{C}_{\eta,t}.$$

Therefore, for $A \subseteq \mathbb{R}^d$ measurable and $\delta > 0$,

$$\begin{aligned}\mathbb{P}\Big(\mu^{(N)}(A,t) - \int_A u(x,t)\,\mathrm{d}x \geq \delta\Big) &\leq \mathbb{P}\big(\exists s \in [\eta,t] : M_s^{(N)} > R_s + \eta\big)\\ &\quad + \mathbb{P}\Big(\frac{1}{N}|\mathcal{C}_{\eta,t} \cap A| - \int_A u(x,t)\,\mathrm{d}x \geq \delta\Big) \qquad (49)\\ &\leq N^{-1-c_2} + \mathbb{P}\Big(\frac{1}{N}|\mathcal{C}_{\eta,t} \cap A| - \int_A u(x,t)\,\mathrm{d}x \geq \delta\Big)\end{aligned}$$

for N sufficiently large (depending on d, μ_0, η and t) by Proposition 1.6. We now focus on the second term on the right-hand side.

LEMMA 5.1. *For any $t > 0$ and $A \subseteq \mathbb{R}^d$ measurable,*

$$\lim_{\eta \searrow 0} \mathbb{E}\left[\frac{1}{N}|\mathcal{C}_{\eta,t} \cap A|\right] = \int_A u(x,t)\,\mathrm{d}x \quad \textit{uniformly in } N.$$

PROOF. We claim that, for $y \in \mathbb{R}^d$,

$$\mathbb{P}_y(B_t \in A, \|B_s\| \le R_s \ \forall s \in (0,t]) = \mathbb{P}_y\big(B_t \in A, \|B_s\| < R_s \ \forall s \in (0,t)\big). \tag{50}$$

(In words: the probability that the Brownian motion touches the moving boundary R at a positive time without crossing it is zero.) We shall begin by showing that the lemma follows from (50) and then prove the claim (50).

For $\eta \in (0,t)$, by the many-to-one lemma and since at time 0 the BBM consists of N particles with locations which are random variables with distribution μ_0,

$$\mathbb{E}\big[|\mathcal{C}_{\eta,t} \cap A|\big] = Ne^t \int_{\mathbb{R}^d} \mu_0(\mathrm{d}y)\,\mathbb{P}_y\big(B_t \in A,\ \|B_s\| \le R_s + \eta \ \forall s \in [\eta,t]\big).$$

By dominated convergence and uniformly in N,

$$\begin{aligned}\lim_{\eta \searrow 0} \mathbb{E}\left[\frac{1}{N}|\mathcal{C}_{\eta,t} \cap A|\right] &= e^t \int_{\mathbb{R}^d} \mu_0(\mathrm{d}y)\,\mathbb{P}_y(B_t \in A, \|B_s\| \le R_s \ \forall s \in (0,t]) \\ &= e^t \int_{\mathbb{R}^d} \mu_0(\mathrm{d}y)\,\mathbb{P}_y\big(B_t \in A, \|B_s\| < R_s \ \forall s \in (0,t)\big) \\ &= \int_A u(x,t)\,\mathrm{d}x,\end{aligned}$$

where the second equality holds by (50) and the last equality follows from (13).

It remains to prove (50); we shall use the following claim. Suppose $(\xi_s)_{s\in[0,t]}$ is a continuous path in $\mathbb{R}^d$ with $\xi_0 = y$ and $\xi_t = 0$, and suppose $\mathbf{e} \in \mathbb{R}^d$ is a unit vector. We claim that there are at most two values of $r \in \mathbb{R}$ such that

$$\min_{s\in(0,t)}\left(R_s - \left\|\xi_s + \frac{s}{t}r\mathbf{e}\right\|\right) = 0, \tag{51}$$

which we write to mean that the min exists in $(0,t)$ and is equal to 0; in other words, $R_s \ge \|\xi_s + \frac{s}{t}r\mathbf{e}\|$ $\forall s \in (0,t)$ and $\exists s \in (0,t)$ such that $R_s = \|\xi_s + \frac{s}{t}r\mathbf{e}\|$. Indeed, we have that

$$\left\|\xi_s + \frac{s}{t}r\mathbf{e}\right\|^2 = \|\xi_s - (\xi_s\cdot\mathbf{e})\mathbf{e}\|^2 + \left(\xi_s\cdot\mathbf{e} + \frac{s}{t}r\right)^2,$$

and so if (51) holds, then the inequality $R_s \ge \|\xi_s + \frac{s}{t}r\mathbf{e}\|$ implies that, for each $s \in (0,t)$,

$$-\frac{t}{s}\big((R_s^2 - \|\xi_s - (\xi_s\cdot\mathbf{e})\mathbf{e}\|^2)^{1/2} + \xi_s\cdot\mathbf{e}\big) \le r \le \frac{t}{s}\big((R_s^2 - \|\xi_s - (\xi_s\cdot\mathbf{e})\mathbf{e}\|^2)^{1/2} - \xi_s\cdot\mathbf{e}\big). \tag{52}$$

Moreover, for any value of $s \in (0,t)$ such that $R_s = \|\xi_s + \frac{s}{t}r\mathbf{e}\|$, one of the two inequalities in (52) must be an equality. Therefore,

$$\begin{aligned} r \in \Bigg\{ &\inf_{s\in(0,t)}\left(\frac{t}{s}\big((R_s^2 - \|\xi_s - (\xi_s\cdot\mathbf{e})\mathbf{e}\|^2)^{1/2} - \xi_s\cdot\mathbf{e}\big)\right), \\ &\sup_{s\in(0,t)}\left(-\frac{t}{s}\big((R_s^2 - \|\xi_s - (\xi_s\cdot\mathbf{e})\mathbf{e}\|^2)^{1/2} + \xi_s\cdot\mathbf{e}\big)\right)\Bigg\}, \end{aligned}$$

which establishes the claim that (51) holds for at most two values of r.

Now, for $y \in \mathbb{R}^d$ under the probability measure $\mathbb{P}_y$, let $(\xi_s)_{s\in[0,t]}$ denote a d-dimensional Brownian bridge with diffusivity $\sqrt{2}$ from y to 0 in time t. Then,

$$\begin{aligned}
&\mathbb{P}_y(\{\|B_s\| \le R_s \quad \forall s \in (0,t]\} \cap \{\exists s \in (0,t) : \|B_s\| = R_s\}) \\
&\quad = \mathbb{E}_y[\mathbb{P}_y(\{\|B_s\| \le R_s \ \forall s \in (0,t]\} \cap \{\exists s \in (0,t) : \|B_s\| = R_s\} | B_t)] \\
&\quad = \int_{\mathbb{R}^d} \mathrm{d}z \Phi_t(y-z)\, \mathbb{P}_y\Big(\min_{s\in(0,t)}\Big(R_s - \Big\|\xi_s + \frac{s}{t}z\Big\|\Big) = 0\Big) \\
&\quad = \mathbb{E}_y\Big[\int_{\mathbb{R}^d} \mathrm{d}z \Phi_t(y-z) \mathbb{1}_{\{\min_{s\in(0,t)}(R_s - \|\xi_s + \frac{s}{t}z\|)=0\}}\Big]
\end{aligned}$$

by Fubini's theorem and where $\Phi_t(x) = (4\pi t)^{-d/2} e^{-\|x\|^2/(4t)}$ is the heat kernel. By (51) we have that $\min_{s\in(0,t)}(R_s - \|\xi_s + \frac{s}{t}z\|) \neq 0$ for almost every z, and (50) follows. □

LEMMA 5.2. *For N large enough ($N \ge 48$ is sufficient), for any $A \subseteq \mathbb{R}^d$ measurable, and any $0 < \eta < t$,*

$$\mathbb{E}\Big[\Big(\frac{1}{N}|\mathcal{C}_{\eta,t} \cap A| - \mathbb{E}\Big[\frac{1}{N}|\mathcal{C}_{\eta,t} \cap A|\Big]\Big)^4\Big] \le 13 e^{4t} N^{-2}.$$

PROOF. Recall that we let $X^+(0) = (X_i^+(0))_{i=1}^N$, where $(X_i^+(0))_{i=1}^N$ are i.i.d. with distribution given by μ_0. As in the proof of Lemma 4.2, denote by $X^{+,i}$ the family of particles descended from the ith particle in the initial configuration $X^+(0)$. The $X^{+,i}$ form a family of independent BBMs, and for each i the process $X^{+,i}$ is started from a single particle at location $X_i^+(0)$. Fix $0 < \eta < t$, write $n_i = |X^{+,i}(t)|$ for the number of particles descended from $X_i^+(0)$ at time t, and introduce $n_{i,A}$ as the number of particles in $\mathcal{C}_{\eta,t} \cap A$, which are descendants of particle $X_i^+(0)$,

$$n_{i,A} = |\mathcal{C}_{\eta,t} \cap A \cap X^{+,i}(t)|.$$

Then, $|\mathcal{C}_{\eta,t} \cap A| = \sum_{i=1}^N n_{i,A}$ and $(n_{i,A})_{i=1}^N$ are i.i.d., so

$$\begin{aligned}
&\mathbb{E}[(|\mathcal{C}_{\eta,t} \cap A| - \mathbb{E}[|\mathcal{C}_{\eta,t} \cap A|])^4] \\
&\quad = \mathbb{E}\Big[\Big(\sum_{i=1}^N (n_{i,A} - \mathbb{E}[n_{i,A}])\Big)^4\Big] \\
&\quad = \sum_{i=1}^N \mathbb{E}[(n_{i,A} - \mathbb{E}[n_{i,A}])^4] + 6 \sum_{\substack{i,j=1\\ i<j}}^N \operatorname{Var}(n_{i,A}) \operatorname{Var}(n_{j,A}).
\end{aligned}$$

By the same argument as in (28), (29), and (30) in the proof of Lemma 4.2,

$$\mathbb{E}[(n_{i,A} - \mathbb{E}[n_{i,A}])^4] \le 2\,\mathbb{E}[n_i^4] \le 48 e^{4t} \quad \text{and} \quad \operatorname{Var}(n_{i,A}) \le \mathbb{E}[n_i^2] \le 2e^{2t}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[(|\mathcal{C}_{\eta,t} \cap A| - \mathbb{E}[|\mathcal{C}_{\eta,t} \cap A|])^4] &\le N \cdot 48 e^{4t} + 3N(N-1)(2e^{2t})^2 \\
&\le 13 e^{4t} N^2,
\end{aligned}$$

where the second inequality holds for any $N \ge 48$. □

We can now conclude; for fixed $t > 0$, $A \subseteq \mathbb{R}^d$ measurable, and $\delta > 0$, let $\eta > 0$ be sufficiently small that, by Lemma 5.1,

$$\left| \frac{1}{N} \mathbb{E}\big[|\mathcal{C}_{\eta,t} \cap A|\big] - \int_A u(x,t)\,\mathrm{d}x \right| < \frac{\delta}{2} \quad \forall N \in \mathbb{N}.$$

Then, for $N \geq 48$,

$$\begin{aligned} \mathbb{P}\bigg(\frac{1}{N}|\mathcal{C}_{\eta,t} \cap A| - \int_A u(x,t)\,\mathrm{d}x \geq \delta \bigg) &\leq \mathbb{P}\bigg(\frac{1}{N}|\mathcal{C}_{\eta,t} \cap A| - \frac{1}{N}\mathbb{E}\big[|\mathcal{C}_{\eta,t} \cap A|\big] > \frac{\delta}{2} \bigg) \\ &\leq \frac{16}{\delta^4} \cdot 13 e^{4t} N^{-2} \end{aligned}$$

by Lemma 5.2 and Markov's inequality. By (49) it follows that, for N sufficiently large,

$$\begin{aligned} \mathbb{P}\bigg(\mu^{(N)}(A,t) - \int_A u(x,t)\,\mathrm{d}x \geq \delta \bigg) &\leq N^{-1-c_2} + 16\delta^{-4} \cdot 13 e^{4t} N^{-2} \\ &\leq N^{-1-\frac{1}{2}c_2} \end{aligned}$$

for N large enough, which establishes (47) with $c_5 = \frac{1}{2}c_2$, and completes the proof of Theorem 1.1.

