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On a model of directed compact animals

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Résumé. — On étudie un modèle d'animaux compacts dirigés qui se pose dans plusieurs problèmes tels : le nombre de partitions d'un entier, marches en spirales auto-évitantes, électrons en interactions en champ magnétique fort. Des formules asymptotiques, pour le nombre de ces animaux de N sites et de longueur L , et pour la longueur moyenne $\langle L \rangle$ à N fixé, sont obtenues analytiquement.

Abstract. — We study a model of directed compact animals which arises in various fields : partition of an integer, spiralling self avoiding walks, interacting electrons in a strong magnetic field. We give asymptotic formulae for the number of these animals with N sites and length L , and for the mean length $\langle L \rangle$ at a given N .

1. Introduction.

In this work we present some results on a model of compact directed animals. The problem consists in enumerating the number $\Omega(N, L)$ of clusters of N sites and length L , which can be constructed on the square lattice with the following rule (Fig. 1) : these clusters are piles such that on each column the height is at most equal to the height of the previous column. Such compact clusters have a preferred direction along the diagonal. These animals are also the so-called Ferrers diagrams in group theory, and this problem is equivalent to the enumeration of the partitions of an integer N into L non-increasing parts :

$$\Omega(N, L) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{n_1} \cdots \sum_{n_L=1}^{n_{L-1}} \delta_{n_1+n_2+\cdots+n_L, N} \quad (1)$$

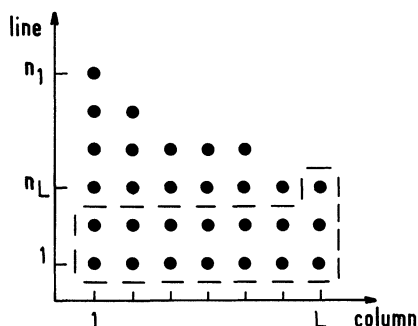


Fig. 1. — Compact directed animal of length L . The heights n_1 and n_L of the first and last columns are indicated. These animals are directed along the diagonal, if we consider the site $(1,1)$ as a source. By removing the sites which are surrounded by the dashed line, one obtains a cluster of length (exactly) $L - 1$ from a cluster of length L . This procedure is the basis of the recursion relation (6).

Old but astonishing results on the partitions of an integer have been obtained in particular by Ramanujan, Hardy and Andrews [1]. This problem is of physical interest because it arises in other various fields : a model of spiralling self-avoiding walks recently studied [2-5] ; some problems of interacting electrons in a strong magnetic field [6]. Known exact results [1] give the total number of partitions $p(N) = \sum_L \Omega(N, L)$, and show that for large N , $\log p(N)$ grows like \sqrt{N} (and not like N as for usual random walk or random cluster problems).

The result that we present here is an asymptotic formula for $S(N, L) = \log \Omega(N, L)$ when N and L go to infinity :

$$S = \frac{1}{\varepsilon} \int_0^{L\varepsilon} \frac{u e^{-u}}{1 - e^{-u}} du - \frac{1}{\varepsilon} \int_0^{L\varepsilon} du \log(1 - e^{-u}) \quad (2.a)$$

where ε is given by

$$N = \frac{1}{\varepsilon^2} \int_0^{L\varepsilon} \frac{u e^{-u}}{1 - e^{-u}} du. \quad (2.b)$$

2. Derivation of the formulae.

Let us now derive these expressions. First one introduces the generating function

$$G_L(z) = \sum_{N=1}^{\infty} \Omega(N, L) z^N \quad (3)$$

which can be written as

$$G_L(z) = \frac{z^L}{\prod_{n=1}^L (1 - z^n)}. \quad (4)$$

One way of proving (4) is to discriminate the clusters according to the height n_L on the last column. If $n_L = 1$, the cluster obtained by removing this site from the last column is a new directed compact cluster with length $(L - 1)$. Then clusters with $n_L = 1$ contribute in (3) to a factor