6. Proof of Proposition 1.7 and of Theorems 1.2, 1.3, and 1.4. We begin by proving Proposition 1.7. We shall first describe heuristically how the proof works. Recall from (3) that, for $K > 0$ and $c \in (0, 1]$,

$$\Gamma(K,c) = \bigg\{ \mathcal{X} \in (\mathbb{R}^d)^N : \frac{1}{N}\big|\{i : \|\mathcal{X}_i\| < K\}\big| \geq c \bigg\}$$

is the set of "good" initial particle configurations that have at least a fraction c of particles within distance K of the origin (the dependence on N is implicit). Recall from (25) that $F^{(N)}(\cdot, t)$ is the empirical cumulative distribution of the particles at time t. Let us explain how we go about proving that, for $\epsilon > 0$, there exist $N_\epsilon = N_\epsilon(K,c)$, $T_\epsilon = T_\epsilon(K,c)$ such that

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{r \geq 0} \big| F^{(N)}(r,t) - V(r) \big| \geq \epsilon \Big) < \epsilon$$

holds for any $N \geq N_\epsilon$, $t \geq T_\epsilon$ and $\mathcal{X} \in \Gamma(K,c)$.

Let $K_0 = 3$, and take c_0, as in Proposition 2.5, and $t_\epsilon = t_\epsilon(c_0, K_0)$, as in Proposition 2.4. We are going to show that there exists a time $t_a = t_a(c,K)$ independent of N such that, for t large enough,

$$\begin{aligned} \mathcal{X} \in \Gamma(K,c) \quad &\Longrightarrow \quad X^{(N)}(t_a) \in \Gamma(K_0, c_0) \quad \text{with high probability} \\ &\Longrightarrow \quad X^{(N)}(t - t_\epsilon) \in \Gamma(K_0, c_0) \quad \text{with high probability} \\ &\Longrightarrow \quad F^{(N)}(\cdot, t) \quad \text{is close to } V \text{ with high probability.} \end{aligned}$$

For the first step we use Proposition 2.5 to choose t_a such that $v^{(N)}(r, t_a) \geq 2c_0 \mathbb{1}_{\{r \geq K_0\}}$ $\forall r \geq 0$, where $v^{(N)}$ denotes the solution of the obstacle problem (7) with initial condition $v_0(r) = F^{(N)}(r, 0) = N^{-1}|\mathcal{X} \cap \mathcal{B}(r)|$. Then, Proposition 1.5 tells us that $X^{(N)}(t_a) \in \Gamma(K_0, c_0)$ with high probability for N large enough.

The second step will be provided by Proposition 6.2 below, with Proposition 6.1 as an intermediate result.

For the third step, let $\tilde{v}$ denote the solution of the obstacle problem (7) with initial condition $\tilde{v}_0(r) = F^{(N)}(r, t - t_\epsilon)$. From the second step, $X^{(N)}(t - t_\epsilon) \in \Gamma(K_0, c_0)$ with high probability which is equivalent to $\tilde{v}_0(K_0) \geq c_0$. Proposition 2.4 then gives us that $\tilde{v}(\cdot, t_\epsilon)$ is close to $V(\cdot)$. Furthermore, Proposition 1.5, together with the Markov property at time $t - t_\epsilon$, implies that $F^{(N)}(\cdot, t)$ is close to $\tilde{v}(\cdot, t_\epsilon)$, which, in turn, is close to $V(\cdot)$.

We start by proving the two propositions needed for the second step, and then we prove Proposition 1.7. The first proposition implies that if $X^{(N)}(0) \in \Gamma(m, c)$ for some m and c, then, for j large and t not too large, with high probability $X^{(N)}(t) \in \Gamma(m + j, c)$.

PROPOSITION 6.1. *For any $c \in (0, 1)$, any $N \in \mathbb{N}$, any $t \geq 0$, and any $m > 0$, if $\mathcal{X} \in \Gamma(m, c)$, then*

$$\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t) \notin \Gamma(m + j, c)\big) \leq 4dNe^t e^{-j^2/(36dt)} \quad \textit{for all } j > 0.$$

PROOF. The argument, in particular the final Gaussian tail estimate, is very similar to the one used in the proof of Lemma 4.12. Recall the coupling in Section 3 between the N-BBM $X^{(N)}$ and the BBM X^+ such that under the coupling, $X^{(N)}(0) = X^+(0)$ and for $t \geq 0$, almost surely

$$\begin{aligned} &\{X_k^{(N)}(t)\}_{k=1}^N = \{X_u^+(t)\}_{u \in \mathcal{N}_t^{(N)}}, \\ &\text{where } \mathcal{N}_t^{(N)} = \{u \in \mathcal{N}_t^+ : \|X_u^+(s)\| \leq M_s^{(N)} \ \forall s \in [0, t]\}. \end{aligned} \tag{53}$$

Fix $t \geq 0$ and $j > 0$, and define the event

$$E = \left\{ \max_{u \in \mathcal{N}_t^+} \sup_{s \in [0,t]} \|X_u^+(s) - X_u^+(0)\| < \frac{1}{3} j \right\}.$$

We claim that, on the event E, if $X^{(N)}(0) \in \Gamma(m, c)$ for some $c \in (0, 1)$ and $m > 0$, then $X^{(N)}(t) \in \Gamma(m + j, c)$. The proof of the claim is split into two cases, depending on the stopping time

$$\tau = \inf\left\{ s \geq 0 : M_s^{(N)} \leq m + \frac{1}{3} j \right\}.$$

We first consider the case $\tau > t$, meaning that $M_s^{(N)} > m + \frac{1}{3} j \ \forall s \leq t$. By (53) and on the event $E \cap \{\tau > t\}$, we have

$$\begin{aligned} X^{(N)}(t) &\supseteq \left\{ X_u^+(t) : u \in \mathcal{N}_t^+, \ \|X_u^+(s)\| < m + \frac{1}{3} j \ \forall s \in [0, t] \right\} \\ &\supseteq \{X_u^+(t) : u \in \mathcal{N}_t^+, \ \|X_u^+(0)\| < m\}, \end{aligned}$$

where we used $\|X_u^+(s) - X_u^+(0)\| < \frac{1}{3} j$ in the last line. Now, using that $\|X_u^+(t) - X_u^+(0)\| < j$ on the event E, it follows that if $X^{(N)}(0) \in \Gamma(m, c)$, then $X^{(N)}(t) \in \Gamma(m + j, c)$.

We now consider the second case, $\tau \leq t$. Take $u \in \mathcal{N}_t^{(N)}$. On the event $E \cap \{\tau \leq t\}$, $\|X_u^+(\tau)\| \leq M_\tau^{(N)} \leq m + \frac{1}{3} j$, and by the triangle inequality,

$$\|X_u^+(t)\| \leq \|X_u^+(t) - X_u^+(0)\| + \|X_u^+(\tau) - X_u^+(0)\| + \|X_u^+(\tau)\| < m + j$$

by the definition of the event E. By (53) this shows that $\|X_k^{(N)}(t)\| < m + j \ \forall k \in \{1, \ldots, N\}$, and, in particular, $X^{(N)}(t) \in \Gamma(m + j, c)$. This completes the proof of the claim.

Hence, for $\mathcal{X} \in \Gamma(m, c)$,

$$\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t) \notin \Gamma(m + j, c)\big) \leq \mathbb{P}_{\mathcal{X}}\big(E^c\big) \leq Ne^t \, \mathbb{P}_0\Big(\sup_{s \in [0,t]} \|B_s\| \geq \frac{1}{3} j \Big)$$

by the many-to-one lemma. Letting $\xi_{1,s}, \dots, \xi_{d,s}$ denote the coordinates of B_s, we have

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t) \notin \Gamma(m+j,c)\big) &\le Ne^t\, \mathbb{P}_0\Big(\sup_{s\in[0,t]} \xi_{1,s}^2 \ge \frac{1}{9d}j^2 \text{ or } \dots \text{ or } \sup_{s\in[0,t]} \xi_{d,s}^2 \ge \frac{1}{9d}j^2\Big) \\ &\le Ne^t d\, \mathbb{P}_0\Big(\sup_{s\in[0,t]} |\xi_{1,s}| \ge \frac{1}{3\sqrt{d}}j\Big) \\ &\le Ne^t \cdot 4d\, \mathbb{P}_0\Big(\xi_{1,t} \ge \frac{1}{3\sqrt{d}}j\Big) \\ &\le 4dNe^t e^{-j^2/(36dt)},\end{aligned}$$

where the third inequality follows by the reflection principle and the fourth by a Gaussian tail estimate. □

We now show that, for $c_0 \in (0,1)$ and $K_0 > 0$ appropriately chosen, if N and t are sufficiently large, if $X^{(N)}(0) \in \Gamma(K_0,c_0)$, then $X^{(N)}(t) \in \Gamma(K_0,c_0)$ with high probability.

PROPOSITION 6.2. *Take $t_0 > 1$ and $c_0 > 0$, as in Proposition 2.5, and fix $K_0 = 3$. For $\epsilon > 0$, there exist $N'_\epsilon < \infty$ and $t'_\epsilon < \infty$ such that, for $N \ge N'_\epsilon$, the following holds. For $t \ge t'_\epsilon$,*

$$\inf_{\mathcal{X}\in\Gamma(K_0,c_0)} \mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t) \in \Gamma(K_0,c_0)\big) \ge 1-\epsilon. \tag{54}$$

Furthermore, for $t_1 \in [t_0, 2t_0]$ and any $K > 0$,

$$\sup_{\mathcal{X}\in\Gamma(K,c_0)} \mathbb{E}_{\mathcal{X}}\big[\inf\{n \ge 1 : X^{(N)}(nt_1) \in \Gamma(K_0,c_0)\}\big] < \infty. \tag{55}$$

Note that for any initial condition $\mathcal{X} \in (\mathbb{R}^d)^N$, we have that $\mathcal{X} \in \Gamma(K,c_0)$ where $K = \max_{i\le N} \|\mathcal{X}_i\| + 1$, and so it follows immediately from (55) that for N sufficiently large, for $\mathcal{X} \in (\mathbb{R}^d)^N$, and $t_1 \in [t_0, 2t_0]$,

$$\mathbb{P}_{\mathcal{X}}\big(\inf\{n \ge 1 : X^{(N)}(nt_1) \in \Gamma(K_0,c_0)\} < \infty\big) = 1. \tag{56}$$

PROOF. The proof uses Propositions 1.5, 6.1, and 2.5 to establish a coupling with a Markov chain. Take $\delta \in (0, 1/16)$ sufficiently small that $\frac{1}{1+15\delta} \ge 1 - \frac{1}{2}\epsilon$. Suppose N'_ϵ is sufficiently large that, for $N \ge N'_\epsilon$ we have

$$4dNe^{2t_0}e^{-(\log N)^{4/3}/(144dt_0)} < \frac{1}{2}\delta \quad \text{and} \quad \frac{(\log N)^{2/3}}{144dt_0} > \log 2. \tag{57}$$

Also, recalling the definition of c_1 in Proposition 1.5, suppose that N'_ϵ is sufficiently large that, for $N \ge N'_\epsilon$, Proposition 1.5 holds,

$$e^{4t_0}N^{-c_1} \le c_0 \quad \text{and} \quad e^{2t_0}N^{-1-c_1} \le \delta 2^{-(\log N)^{2/3}-1}. \tag{58}$$

Take $N \ge N'_\epsilon$ and $t_1 \in [t_0, 2t_0]$. We first show that

$$\begin{aligned}&\mathcal{X} \in \Gamma(m,c_0) \quad \Longrightarrow \quad \mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t_1) \notin \Gamma(m+j-1,c_0)\big) \le \delta 2^{-j} \\ &\quad \forall j, m \in \mathbb{N}_0 \text{ with } m \ge K_0.\end{aligned} \tag{59}$$

Take $m \in \mathbb{N}$ with $m \ge K_0$, and suppose $\mathcal{X} \in \Gamma(m,c_0)$. We first assume that $j \in \mathbb{N}$ with $j \ge (\log N)^{2/3} + 1$. Then, by Proposition 6.1 the fact that $t_1 \in [t_0, 2t_0]$ and then by (57) we have

$$\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t_1) \notin \Gamma(m+j-1,c_0)\big) \le 4dNe^{2t_0}e^{-(j-1)^2/(72dt_0)} \le \frac{1}{2}\delta e^{-(j-1)^2/(144dt_0)} \le \delta 2^{-j}.$$

We now consider the case $j \le (\log N)^{2/3} + 1$. By Proposition 2.5 for $v^{(N)}$ the solution of (7) with $v_0(r) = N^{-1}|\mathcal{X} \cap \mathcal{B}(r)|$, we have that $v^{(N)}(m-1, t_1) \ge 2c_0$. Therefore, by Proposition 1.5, since by (58), $e^{2t_1} N^{-c_1} \le c_0$,

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t_1) \notin \Gamma(m-1, c_0)\big) &\le \mathbb{P}_{\mathcal{X}}\big(\big|F^{(N)}(m-1, t_1) - v^{(N)}(m-1, t_1)\big| \ge c_0\big) \\ &\le e^{t_1} N^{-1-c_1}.\end{aligned}$$

In particular, this and the condition (58) imply that, for $j \in \mathbb{N}_0$ with $j \le (\log N)^{2/3} + 1$,

$$\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t_1) \notin \Gamma(m+j-1, c_0)\big) \le \mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t_1) \notin \Gamma(m-1, c_0)\big) \le e^{t_1} N^{-1-c_1} \le \delta 2^{-j},$$

and (59) is proved.

Let us define the sequence of random variables $(\theta_n)_{n=0}^{\infty}$ by

$$\theta_n = \min\{i \in \mathbb{N}_0 : X^{(N)}(nt_1) \in \Gamma(K_0 + i, c_0)\} = \min\{i \in \mathbb{N}_0 : F^{(N)}(K_0 + i, nt_1) \ge c_0\}.$$

Although θ_n itself is not a Markovian process, (59) and the Markov property applied to $X^{(N)}$ implies that, for all $\mathcal{X} \in (\mathbb{R}^d)^N$ and $n, i, j \in \mathbb{N}_0$,

$$\theta_n \le i \quad \Longrightarrow \quad \mathbb{P}_{\mathcal{X}}(\theta_{n+1} \ge i + j | \mathcal{F}_{nt_1}) \le \delta 2^{-j}. \tag{60}$$

Define a Markov chain $(Y_n)_{n=0}^{\infty}$ on $\mathbb{N}_0$ as follows. For $n \in \mathbb{N}_0$ and $i, j \in \mathbb{N}_0$, let

$$\mathbb{P}(Y_{n+1} = j | Y_n = i) = p_{i,j},$$

where

$$p_{0,j} = \begin{cases} 1-\delta & \text{if } j = 0, \\ \delta 2^{-j} & \text{if } j \ge 1, \end{cases} \quad \text{and, for } i \ge 1, \quad p_{i,i+j} = \begin{cases} 1-2\delta & \text{if } j = -1, \\ \delta 2^{-j} & \text{if } j \ge 0. \end{cases}$$

Suppose for $K > 0$ that $\mathcal{X} \in \Gamma(K, c_0)$. Then, by (60), conditional on $X^{(N)}(0) = \mathcal{X}$ and $Y_0 = \max(0, \lceil K - K_0 \rceil)$, we can couple $(X^{(N)}(nt_1))_{n=0}^{\infty}$ and $(Y_n)_{n=0}^{\infty}$ in such a way that, almost surely, $\theta_n \le Y_n$ holds for each $n \in \mathbb{N}_0$ which means that

$$X^{(N)}(nt_1) \in \Gamma(K_0 + Y_n, c_0).$$

For $j \in \mathbb{N}_0$, introduce $m_j \ge 1$ as the expected number of steps needed for Y_n to reach zero starting from $Y_0 = j$,

$$m_j := \mathbb{E}\big[\inf\{n \ge 1 : Y_n = 0\} | Y_0 = j\big].$$

Then, for $n \in \mathbb{N}_0$ and $\mathcal{X} \in \Gamma(K, c_0)$, the coupling implies

$$\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(nt_1) \notin \Gamma(K_0, c_0)\big) = \mathbb{P}_{\mathcal{X}}(\theta_n > 0) \le \mathbb{P}\big(Y_n \ne 0 | Y_0 = \lceil K - K_0 \rceil \vee 0\big) \tag{61}$$

and

$$\mathbb{E}_{\mathcal{X}}\big[\inf\{n \ge 1 : X^{(N)}(nt_1) \in \Gamma(K_0, c_0)\}\big] \le m_{\lceil K - K_0 \rceil \vee 0}. \tag{62}$$

Note also that

$$m_0 = 1 - \delta + \sum_{j=1}^{\infty} \delta 2^{-j} m_j. \tag{63}$$

We now bound m_j for $j \in \mathbb{N}$. Let $(A_i)_{i=1}^{\infty}$ be i.i.d. with $A_i \sim \text{Bernoulli}(2\delta)$, and let $(G_i)_{i=1}^{\infty}$ be i.i.d. geometric random variables with $\mathbb{P}(G_1 = k) = 2^{-k-1}$ for $k \in \mathbb{N}_0$. Then, for $j \in \mathbb{N}$,

$$
\begin{aligned}
m_j &= \mathbb{E}\left[\inf\left\{n \geq 1 : j + \sum_{i=1}^{n}\big(A_i G_i - (1 - A_i)\big) \leq 0\right\}\right] \\
&= \mathbb{E}\left[\inf\left\{n \geq 1 : \sum_{i=1}^{n} A_i(G_i + 1) \leq n - j\right\}\right] \\
&\leq 1 + \sum_{k=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{k} A_i(G_i + 1) > k - j\right),
\end{aligned}
\tag{64}
$$

since for a random variable Z taking values in $\mathbb{N}_0$, we have that $\mathbb{E}[Z] = \sum_{k=0}^{\infty} \mathbb{P}(Z > k)$, and since for $k \geq 1$,

$$
\mathbb{P}\left(\inf\left\{n \geq 1 : \sum_{i=1}^{n} A_i(G_i + 1) \leq n - j\right\} > k\right) \leq \mathbb{P}\left(\sum_{i=1}^{k} A_i(G_i + 1) > k - j\right).
$$

For $k \in \mathbb{N}$ and $\lambda > 0$, by Markov's inequality,

$$
\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{k} A_i(G_i + 1) > k - j\right) &\leq e^{-\lambda(k-j)}\, \mathbb{E}\big[e^{\lambda \sum_{i=1}^{k} A_i(G_i+1)}\big] \\
&= e^{-\lambda(k-j)}\, \mathbb{E}\big[e^{\lambda A_1(G_1+1)}\big]^k.
\end{aligned}
$$

For $\lambda \in (0, \log 2)$,

$$
\mathbb{E}\big[e^{\lambda A_1(G_1+1)}\big] = 2\delta \frac{\frac{1}{2}e^{\lambda}}{1 - \frac{1}{2}e^{\lambda}} + 1 - 2\delta.
$$

Hence, letting $\lambda = \log(3/2)$,

$$
\mathbb{E}\big[e^{\log(3/2) A_1(G_1+1)}\big] = 1 + 4\delta < \frac{5}{4},
$$

since we chose $\delta < \frac{1}{16}$. It follows that, for $k \in \mathbb{N}$,

$$
\mathbb{P}\left(\sum_{i=1}^{k} A_i(G_i + 1) > k - j\right) \leq \left(\frac{3}{2}\right)^j \left(\frac{5}{6}\right)^k.
$$

Hence, by (64),

$$
m_j \leq 1 + \left(\frac{3}{2}\right)^j \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^k = 1 + 5\left(\frac{3}{2}\right)^j.
\tag{65}
$$

Therefore, by (63),

$$
m_0 \leq 1 - \delta + \sum_{j=1}^{\infty} \delta 2^{-j}\left(1 + 5\left(\frac{3}{2}\right)^j\right) = 1 + 15\delta.
$$

It follows that $(Y_n)_{n=0}^{\infty}$ is positive recurrent, and since it is also irreducible and aperiodic, by convergence to equilibrium for Markov chains we have that, as $n \to \infty$,

$$
\mathbb{P}(Y_n = 0 | Y_0 = 0) \to \frac{1}{m_0} \geq \frac{1}{1 + 15\delta}.
$$

Since $\frac{1}{1+15\delta} \geq 1 - \frac{1}{2}\epsilon$, there exists $n_\epsilon < \infty$ such that, for $n \geq n_\epsilon$,

$$\mathbb{P}(Y_n \neq 0 | Y_0 = 0) < \epsilon,$$

and so by (61), for $\mathcal{X} \in \Gamma(K_0, c_0)$ and $n \geq n_\epsilon$,

$$\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(nt_1) \notin \Gamma(K_0, c_0)\big) \leq \epsilon. \tag{66}$$

Let $t'_\epsilon = \max(n_\epsilon t_0, 2t_0)$. Then, for $t \geq t'_\epsilon$, we have that $\lfloor t/t_0 \rfloor \geq n_\epsilon$ and $t/\lfloor t/t_0 \rfloor \in [t_0, 2t_0]$. Therefore, (54) follows from (66) with $t_1 = t/\lfloor t/t_0 \rfloor$ and $n = \lfloor t/t_0 \rfloor$.

Finally, note that by (62) and (65), for $K > 0$, if $\mathcal{X} \in \Gamma(K, c_0)$; then,

$$\mathbb{E}_{\mathcal{X}}\big[\inf\{n \geq 1 : X^{(N)}(nt_1) \in \Gamma(K_0, c_0)\}\big] \leq 1 + 5\left(\frac{3}{2}\right)^{\lceil K - K_0 \rceil \vee 0}$$

which establishes (55) and completes the proof. □

PROOF OF PROPOSITION 1.7. Take t_0 and c_0, as in Proposition 2.5, and fix $K_0 = 3$. Take $\epsilon > 0$, and take N'_ϵ and t'_ϵ, as in Proposition 6.2. Take $t_\epsilon = t_\epsilon(c_0, K_0)$, as defined in Proposition 2.4. Take $c \in (0, c_0]$ and $K \geq K_0$, and let

$$L = \lceil K - K_0 \rceil + 1 + \left\lceil \frac{\log(c_0/c)}{\log 2} \right\rceil. \tag{67}$$

Recall the definition of c_1 in Proposition 1.5, and suppose $N \geq N'_\epsilon$ is sufficiently large that Proposition 1.5 holds and that

$$e^{2Lt_0} N^{-c_1} \leq c_0, \qquad e^{Lt_0} N^{-1-c_1} \leq \epsilon, \qquad e^{2t_\epsilon} N^{-c_1} < \epsilon \quad \text{and} \quad e^{t_\epsilon} N^{-1-c_1} < \epsilon.$$

Take $\mathcal{X} \in \Gamma(K, c)$, and let $v^{(N)}$ denote the solution of (7) with initial condition $v_0(r) = N^{-1}|\mathcal{X} \cap \mathcal{B}(r)|$ which satisfies $v_0(r) \geq c\mathbb{1}_{\{r \geq K\}}$. By Proposition 2.5, recalling the definition of L in (67), we have the lower bound

$$v^{(N)}(r, Lt_0) \geq 2c_0 \mathbb{1}_{\{r \geq K_0\}}, \quad \forall r \geq 0. \tag{68}$$

(The time Lt_0 is the same as the time t_a mentioned in the outline at the start of Section 6.)

Now, we compare $F^{(N)}(\cdot, Lt_0)$ with $v^{(N)}(\cdot, Lt_0)$, to show that $F^{(N)}(K_0, Lt_0) \geq c_0$ with high probability. By (68) and by Proposition 1.5, since N is sufficiently large that $e^{2Lt_0}N^{-c_1} \leq c_0$ and $e^{Lt_0}N^{-1-c_1} \leq \epsilon$, we now have that

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}\big(F^{(N)}(K_0, Lt_0) \leq c_0\big) &\leq \mathbb{P}_{\mathcal{X}}\Big(\sup_{r \geq 0}\big|F^{(N)}(r, Lt_0) - v^{(N)}(r, Lt_0)\big| \geq c_0\Big) \\ &\leq \epsilon.\end{aligned} \tag{69}$$

Take $t \geq t'_\epsilon + Lt_0$. Then,

$$\begin{aligned}&\mathbb{P}_{\mathcal{X}}\big(F^{(N)}(K_0, t) < c_0\big) \\ &\quad \leq \mathbb{P}_{\mathcal{X}}\big(F^{(N)}(K_0, Lt_0) \leq c_0\big) + \mathbb{P}_{\mathcal{X}}\big(F^{(N)}(K_0, Lt_0) \geq c_0, F^{(N)}(K_0, t) < c_0\big) \\ &\quad \leq \epsilon + \mathbb{E}_{\mathcal{X}}\big[\mathbb{P}_{X^{(N)}(Lt_0)}\big(X^{(N)}(t - Lt_0) \notin \Gamma(K_0, c_0)\big)\mathbb{1}_{\{X^{(N)}(Lt_0) \in \Gamma(K_0, c_0)\}}\big] \\ &\quad \leq 2\epsilon,\end{aligned} \tag{70}$$

where the second inequality follows by (69) and the last inequality follows by (54) in Proposition 6.2.

Now, note that for any configuration $\tilde{\mathcal{X}} \in \Gamma(K_0, c_0)$, letting $\tilde{v}$ denote the solution of (7) with initial condition $v_0(r) = N^{-1}|\tilde{\mathcal{X}} \cap \mathcal{B}(r)|$, we have by Proposition 2.4 that $\sup_{r\geq 0} |\tilde{v}(r, t_\epsilon) - V(r)| < \epsilon$. Hence,

$$\mathbb{P}_{\tilde{\mathcal{X}}}\Big(\sup_{r\geq 0}\big|F^{(N)}(r, t_\epsilon) - V(r)\big| \geq 2\epsilon\Big) \leq \mathbb{P}_{\tilde{\mathcal{X}}}\Big(\sup_{r\geq 0}\big|F^{(N)}(r, t_\epsilon) - \tilde{v}(r, t_\epsilon)\big| \geq \epsilon\Big) \leq \epsilon \tag{71}$$

by Proposition 1.5, since N is large enough that $e^{2t_\epsilon} N^{-c_1} \leq \epsilon$ and $e^{t_\epsilon} N^{-1-c_1} \leq \epsilon$.

Hence, for $t \geq t'_\epsilon + Lt_0 + t_\epsilon$ and $\mathcal{X} \in \Gamma(K, c)$,

$$\begin{aligned}
&\mathbb{P}_{\mathcal{X}}\Big(\sup_{r\geq 0}\big|F^{(N)}(r, t) - V(r)\big| \geq 2\epsilon\Big) \\
&\quad \leq \mathbb{P}_{\mathcal{X}}\Big(\sup_{r\geq 0}\big|F^{(N)}(r, t) - V(r)\big| \geq 2\epsilon, F^{(N)}(K_0, t - t_\epsilon) \geq c_0\Big) \\
&\qquad + \mathbb{P}_{\mathcal{X}}\big(F^{(N)}(K_0, t - t_\epsilon) < c_0\big) \\
&\quad \leq \epsilon + 2\epsilon,
\end{aligned} \tag{72}$$

using (70) for the second term and (71) with the Markov property and $\tilde{\mathcal{X}} := X^{(N)}(t - t_\epsilon)$ for the first term. (Indeed, $F^{(N)}(K_0, t - t_\epsilon) \geq c_0$ is equivalent to $X^{(N)}(t - t_\epsilon) \in \Gamma(K_0, c_0)$.) This concludes the proof of (10), the first statement of Proposition 1.7. We now turn to proving (12), the third statement.

Assume that $t \geq t'_\epsilon + Lt_0 + t_\epsilon + 1$ and $\mathcal{X} \in \Gamma(K, c)$, and introduce $\lambda = N^{-c_1/3}$. For any family $(E_k)_{k=0}^\infty$ of events, using (70), we have that

$$\begin{aligned}
&\mathbb{P}_{\mathcal{X}}\Big(\sup_{s\in[0,1]} M_{t+s}^{(N)} > R_\infty + 2\epsilon\Big) \\
&\quad \leq 2\epsilon + \sum_{k=0}^{\lfloor 1/\lambda\rfloor} \mathbb{P}_{\mathcal{X}}\Big(E_k, \sup_{s\in[\lambda,2\lambda]} M_{t+(k-1)\lambda+s}^{(N)} > R_\infty + 2\epsilon\Big) \\
&\qquad + \sum_{k=0}^{\lfloor 1/\lambda\rfloor} \mathbb{P}_{\mathcal{X}}\big(E_k^c, F^{(N)}(K_0, t - t_\epsilon - \lambda) \geq c_0\big).
\end{aligned} \tag{73}$$

We use this expression with the events

$$E_k = \big\{F^{(N)}\big(R_\infty + \epsilon, t + (k-1)\lambda\big) \geq 1 - N^{-c_1/2}\big\}.$$

For $\tilde{\mathcal{X}} \in \Gamma(K_0, c_0)$, let $\tilde{v}$ denote the solution of (7) with initial condition

$$v_0(r) = N^{-1}\big|\tilde{\mathcal{X}} \cap \mathcal{B}(r)\big|,$$

and let $\tilde{R}_t = \inf\{r \geq 0 : \tilde{v}(r, t) = 1\}$ for $t > 0$. Then, by Proposition 2.4, for $t \geq t_\epsilon$, $|\tilde{R}_t - R_\infty| < \epsilon$ and so $\tilde{v}(R_\infty + \epsilon, t) = 1$. Hence, by Proposition 1.5, for $k \in \{0, \dots, \lfloor 1/\lambda\rfloor\}$,

$$\mathbb{P}_{\tilde{\mathcal{X}}}\big(F^{(N)}(R_\infty + \epsilon, t_\epsilon + k\lambda) \leq 1 - e^{2(t_\epsilon+1)} N^{-c_1}\big) \leq e^{t_\epsilon+1} N^{-1-c_1}.$$

For N sufficiently large that $e^{2(t_\epsilon+1)} N^{-c_1} < N^{-c_1/2}$, by the Markov property at time $t - t_\epsilon - \lambda$ this implies that

$$\mathbb{P}_{\mathcal{X}}\big(E_k^c, F^{(N)}(K_0, t - t_\epsilon - \lambda) \geq c_0\big) \leq e^{t_\epsilon+1} N^{-1-c_1}.$$

By Lemma 4.12 with $b = c_1/3$, for N sufficiently large for $t' \geq 0$,

$$\mathbb{P}_{\mathcal{X}}\Big(F^{(N)}\big(R_\infty + \epsilon, t'\big) \geq 1 - \frac{1}{4}\lambda, \sup_{s\in[\lambda,2\lambda]} M_{t'+s}^{(N)} > R_\infty + \epsilon + \lambda^{1/3}\Big) \leq e^{-\lambda^{-1/4}}.$$

For N sufficiently large that $\frac{1}{4}\lambda \geq N^{-c_1/2}$ and $\lambda^{1/3} = N^{-c_1/9} < \epsilon$, choosing $t' = t + (k-1)\lambda$, this implies that

$$\mathbb{P}_{\mathcal{X}}\Big(E_k, \sup_{s\in[\lambda,2\lambda]} M^{(N)}_{t+(k-1)\lambda+s} > R_\infty + 2\epsilon\Big) \leq e^{-\lambda^{-1/4}} = e^{-N^{c_1/12}}.$$

By (73) it follows that, for N sufficiently large,

$$\mathbb{P}_{\mathcal{X}}\Big(\sup_{s\in[0,1]} M^{(N)}_{t+s} > R_\infty + 2\epsilon\Big) \leq 2\epsilon + (N^{c_1/3}+1)(e^{-N^{c_1/12}} + e^{t_\epsilon+1}N^{-1-c_1}) \leq 3\epsilon$$

which concludes the proof of (12) in Proposition 1.7. It remains to show (11).

Recall from (9) that V is strictly increasing and continuous on $[0, R_\infty]$, with $V(0) = 0$ and $V(R_\infty) = 1$. If $\sup_{r\geq 0}|F^{(N)}(r,t) - V(r)| < 2\epsilon$, then $M^{(N)}_t > V^{-1}(1-2\epsilon)$, and so, for $t \geq t'_\epsilon + Lt_0 + t_\epsilon$ and $\mathcal{X} \in \Gamma(K,c)$, by (72),

$$\mathbb{P}_{\mathcal{X}}\big(M^{(N)}_t \leq V^{-1}(1-2\epsilon)\big) \leq 3\epsilon. \tag{74}$$

The statement (11) now follows from (74) and (12) which completes the proof. □

For $\epsilon > 0$, let

$$\mathcal{C}_\epsilon = \{X^+_u(\epsilon^{-1/2}) : u \in \mathcal{N}^+_{\epsilon^{-1/2}}, \|X^+_u(s)\| \leq R_\infty + \epsilon \ \forall s \in [\epsilon, \epsilon^{-1/2}]\}.$$

The following lemma is the main remaining step in the proof of Theorem 1.3.

LEMMA 6.3. *Take $\delta > 0$. Then, there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ such that, for all $\epsilon \in (0, \epsilon_0]$, there exists $N_0 = N_0(\epsilon,\delta)$ such that, for all $N \geq N_0$, the following holds: if the initial particle configuration $\mathcal{X} \in (\mathbb{R}^d)^N$ satisfies*

$$\sup_{r\geq 0}\big|N^{-1}|\mathcal{X}\cap\mathcal{B}(r)| - V(r)\big| \leq \epsilon,$$

then, for any $A \subseteq \mathbb{R}^d$ measurable,

$$\mathbb{P}_{\mathcal{X}}\Big(\frac{1}{N}|\mathcal{C}_\epsilon \cap A| - \int_A U(x)\,dx \geq \delta\Big) < \frac{1}{2}\delta.$$

PROOF. First, we estimate the mean of $\frac{1}{N}|\mathcal{C}_\epsilon \cap A|$. By the many-to-one lemma,

$$\begin{aligned}
&\mathbb{E}_{\mathcal{X}}\Big[\frac{1}{N}|\mathcal{C}_\epsilon \cap A|\Big]\\
&= \frac{1}{N}e^{\epsilon^{-1/2}}\sum_{k=1}^N \mathbb{P}_{\mathcal{X}_k}\big(B_{\epsilon^{-1/2}} \in A, \|B_s\| \leq R_\infty + \epsilon \ \forall s \in [\epsilon, \epsilon^{-1/2}]\big)\\
&= \frac{1}{N}e^{\epsilon^{-1/2}}\sum_{k=1}^N \mathbb{E}_{\mathcal{X}_k}\big[\mathbb{P}_{B_\epsilon}\big(B_{\epsilon^{-1/2}-\epsilon} \in A, \|B_s\| \leq R_\infty + \epsilon \ \forall s \in [0, \epsilon^{-1/2}-\epsilon]\big)\big]\\
&= e^{\epsilon^{-1/2}}\int_A w(y, \epsilon^{-1/2}-\epsilon)\,dy,
\end{aligned} \tag{75}$$

where $w(y,s)$ is the unique solution to

$$\begin{cases} \partial_s w = \Delta w & s > 0, \|y\| < R_\infty + \epsilon,\\ w(y,s) = 0 & s \geq 0, \|y\| \geq R_\infty + \epsilon,\\ w(y,0) = (\Phi_\epsilon * \mu_{\mathcal{X}})(y) & \|y\| < R_\infty + \epsilon, \end{cases} \tag{76}$$

continuous in (y, s) for $s > 0$, where $\Phi_\epsilon(y)$ is the heat kernel and $\mu_{\mathcal{X}}(\mathrm{d}y)$ is the empirical measure on $\mathbb{R}^d$ determined by the points $\{\mathcal{X}_k\}_{k=1}^N$,

$$\mu_{\mathcal{X}}(\mathrm{d}y) = \frac{1}{N}\sum_{k=1}^{N}\delta_{\mathcal{X}_k}, \qquad \Phi_\epsilon(y) = (4\pi\epsilon)^{-d/2}e^{-\frac{\|y\|^2}{4\epsilon}}.$$

The function $w(y, s)$ can be expanded as a series in terms of the Dirichlet eigenfunctions of the Laplacian on $\mathcal{B}(R_\infty + \epsilon)$,

$$w(x, s) = \sum_{k=1}^{\infty} a_k e^{-s\lambda_k^\epsilon} U_k^\epsilon(x), \quad s \ge 0, \|x\| \le R_\infty + \epsilon, \tag{77}$$

where the partial sums converge weakly in $L^2([0, T]; H_0^1)$ for any $T > 0$ and $s \mapsto w(\cdot, s)$ is continuous from $[0, \infty)$ to $L^2(\mathcal{B}(R_\infty + \epsilon))$; see Theorem 3 in Section 7.1 of [16] and the remark on p. 374, therein. Here, $\{(U_k^\epsilon(x), \lambda_k^\epsilon)\}_{k\ge 1}$ denote the Dirichlet eigenfunctions and eigenvalues for $-\Delta$ on the ball $\{\|x\| \le R_\infty + \epsilon\}$,

$$2 - \Delta U_k^\epsilon = \lambda_k^\epsilon U_k^\epsilon, \quad \text{for } \|x\| < R_\infty + \epsilon,$$
$$U_k^\epsilon(x) = 0, \quad \text{for } \|x\| \ge R_\infty + \epsilon.$$

By scaling we have

$$\lambda_k^\epsilon = \left(\frac{R_\infty}{R_\infty + \epsilon}\right)^2 \lambda_k^0 \quad \text{and} \quad U_k^\epsilon(x) = \left(\frac{R_\infty}{R_\infty + \epsilon}\right)^{d/2} U_k^0\left(x\frac{R_\infty}{R_\infty + \epsilon}\right).$$

The eigenvalues satisfy $\lambda_1^0 = 1 < \lambda_2^0 \le \dots$ and the eigenfunctions $\{U_k^\epsilon\}_{k=1}^\infty$ form an orthonormal basis in $L^2(\mathcal{B}(R_\infty + \epsilon))$. Furthermore, $U_1^0(x) = \|U\|_{L^2}^{-1} U(x)$; see (2).

Define $\tilde{w}(y, s) = \sum_{k=2}^\infty a_k e^{-s\lambda_k^\epsilon} U_k^\epsilon(y)$. Thus,

$$w(y, s) = a_1 e^{-s\lambda_1^\epsilon} U_1^\epsilon(y) + \tilde{w}(y, s), \tag{78}$$

and $\tilde{w}(\cdot, s)$ is orthogonal to U_1^ϵ in L^2 for all $s \ge 0$. Observe that

$$\begin{aligned}
\|w(\cdot, 0)\|_{L^2}^2 &\le \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\Phi_\epsilon(y - x)\mu_{\mathcal{X}}(\mathrm{d}x)\right)^2 \mathrm{d}y \\
&\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Phi_\epsilon(y - x)^2\mu_{\mathcal{X}}(\mathrm{d}x)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\Phi_\epsilon(y - x)^2\,\mathrm{d}y\right)\mu_{\mathcal{X}}(\mathrm{d}x) = (8\pi\epsilon)^{-d/2},
\end{aligned}$$

where the second line follows by Jensen's inequality. Therefore, for $s \ge 0$,

$$\|\tilde{w}(\cdot, s)\|_{L^2}^2 \le e^{-2\lambda_2^\epsilon s}\|\tilde{w}(\cdot, 0)\|_{L^2}^2 \le e^{-2\lambda_2^\epsilon s}\|w(\cdot, 0)\|_{L^2}^2 \le e^{-2\lambda_2^\epsilon s}(8\pi\epsilon)^{-d/2}. \tag{79}$$

Now, we estimate a_1. Since $\|U_1^\epsilon\|_{L^2} = \|U_1^0\|_{L^2} = 1$ and since U_1^ϵ is spherically symmetric, writing $\mathbf{e}$ for a unit vector, we have

$$\begin{aligned}
a_1 &= \int_{\mathcal{B}(R_\infty+\epsilon)} U_1^\epsilon(x) w(x, 0)\,\mathrm{d}x \\
&= \int_0^{R_\infty+\epsilon} U_1^\epsilon(r\mathbf{e})\left(\int_{\partial\mathcal{B}(r)}(\Phi_\epsilon * \mu_{\mathcal{X}})(y)\,\mathrm{d}S(y)\right)\mathrm{d}r \qquad (80)\\
&= -\int_0^{R_\infty+\epsilon} \partial_r U_1^\epsilon(r\mathbf{e})\left(\int_{\mathcal{B}(r)}(\Phi_\epsilon * \mu_{\mathcal{X}})(y)\,\mathrm{d}y\right)\mathrm{d}r
\end{aligned}$$

by integration by parts. Also, for $r \geq 0$,

$$-\partial_r U_1^\epsilon(r\mathbf{e}) = -\left(\frac{R_\infty}{R_\infty+\epsilon}\right)^{d/2+1} \|U\|_{L^2}^{-1} \partial_r U\left(\frac{rR_\infty \mathbf{e}}{R_\infty+\epsilon}\right). \tag{81}$$

In particular, since $U(r\mathbf{e})$ is nonincreasing in r for $r \geq 0$, $-\partial_r U_1^\epsilon(r\mathbf{e}) \geq 0$.

Now, recall that we assume $|N^{-1}|\mathcal{X} \cap \mathcal{B}(r)| - V(r)| < \epsilon$ for all $r \geq 0$, and so

$$\sup_{r\geq 0} \left|\mu_{\mathcal{X}}(\mathcal{B}(r)) - V(r)\right| < \epsilon. \tag{82}$$

Then, for $r \geq 0$,

$$\int_{\mathcal{B}(r)} \int_{\|x\|<r+\epsilon^{1/3}} \Phi_\epsilon(y-x)\mu_{\mathcal{X}}(\mathrm{d}x)\,\mathrm{d}y \leq \mu_{\mathcal{X}}\left(\mathcal{B}\left(r+\epsilon^{1/3}\right)\right) \leq V\left(r+\epsilon^{1/3}\right)+\epsilon,$$

where the first inequality follows since $\int_{\mathbb{R}^d} \Phi_\epsilon(y-x)\,\mathrm{d}y = 1$. Furthermore, writing $y = x+z$,

$$\begin{aligned}\int_{\mathcal{B}(r)} \int_{\|x\|\geq r+\epsilon^{1/3}} \Phi_\epsilon(y-x)\mu_{\mathcal{X}}(\mathrm{d}x)\,\mathrm{d}y &= \int_{\|x\|\geq r+\epsilon^{1/3}} \int_{\mathbb{R}^d} \mathbb{1}_{\{\|x+z\|<r\}}\Phi_\epsilon(z)\,\mathrm{d}z\mu_{\mathcal{X}}(\mathrm{d}x)\\ &\leq \int_{\|x\|\geq r+\epsilon^{1/3}} \int_{\|z\|>\epsilon^{1/3}} \Phi_\epsilon(z)\,\mathrm{d}z\mu_{\mathcal{X}}(\mathrm{d}x)\\ &\leq \int_{\|z\|>\epsilon^{1/3}} \Phi_\epsilon(z)\,\mathrm{d}z\\ &\leq 2de^{-\epsilon^{-1/3}/(4d)},\end{aligned}$$

where we used $\mu_{\mathcal{X}}(\mathbb{R}^d) = 1$ in the third line and a Gaussian tail bound in the last line. This implies that

$$\int_{\mathcal{B}(r)} (\Phi_\epsilon * \mu_{\mathcal{X}})(y)\,\mathrm{d}y \leq V\left(r+\epsilon^{1/3}\right)+\epsilon+2de^{-\epsilon^{-1/3}/(4d)}.$$

Therefore, by (80) and (81),

$$\begin{aligned}a_1 &\leq -\int_0^{R_\infty+\epsilon}\left(\frac{R_\infty}{R_\infty+\epsilon}\right)^{d/2+1} \|U\|_{L^2}^{-1} \partial_r U\left(\frac{rR_\infty \mathbf{e}}{R_\infty+\epsilon}\right)\\ &\qquad \times \left(V\left(r+\epsilon^{1/3}\right)+\epsilon+2de^{-\epsilon^{-1/3}/(4d)}\right)\mathrm{d}r\\ &\leq \|U\|_{L^2}^{-1}\int_0^{R_\infty+\epsilon}\left(-\partial_r U(r\mathbf{e})+\epsilon\left\|\partial_r^2 U\right\|_\infty\right)\\ &\qquad \times \left(V(r)+\epsilon^{1/3}\left\|V'\right\|_\infty+\epsilon+2de^{-\epsilon^{-1/3}/(4d)}\right)\mathrm{d}r\\ &\leq -\|U\|_{L^2}^{-1}\int_0^{R_\infty} \partial_r U(r\mathbf{e})V(r)\,\mathrm{d}r + \mathcal{O}\left(\epsilon^{1/3}\right).\end{aligned}$$

Note that by integration by parts,

$$-\int_0^{R_\infty} \partial_r U(r\mathbf{e})V(r)\,\mathrm{d}r = \int_0^{R_\infty} U(r\mathbf{e})V'(r)\,\mathrm{d}r = \|U\|_{L^2}^2,$$

since $V(r) = \int_{\mathcal{B}(r)} U(y)\,\mathrm{d}y$. Hence, for $\delta > 0$ and for ϵ sufficiently small, we have that

$$a_1 \leq \|U\|_{L^2}\left(1+\frac{1}{4}\delta\right). \tag{83}$$

By (78) we have now shown that for ϵ sufficiently small (depending on δ), for $s \geq 0$ and $\|x\| \leq R_\infty + \epsilon$, (using that $U_1^0 = U/\|U\|_{L^2}$)

$$w(x,s) \leq \left(1 + \frac{1}{4}\delta\right) e^{-s(\frac{R_\infty}{R_\infty+\epsilon})^2} \left(\frac{R_\infty}{R_\infty+\epsilon}\right)^{d/2} U\left(\frac{xR_\infty}{R_\infty+\epsilon}\right) + \tilde{w}(x,s), \tag{84}$$

where $\tilde{w}$ satisfies (79). Then,

$$\int_A w(x,s)\,dx \leq \left(1 + \frac{1}{4}\delta + \mathcal{O}(\epsilon)\right) e^{-s(\frac{R_\infty}{R_\infty+\epsilon})^2} \int_A U(x)\,dx + \int_{\mathcal{B}(R_\infty+\epsilon)} |\tilde{w}(x,s)|\,dx. \tag{85}$$

By Jensen's inequality and then by (79),

$$\int_{\mathcal{B}(R_\infty+\epsilon)} |\tilde{w}(x,s)|\,dx \leq |\mathcal{B}(R_\infty+\epsilon)|^{1/2} \|\tilde{w}(\cdot,s)\|_{L^2} \leq |\mathcal{B}(R_\infty+\epsilon)|^{1/2} e^{-\lambda_2^\epsilon s}(8\pi\epsilon)^{-d/4}.$$

Take $c \in (0, \lambda_2^0 - 1)$, and suppose ϵ is sufficiently small that $\lambda_2^\epsilon = (\frac{R_\infty}{R_\infty+\epsilon})^2 \lambda_2^0 > 1 + c$. Then,

$$\int_{\mathcal{B}(R_\infty+\epsilon)} |\tilde{w}(x,s)|\,dx \leq |\mathcal{B}(R_\infty+\epsilon)|^{1/2} e^{-(1+c)s}(8\pi\epsilon)^{-d/4}.$$

By (85) and (75), it follows that, for all N,

$$\begin{aligned} \mathbb{E}_{\mathcal{X}}\left[\frac{1}{N}|\mathcal{C}_\epsilon \cap A|\right] &\leq \left(1 + \frac{1}{4}\delta + \mathcal{O}(\epsilon)\right) e^{\epsilon^{-1/2}[1-(\frac{R_\infty}{R_\infty+\epsilon})^2]} \int_A U(x)\,dx \\ &\quad + |\mathcal{B}(R_\infty+\epsilon)|^{1/2} e^{-c\epsilon^{-1/2}+(1+c)\epsilon}(8\pi\epsilon)^{-d/4} \\ &\leq \left(1 + \frac{1}{4}\delta + \mathcal{O}(\epsilon^{1/2})\right) \int_A U(x)\,dx + \mathcal{O}(\epsilon). \end{aligned} \tag{86}$$

Therefore, for $\delta > 0$, for $\epsilon > 0$ sufficiently small (depending only on δ), if $\mathcal{X} \in (\mathbb{R}^d)^N$ satisfies (82), then for $A \subseteq \mathbb{R}^d$ measurable,

$$\mathbb{E}_{\mathcal{X}}\left[\frac{1}{N}|\mathcal{C}_\epsilon \cap A|\right] \leq \int_A U(x)\,dx + \frac{1}{2}\delta.$$

By the same argument as for Lemma 5.2, for $N \geq 48$, for $A \subseteq \mathbb{R}^d$ measurable and $\epsilon > 0$,

$$\mathbb{E}_{\mathcal{X}}\left[\left(\frac{1}{N}|\mathcal{C}_\epsilon \cap A| - \mathbb{E}_{\mathcal{X}}\left[\frac{1}{N}|\mathcal{C}_\epsilon \cap A|\right]\right)^4\right] \leq 13e^{4\epsilon^{-1/2}} N^{-2}. \tag{87}$$

So for $\delta > 0$, for $\epsilon > 0$ sufficiently small (depending only on δ), and $N \geq 48$, if $\mathcal{X} \in (\mathbb{R}^d)^N$ satisfies (82), then for $A \subseteq \mathbb{R}^d$ measurable,

$$\begin{aligned} \mathbb{P}_{\mathcal{X}}\left(\frac{1}{N}|\mathcal{C}_\epsilon \cap A| - \int_A U(x)\,dx \geq \delta\right) &\leq \mathbb{P}_{\mathcal{X}}\left(\left|\frac{1}{N}\left|\mathcal{C}_\epsilon \cap A\right| - \mathbb{E}_{\mathcal{X}}\left[\frac{1}{N}|\mathcal{C}_\epsilon \cap A|\right]\right| \geq \frac{1}{2}\delta\right) \\ &\leq 16\delta^{-4} \cdot 13e^{4\epsilon^{-1/2}} N^{-2} \end{aligned}$$

by (87) and Markov's inequality. The result follows by taking N sufficiently large that $16\delta^{-4} \cdot 13e^{4\epsilon^{-1/2}} N^{-2} < \frac{1}{2}\delta$. $\square$

The following result is an immediate consequence of Lemma 6.3, and the coupling between the BBM $X^+(t)$ and the N-BBM $X^{(N)}(t)$ described in Section 3.

COROLLARY 6.4. *Take $\delta > 0$. Then, there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ such that, for all $\epsilon \in (0, \epsilon_0]$, there exists $N_0 = N_0(\epsilon, \delta)$ such that, for all $N \geq N_0$, the following holds: if the initial particle configuration $\mathcal{X} \in (\mathbb{R}^d)^N$ satisfies*

$$\sup_{r \geq 0} |N^{-1}|\mathcal{X} \cap \mathcal{B}(r)| - V(r)| \leq \epsilon,$$

then, for any $A \subseteq \mathbb{R}^d$ measurable,

$$\mathbb{P}_{\mathcal{X}}\left(M_s^{(N)} \le R_\infty + \epsilon \ \forall s \in [0, \epsilon^{-1/2}], \mu^{(N)}(A, \epsilon^{-1/2}) - \int_A U(x)\,\mathrm{d}x \ge \delta\right) < \frac{1}{2}\delta.$$

PROOF. Under the coupling between the BBM X^+ and the N-BBM $X^{(N)}$ described in Section 3, we have that, for $\epsilon > 0$, if $M_s^{(N)} \le R_\infty + \epsilon \ \forall s \in [0, \epsilon^{-1/2}]$, then by (24), almost surely,

$$\begin{aligned}&\{X_k^{(N)}(\epsilon^{-1/2}) : k \in \{1, \dots, N\}\}\\ &\quad \subseteq \{X_u^+(\epsilon^{-1/2}) : u \in \mathcal{N}^+_{\epsilon^{-1/2}}, \|X_u^+(s)\| \le R_\infty + \epsilon \ \forall s \in [0, \epsilon^{-1/2}]\} \subseteq \mathcal{C}_\epsilon,\end{aligned}$$

where $\mathcal{C}_\epsilon$ is defined just before Lemma 6.3. Hence, for any $A \subseteq \mathbb{R}^d$ measurable, $\mu^{(N)}(A, \epsilon^{-1/2}) \le \frac{1}{N}|\mathcal{C}_\epsilon \cap A|$. The result follows by Lemma 6.3. □

We can now use Proposition 1.7 and Corollary 6.4 to prove Theorem 1.3.

PROOF OF THEOREM 1.3. Take $K > 0$, $c \in (0, 1]$ and $\delta > 0$. As the second statement of the theorem was already proved in Proposition 1.7, it remains to prove that for N and t large enough, for $\mathcal{X} \in \Gamma(K, c)$ and $A \subseteq \mathbb{R}^d$ measurable,

$$\mathbb{P}_{\mathcal{X}}\left(\left|\mu^{(N)}(A, t) - \int_A U(x)\,\mathrm{d}x\right| \ge \delta\right) < \delta.$$

Take $\epsilon \in (0, \epsilon_0(\delta)]$ so that Corollary 6.4 holds and sufficiently small that $\lceil \epsilon^{-1/2} \rceil \epsilon + \epsilon < \frac{1}{2}\delta$. Take $N_\epsilon = N_\epsilon(K, c)$, $T_\epsilon = T_\epsilon(K, c)$, as defined in Proposition 1.7, and take $N_0 = N_0(\epsilon, \delta)$, as defined in Corollary 6.4. Let $N \ge \max(N_\epsilon, N_0)$, and take $t \ge T_\epsilon + \epsilon^{-1/2}$. Let $t_0 = t - \epsilon^{-1/2}$.

For an initial particle configuration $\mathcal{X} \in \Gamma(K, c)$ and $A \subseteq \mathbb{R}^d$ measurable, we have by a union bound that

$$\begin{aligned}&\mathbb{P}_{\mathcal{X}}\left(\mu^{(N)}(A, t) - \int_A U(x)\,\mathrm{d}x \ge \delta\right)\\ &\quad \le \mathbb{P}_{\mathcal{X}}\biggl(M_s^{(N)} \le R_\infty + \epsilon \forall s \in [t_0, t], \sup_{r \ge 0}\left|F^{(N)}(r, t_0) - V(r)\right| \le \epsilon,\\ &\qquad\qquad\qquad \mu^{(N)}(A, t) - \int_A U(x)\,\mathrm{d}x \ge \delta\biggr)\\ &\qquad + \mathbb{P}_{\mathcal{X}}\left(\sup_{s \in [t_0, t]} M_s^{(N)} > R_\infty + \epsilon\right) + \mathbb{P}_{\mathcal{X}}\left(\sup_{r \ge 0}\left|F^{(N)}(r, t_0) - V(r)\right| \ge \epsilon\right)\\ &\quad \le \frac{1}{2}\delta + \lceil t - t_0 \rceil \epsilon + \epsilon,\end{aligned}$$

by (12) and (10) in Proposition 1.7 for the last two terms (since $N \ge N_\epsilon$ and $t_0 = t - \epsilon^{-1/2} \ge T_\epsilon$) and by Corollary 6.4 and the Markov property at time t_0 for the first term. Therefore, since $\lceil t - t_0 \rceil \epsilon + \epsilon = \lceil \epsilon^{-1/2} \rceil \epsilon + \epsilon < \frac{1}{2}\delta$, it follows that, for any $A \subseteq \mathbb{R}^d$ measurable,

$$\mathbb{P}_{\mathcal{X}}\left(\mu^{(N)}(A, t) - \int_A U(x)\,\mathrm{d}x \ge \delta\right) \le \delta.$$

As in (48), since $\mu^{(N)}(A, t) + \mu^{(N)}(\mathbb{R}^d \setminus A, t) = 1$ and $\int_A U(x)\,\mathrm{d}x + \int_{\mathbb{R}^d \setminus A} U(x)\,\mathrm{d}x = 1$, we have that

$$\mathbb{P}_{\mathcal{X}}\left(\mu^{(N)}(A, t) - \int_A U(x)\,\mathrm{d}x \le -\delta\right) = \mathbb{P}_{\mathcal{X}}\left(\mu^{(N)}(\mathbb{R}^d \setminus A, t) - \int_{\mathbb{R}^d \setminus A} U(x)\,\mathrm{d}x \ge \delta\right) \le \delta$$

which completes the proof. □

It remains to prove Theorems 1.2 and 1.4, which will follow easily from the following proposition.

PROPOSITION 6.5. *Take $t_0 > 1$, as in Proposition 6.2. For N sufficiently large for any $t_1 \in (0, t_0]$, the Markov chain $(X^{(N)}(t_1 n))_{n=0}^{\infty}$ is a positive recurrent strongly aperiodic Harris chain.*

REMARK. This proposition will be used in the proof of Theorem 1.2 in combination with Theorems 6.1 and 4.1 of [3], which say that a positive recurrent strongly aperiodic Harris chain admits a unique invariant probability measure and that the distribution of the state of the Harris chain after n steps converges to that invariant probability measure as $n \to \infty$.

PROOF. For $n \in \mathbb{N}_0$, let $Y_n = X^{(N)}(t_1 n)$. We use a similar strategy to the proof of Proposition 3.1 in [15]. By [3] and to show that $(Y_n)_{n=0}^{\infty}$ is a recurrent strongly aperiodic Harris chain, it suffices to show that there exists a set $\Lambda \subseteq (\mathbb{R}^d)^N$ such that:

1. $\mathbb{P}_{\mathcal{X}}(\tau_\Lambda < \infty) = 1 \; \forall \mathcal{X} \in (\mathbb{R}^d)^N$, where $\tau_\Lambda = \inf\{n \geq 1 : Y_n \in \Lambda\}$.
2. There exist $\epsilon > 0$ and a probability measure q on Λ such that $\mathbb{P}_{\mathcal{X}}(Y_1 \in C) \geq \epsilon q(C)$ for any $\mathcal{X} \in \Lambda$ and $C \subseteq \Lambda$.

Furthermore, to prove that the Harris chain is positive recurrent, we also need to show that:

3. $\sup_{\mathcal{X} \in \Lambda} \mathbb{E}_{\mathcal{X}}[\tau_\Lambda] < \infty$.

We prove the proposition with the set

$$\Lambda = \big(\mathcal{B}(R_\infty + 1)\big)^{\otimes N}.$$

We start with the third point, showing that $\sup_{\mathcal{X} \in \Lambda} \mathbb{E}_{\mathcal{X}}[\tau_\Lambda] < \infty$.

Take $K_0 = 3$ and $c_0 > 0$, as in Proposition 6.2; let $\Lambda' = \Gamma(K_0 \vee (R_\infty + 1), c_0) \supset \Lambda$. By (11) in Proposition 1.7, there exist $n_1, N_1 < \infty$ such that, for $N \geq N_1$, for $\mathcal{X} \in \Lambda'$,

$$\mathbb{P}_{\mathcal{X}}\big(M_{n_1 t_1}^{(N)} \geq R_\infty + 1\big) \leq \frac{1}{2}.$$

Hence,

$$\sup_{\mathcal{X} \in \Lambda'} \mathbb{P}_{\mathcal{X}}(Y_{n_1} \notin \Lambda) \leq \frac{1}{2}. \tag{88}$$

Let $\tau_{\Lambda'} = \inf\{n \geq 1 : Y_n \in \Lambda'\}$. Note that letting $t_2 = \lceil t_0/t_1 \rceil t_1$, we have that

$$\tau_{\Lambda'} \leq \left\lceil \frac{t_0}{t_1} \right\rceil \inf\big\{n \geq 1 : X^{(N)}(nt_2) \in \Gamma\big(K_0 \vee (R_\infty + 1), c_0\big)\big\}.$$

Hence, since $t_2 \in [t_0, 2t_0]$, if N is sufficiently large, then by (56) and (55) in Proposition 6.2,

$$\mathbb{P}_{\mathcal{X}}(\tau_{\Lambda'} < \infty) = 1 \quad \forall \mathcal{X} \in \big(\mathbb{R}^d\big)^N \quad \text{and} \quad \sup_{\mathcal{X} \in \Lambda'} \mathbb{E}_{\mathcal{X}}[\tau_{\Lambda'}] < \infty. \tag{89}$$

Let $\tau(0) = 0$, and for $k \in \mathbb{N}$, let

$$\tau(k) = \inf\big\{n \geq 1 + \tau(k-1) : Y_n \in \Lambda'\big\}.$$

Then, by (89), $(\tau(k))_{k=1}^{\infty}$ form an increasing sequence of almost surely finite times such that $Y_{\tau(k)} \in \Lambda'$. Notice that $\tau(1) = \tau_{\Lambda'}$ and that

$$\tau_\Lambda \leq \tau\big(k^*\big) + n_1, \quad \text{where } k^* = \inf\{k \geq 1 : Y_{\tau(k)+n_1} \in \Lambda\}.$$

It is, therefore, sufficient to show that $\sup_{\mathcal{X}\in\Lambda}\mathbb{E}_{\mathcal{X}}[\tau(k^*)]<\infty$ to establish that $\sup_{\mathcal{X}\in\Lambda}\mathbb{E}_{\mathcal{X}}[\tau_\Lambda]<\infty$.

Write $(\mathcal{F}_n)_{n=0}^\infty$ for the natural filtration of the Markov chain $(Y_n)_{n=0}^\infty$. Notice that, for $k\ge n_1$, the event $\{k^*>k-n_1\}=\{Y_{\tau(j)+n_1}\notin\Lambda\ \forall 1\le j\le k-n_1\}$ is measurable in $\mathcal{F}_{\tau(k)}$ since $\tau(k-n_1)+n_1\le\tau(k)$. Therefore, by the strong Markov property, for $k\ge n_1$ and $\mathcal{X}\in\Lambda'$,

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}(k^*>k)&=\mathbb{P}_{\mathcal{X}}(Y_{\tau(j)+n_1}\notin\Lambda\ \forall 1\le j\le k)\le\mathbb{P}_{\mathcal{X}}(k^*>k-n_1,Y_{\tau(k)+n_1}\notin\Lambda)\\&=\mathbb{E}_{\mathcal{X}}\big[\mathbb{1}_{\{k^*>k-n_1\}}\,\mathbb{P}_{Y_{\tau(k)}}(Y_{n_1}\notin\Lambda)\big]\\&\le\frac{1}{2}\mathbb{P}_{\mathcal{X}}(k^*>k-n_1),\end{aligned}$$

where we used (88) in the last step. Then, by an induction argument for $k\in\mathbb{N}$ and $\mathcal{X}\in\Lambda'$,

$$\mathbb{P}_{\mathcal{X}}(k^*>k)\le 2^{-\lfloor k/n_1\rfloor}. \tag{90}$$

In particular, k^* is almost surely finite. For $\mathcal{X}\in\Lambda'$,

$$\begin{aligned}\mathbb{E}_{\mathcal{X}}[\tau(k^*)]&=\sum_{k=1}^\infty\mathbb{E}_{\mathcal{X}}[\tau(k)\mathbb{1}_{\{k^*=k\}}]\\&=\sum_{k=1}^\infty\sum_{\ell=1}^k\mathbb{E}_{\mathcal{X}}\big[(\tau(\ell)-\tau(\ell-1))\mathbb{1}_{\{k^*=k\}}\big]\\&=\sum_{\ell=1}^\infty\mathbb{E}_{\mathcal{X}}\big[(\tau(\ell)-\tau(\ell-1))\mathbb{1}_{\{k^*\ge\ell\}}\big].\end{aligned}\tag{91}$$

Then, for $\mathcal{X}\in\Lambda'$ and $\ell\ge1$, by (90) and the strong Markov property,

$$\begin{aligned}\mathbb{E}_{\mathcal{X}}\big[(\tau(\ell)-\tau(\ell-1))\mathbb{1}_{\{k^*\ge\ell\}}\big]&\le\mathbb{E}_{\mathcal{X}}\big[(\tau(\ell)-\tau(\ell-1))\mathbb{1}_{\{k^*>\ell-1-n_1\}}\big]\\&=\mathbb{E}_{\mathcal{X}}\big[\mathbb{E}_{\mathcal{X}}[\tau(\ell)-\tau(\ell-1)|\mathcal{F}_{\tau(\ell-1)}]\mathbb{1}_{\{k^*>\ell-1-n_1\}}\big]\\&\le\sup_{\mathcal{Y}\in\Lambda'}\mathbb{E}_{\mathcal{Y}}[\tau(1)]\ \mathbb{P}_{\mathcal{X}}(k^*>\ell-1-n_1)\\&\le 2^{-\lfloor(\ell-1-n_1)/n_1\rfloor}\sup_{\mathcal{Y}\in\Lambda'}\mathbb{E}_{\mathcal{Y}}[\tau(1)].\end{aligned}$$

By substituting into (91), we get that $\sup_{\mathcal{X}\in\Lambda'}\mathbb{E}_{\mathcal{X}}[\tau(k^*)]<\infty$, since we have that $\tau(1)=\tau_{\Lambda'}$ and, from (89), that $\sup_{\mathcal{Y}\in\Lambda'}\mathbb{E}_{\mathcal{Y}}[\tau_{\Lambda'}]<\infty$. This implies that $\sup_{\mathcal{X}\in\Lambda'}\mathbb{E}_{\mathcal{X}}[\tau_\Lambda]<\infty$ and, in particular, $\sup_{\mathcal{X}\in\Lambda}\mathbb{E}_{\mathcal{X}}[\tau_\Lambda]<\infty$.

Proving the first of the three points at the beginning of the proof is now straightforward: for $\mathcal{X}\in(\mathbb{R}^d)^N$, we have $\tau_{\Lambda'}<\infty$ a.s. from (89) and, by the strong Markov property at time $\tau_{\Lambda'}$ and since $\mathbb{P}_{\mathcal{X}'}(\tau_\Lambda<\infty)=1$ for $\mathcal{X}'\in\Lambda'$,

$$\mathbb{P}_{\mathcal{X}}(\tau_\Lambda<\infty)=1.$$

Finally, it remains to prove the second of the three points at the beginning of the proof. Note that, conditional on the event that none of the particles in the N-BBM branch in the time interval $[0,t_1]$, the N particles move according to independent Brownian motions. Therefore, for $\mathcal{X}=(\mathcal{X}_1,\dots,\mathcal{X}_N)\in\Lambda$ and $C\subseteq\Lambda$,

$$\begin{aligned}\mathbb{P}_{\mathcal{X}}(Y_1\in C)&\ge e^{-t_1N}\int_C\prod_{i=1}^N\bigg(\frac{1}{(4\pi t_1)^{d/2}}e^{-\frac{1}{4t_1}\|\mathcal{X}_i-y_i\|^2}\bigg)\,\mathrm{d}y_1\dots\mathrm{d}y_N\\&\ge e^{-t_1N}(4\pi t_1)^{-Nd/2}e^{-N(R_\infty+1)^2/t_1}\,\mathrm{Leb}(C),\end{aligned}$$

where $\mathrm{Leb}(\cdot)$ is the Lebesgue measure on $(\mathbb{R}^d)^N$. Indeed, for $y \in \Lambda$, $\|\mathcal{X}_i - y_i\|^2 \le 4(R_\infty+1)^2$ $\forall i \in \{1, \dots, N\}$. Let

$$\epsilon = e^{-t_1 N}(4\pi t_1)^{-Nd/2} e^{-N(R_\infty+1)^2/t_1}\,\mathrm{Leb}(\Lambda),$$

and define a probability measure q on Λ by letting $q(C) = \mathrm{Leb}(C)/\mathrm{Leb}(\Lambda)$ for $C \subseteq \Lambda$. Then, for $\mathcal{X} \in \Lambda$ and $C \subseteq \Lambda$, $\mathbb{P}_{\mathcal{X}}(Y_1 \in C) \ge \epsilon q(C)$. The result follows. □

PROOF OF THEOREM 1.2. By Proposition 6.5, and by Theorems 6.1 and 4.1 in [3] for N sufficiently large, for $t_1 \in (0, t_0]$, $(X^{(N)}(nt_1))_{n=0}^{\infty}$ has a unique invariant measure $\pi_{t_1}^{(N)}$, which is a probability measure on $(\mathbb{R}^d)^N$, and for any $\mathcal{X} \in (\mathbb{R}^d)^N$, the law of $X^{(N)}(nt_1)$ under $\mathbb{P}_{\mathcal{X}}$ converges as $n \to \infty$ to $\pi_{t_1}^{(N)}$ in total variation norm. In particular, if $C \subseteq (\mathbb{R}^d)^N$ is measurable,

$$\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t_1 n) \in C\big) \to \pi_{t_1}^{(N)}(C) \quad \text{as } n \to \infty. \tag{92}$$

Fix N large enough for Proposition 6.5 to hold. We begin by showing that

$$\pi_{t_1}^{(N)} = \pi_{t_0}^{(N)} =: \pi^{(N)} \quad \forall t_1 \in (0, t_0]. \tag{93}$$

Take $\mathcal{X} \in (\mathbb{R}^d)^N$, and $C \subseteq (\mathbb{R}^d)^N$ a closed set. Take $\delta > 0$. For $\epsilon > 0$, let

$$C^\epsilon = \Big\{\mathcal{Y} \in (\mathbb{R}^d)^N : \inf_{\mathcal{Z} \in C} \|\mathcal{Y} - \mathcal{Z}\| < \epsilon\Big\}.$$

Here, $\|\mathcal{Y} - \mathcal{Z}\|$ denotes the Euclidean norm of $\mathcal{Y} - \mathcal{Z}$ regarded as a vector in $\mathbb{R}^{dN}$. Then, $\pi_{t_0}^{(N)}(C^\epsilon) \to \pi_{t_0}^{(N)}(C)$, as $\epsilon \to 0$. Take $\epsilon > 0$ sufficiently small that

$$\pi_{t_0}^{(N)}(C^\epsilon) < \pi_{t_0}^{(N)}(C) + \frac{1}{3}\delta.$$

It is easy to see that if $t_1 > 0$ is small enough, then

$$\mathbb{P}_{\tilde{\mathcal{X}}}\big(X^{(N)}(s) \notin C^\epsilon\big) < \frac{1}{3}\delta \quad \forall \tilde{\mathcal{X}} \in C, s \in [0, t_1]. \tag{94}$$

Indeed, the event that no particle branches on the time interval $[0, t_1]$ has probability e^{-Nt_1} which can be arbitrarily close to 1 if t_1 is small enough. Conditioned on this event, the random process $Y(s) = X^{(N)}(s) - X^{(N)}(0)$ is a Brownian motion in $\mathbb{R}^{dN}$. In particular, $Y(s)$ is almost surely continuous on $[0, t_1]$, with $Y_0 = 0$, and the law of $Y(s)$ does not depend on $X^{(N)}(0)$ or t_1. Then, $\mathbb{P}_{\tilde{\mathcal{X}}}(X^{(N)}(s) \in C^\epsilon) \ge e^{-Nt_1}\,\mathbb{P}(\|Y(s)\| < \epsilon\ \forall s \in [0, t_1])$ which can be made arbitrarily close to 1 by taking t_1 sufficiently small.

When choosing t_1 small enough for (94), we furthermore require that $t_0/t_1 \in \mathbb{N}$. It is then clear from (92) that $\pi_{t_1}^{(N)} = \pi_{t_0}^{(N)}$. Take $n_0 \in \mathbb{N}$ sufficiently large that, for $n \ge n_0$,

$$\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t_1 n) \in C^\epsilon\big) \le \pi_{t_0}^{(N)}(C^\epsilon) + \frac{1}{3}\delta \le \pi_{t_0}^{(N)}(C) + \frac{2}{3}\delta.$$

For $t \ge t_1 n_0$, we have

$$\begin{aligned}
\mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t) \in C\big) &\le \mathbb{P}_{\mathcal{X}}\big(X^{(N)}(\lceil t/t_1\rceil t_1) \in C^\epsilon\big) + \mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t) \in C, X^{(N)}(\lceil t/t_1\rceil t_1) \notin C^\epsilon\big)\\
&\le \pi_{t_0}^{(N)}(C) + \frac{2}{3}\delta + \mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t) \in C, X^{(N)}(\lceil t/t_1\rceil t_1) \notin C^\epsilon\big)\\
&\le \pi_{t_0}^{(N)}(C) + \delta
\end{aligned}$$

by the Markov property at time t and (94). Since $\delta > 0$ was arbitrary, it follows that

$$\limsup_{t\to\infty} \mathbb{P}_{\mathcal{X}}\big(X^{(N)}(t) \in C\big) \le \pi_{t_0}^{(N)}(C). \tag{95}$$

Comparing (95) and (92), we see that $\pi_{t_1}^{(N)}(C) \le \pi_{t_0}^{(N)}(C)$ for all closed sets C and all $t_1 \in (0, t_0]$. Hence, by the Portmanteau theorem, $\pi_{t_1}^{(N)} = \pi_{t_0}^{(N)}$ for all $t_1 \in (0, t_0]$. This proves (93), so we now write $\pi^{(N)}$ for the unique invariant measure of the process $(X^{(N)}(t), t \ge 0)$.

By the Markov property we have that, for any $\mathcal{X}_0 \in (\mathbb{R}^d)^N$, $D \subseteq (\mathbb{R}^d)^N$ and $t > s > 0$,

$$\mathbb{P}_{\mathcal{X}_0}(X^{(N)}(t) \in D) = \int_{(\mathbb{R}^d)^N} \mathbb{P}_{\mathcal{X}_0}(X^{(N)}(s) \in \mathrm{d}\mathcal{X}) \mathbb{P}_{\mathcal{X}}(X^{(N)}(t-s) \in D)$$

$$\text{and} \quad \pi^{(N)}(D) = \int_{(\mathbb{R}^d)^N} \pi^{(N)}(\mathrm{d}\mathcal{X}) \mathbb{P}_{\mathcal{X}}(X^{(N)}(t-s) \in D).$$

By taking the difference between these two equations,

$$\begin{aligned} &\left|\mathbb{P}_{\mathcal{X}_0}(X^{(N)}(t) \in D) - \pi^{(N)}(D)\right| \\ &\quad \le \int_{(\mathbb{R}^d)^N} \left|\mathbb{P}_{\mathcal{X}_0}(X^{(N)}(s) \in \mathrm{d}\mathcal{X}) - \pi^{(N)}(\mathrm{d}\mathcal{X})\right| \mathbb{P}_{\mathcal{X}}(X^{(N)}(t-s) \in D) \\ &\quad \le \int_{(\mathbb{R}^d)^N} \left|\mathbb{P}_{\mathcal{X}_0}(X^{(N)}(s) \in \mathrm{d}\mathcal{X}) - \pi^{(N)}(\mathrm{d}\mathcal{X})\right|, \end{aligned} \tag{96}$$

where the right-hand side is the total variation norm of the difference between $\pi^{(N)}$ and the law of $X^{(N)}(s)$ under $\mathbb{P}_{\mathcal{X}_0}$.

Now, choose $s = \lfloor t/t_0 \rfloor t_0$, and let $t \to \infty$. Since the law of $X^{(N)}(nt_0)$ under $\mathbb{P}_{\mathcal{X}_0}$ converges to $\pi^{(N)}$ as $n \to \infty$ in total variation norm, the right-hand side of (96) converges to zero as $t \to \infty$, and the result follows. □

PROOF OF THEOREM 1.4. Take $\epsilon > 0$ and $A \subseteq \mathbb{R}^d$ measurable. Let

$$D_\epsilon = \left\{ \mathcal{X} \in (\mathbb{R}^d)^N : \left| \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\mathcal{X}_i \in A\}} - \int_A U(x)\,\mathrm{d}x \right| \ge \epsilon \text{ or } \left| \max_{i \le N} \|\mathcal{X}_i\| - R_\infty \right| \ge \epsilon \right\}.$$

Take the initial condition $\mathcal{X} = (0, \dots, 0) \in (\mathbb{R}^d)^N$. By Theorem 1.3 for any $\delta \in (0, \epsilon)$, there exist $N_\delta, T_\delta < \infty$ such that, for $N \ge N_\delta$ and $t \ge T_\delta$,

$$\mathbb{P}_{\mathcal{X}}(X^{(N)}(t) \in D_\epsilon) \le \mathbb{P}_{\mathcal{X}}(X^{(N)}(t) \in D_\delta) < 2\delta.$$

But by Theorem 1.2,

$$\mathbb{P}_{\mathcal{X}}(X^{(N)}(t) \in D_\epsilon) \to \pi^{(N)}(D_\epsilon) \quad \text{as } t \to \infty.$$

It follows that $\pi^{(N)}(D_\epsilon) \le 2\delta$ for $N \ge N_\delta$, and so $\lim_{N\to\infty} \pi^{(N)}(D_\epsilon) = 0$ which completes the proof. □

Acknowledgments. The authors wish to thank Louigi Addario-Berry, Erin Beckman, Nathanaël Berestycki and Pascal Maillard for stimulating discussions at various points of this project.

Funding. The work of JN was partially funded through grant DMS-1351653 from the U.S. National Science Foundation.

REFERENCES

[1] ADDARIO-BERRY, L., LIN, J. and TENDRON, T. (2020). Barycentric Brownian bees. Preprint. Available at arXiv:2006.04743.

[2] ASSELAH, A., FERRARI, P. A., GROISMAN, P. and JONCKHEERE, M. (2016). Fleming–Viot selects the minimal quasi-stationary distribution: The Galton–Watson case. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 647–668. MR3498004 https://doi.org/10.1214/14-AIHP635

[3] ATHREYA, K. B. and NEY, P. (1978). A new approach to the limit theory of recurrent Markov chains. *Trans. Amer. Math. Soc.* **245** 493–501. MR0511425 https://doi.org/10.2307/1998882

[4] BECKMAN, E. (2019). Asymptotic Behavior of Certain Branching Processes Ph.D. thesis Duke Univ.

[5] BÉRARD, J. and GOUÉRÉ, J.-B. (2010). Brunet–Derrida behavior of branching-selection particle systems on the line. *Comm. Math. Phys.* **298** 323–342. MR2669438 https://doi.org/10.1007/s00220-010-1067-y

[6] BERESTYCKI, J., BRUNET, É., NOLEN, J. and PENINGTON, S. (2021). A free boundary problem arising from branching Brownian motion with selection. *Trans. Amer. Math. Soc.* **374** 6269–6329. MR4302161 https://doi.org/10.1090/tran/8370

[7] BERESTYCKI, J., BRUNET, É. and PENINGTON, S. (2019). Global existence for a free boundary problem of Fisher-KPP type. *Nonlinearity* **32** 3912–3939. MR4012576 https://doi.org/10.1088/1361-6544/ab25af

[8] BERESTYCKI, N. and ZHAO, L. Z. (2018). The shape of multidimensional Brunet–Derrida particle systems. *Ann. Appl. Probab.* **28** 651–687. MR3784486 https://doi.org/10.1214/14-AAP1062

[9] BRUNET, E., DERRIDA, B., MUELLER, A. H. and MUNIER, S. (2006). Noisy traveling waves: Effect of selection on genealogies. *Europhys. Lett.* **76** 1–7. MR2299937 https://doi.org/10.1209/epl/i2006-10224-4

[10] BRUNET, É., DERRIDA, B., MUELLER, A. H. and MUNIER, S. (2007). Effect of selection on ancestry: An exactly soluble case and its phenomenological generalization. *Phys. Rev. E* (3) **76** 041104, 20. MR2365627 https://doi.org/10.1103/PhysRevE.76.041104

[11] BURDZY, K., HOŁYST, R. and MARCH, P. (2000). A Fleming–Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.* **214** 679–703. MR1800866 https://doi.org/10.1007/s002200000294

[12] COLLET, P., MARTÍNEZ, S. and SAN MARTÍN, J. (2013). *Quasi-Stationary Distributions: Markov Chains, Diffusions and Dynamical Systems. Probability and Its Applications (New York)*. Springer, Heidelberg. MR2986807 https://doi.org/10.1007/978-3-642-33131-2

[13] DE MASI, A., FERRARI, P. A., PRESUTTI, E. and SOPRANO-LOTO, N. (2019). Hydrodynamics of the N-BBM process. In *Stochastic Dynamics Out of Equilibrium. Springer Proc. Math. Stat.* **282** 523–549. Springer, Cham. MR3986078 https://doi.org/10.1007/978-3-030-15096-9_18

[14] DE MASI, A., FERRARI, P. A., PRESUTTI, E. and SOPRANO-LOTO, N. (2019). Non local branching Brownian motions with annihilation and free boundary problems. *Electron. J. Probab.* **24** Paper No. 63, 30. MR3978213 https://doi.org/10.1214/19-EJP324

[15] DURRETT, R. and REMENIK, D. (2011). Brunet–Derrida particle systems, free boundary problems and Wiener–Hopf equations. *Ann. Probab.* **39** 2043–2078. MR2932664 https://doi.org/10.1214/10-AOP601

[16] EVANS, L. C. (2010). *Partial Differential Equations*, 2nd ed. *Graduate Studies in Mathematics* **19**. Amer. Math. Soc., Providence, RI. MR2597943 https://doi.org/10.1090/gsm/019

[17] GROISMAN, P. and JONCKHEERE, M. (2013). Front propagation and quasi-stationary distributions: The same selection principle? Preprint. Available at arXiv:1304.4847.

[18] GROISMAN, P. and JONCKHEERE, M. (2018). Front propagation and quasi-stationary distributions for one-dimensional Lévy processes. *Electron. Commun. Probab.* **23** Paper No. 93, 11. MR3896831 https://doi.org/10.1214/18-ECP199

[19] HARRIS, S. C. and ROBERTS, M. I. (2017). The many-to-few lemma and multiple spines. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 226–242. MR3606740 https://doi.org/10.1214/15-AIHP714

[20] JAGERS, P. (1989). General branching processes as Markov fields. *Stochastic Process. Appl.* **32** 183–212. MR1014449 https://doi.org/10.1016/0304-4149(89)90075-6

[21] MAILLARD, P. (2016). Speed and fluctuations of N-particle branching Brownian motion with spatial selection. *Probab. Theory Related Fields* **166** 1061–1173. MR3568046 https://doi.org/10.1007/s00440-016-0701-9

[22] MASSART, P. (1990). The tight constant in the Dvoretzky–Kiefer–Wolfowitz inequality. *Ann. Probab.* **18** 1269–1283. MR1062069