$zG_{L-1}(z)$. If $n_L > 1$, the cluster obtained by removing the first $(n_L - 1)$ lines is again a directed compact cluster of length L with only one site on the last column. Then clusters with a given n_L contribute to a factor $(z^L)^{n_L-1} \times (zG_{L-1}(z))$, and finally one has

$$G_L(z) = zG_{L-1}(z) + z^L zG_{L-1}(z) + \dots + (z^L)^p zG_{L-1}(z) + \dots \quad (5)$$

which gives

$$G_L(z) = \frac{z}{1 - z^L} G_{L-1}(z) \quad (6)$$

and, with $G_0(z) = 1$, one gets (4).

Now to obtain $\Omega(N, L)$ one can use the Cauchy formula :

$$\Omega(N, L) = \frac{1}{2\pi i} \oint_C \frac{dz}{z^{N+1}} G_L(z) \quad (7)$$

where C is a circle of radius $r < 1$. Writing $z = e^{-\varepsilon + i\theta}$ (with $r = e^{-\varepsilon}$), one has

$$\Omega(N, L) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \exp \left[(N - L)(\varepsilon - i\theta) - \sum_{n=1}^L \log(1 - e^{-n(\varepsilon - i\theta)}) \right].$$

For N and L large, this integral can be evaluated by the saddle point method. The saddle point is at $\theta = 0$, and at ε given by

$$N - L - \sum_{n=1}^L \frac{n e^{-n\varepsilon}}{1 - e^{-n\varepsilon}} = 0. \quad (8)$$

Then one gets

$$\Omega(N, L) \sim \frac{1}{\sqrt{2\pi B}} \exp \left[(N - L)\varepsilon - \sum_{n=1}^L \log(1 - e^{-n\varepsilon}) \right] \quad (9)$$

where ε is given by (8), and B results from the Gaussian integration near the saddle point :

$$B = \sum_{n=1}^L \frac{n^2 e^{-n\varepsilon}}{(1 - e^{-n\varepsilon})^2}. \quad (10)$$

In the large (N, L) limit one can replace the sums in (8), (9) and (10) by integrals, which gives formula (2). This formula (2. a) gives $\log \Omega(N, L)$ implicitly since we have to solve ε from (2. b). In general (8) is the simplest way of expressing ε in terms of N and L . However in some ranges of values of L , one can get an explicit formula for ε from (8) : these cases correspond to $L^2 \ll N$ (case i below), $N \ll L^2 \ll N^2$ (case ii), $L/\sqrt{N} = \gamma$ with γ finite (case iii) where we make expansions for γ small and γ large.

Case i) $L\varepsilon \ll 1$. (8) gives

$$N - L \sim L/\varepsilon \quad (11)$$

which means $L^2 \ll N$, and

$$\log \Omega(N, L) \sim 2L + L \log \frac{N}{L^2} \quad (12)$$

$$B \sim L/\varepsilon^2 \sim \frac{(N - L)^2}{L}.$$

Case ii) $L\varepsilon \gg 1$, but $N - L$ large enough. In this case one has

$$N - L \sim \frac{1}{\varepsilon^2} \frac{\pi^2}{6} \quad (13)$$

$$\log \Omega(N, L) \sim \frac{2}{\varepsilon} \frac{\pi^2}{6} \sim \sqrt{\frac{2}{3}} \pi \sqrt{N - L} \quad (14)$$

$$B \sim \pi^2/3 \varepsilon^3 \sim \frac{2\sqrt{6}}{\pi} (N - L)^{3/2}.$$

Case iii) $L\varepsilon \sim 1$, that is

$$L = \gamma \sqrt{N}. \quad (15)$$

Here we can expand S in the two limits ; first γ small, which gives

$$S \sim \sqrt{N} \left(-2\gamma \log \gamma + 2\gamma + \frac{\gamma^3}{4} \right) \quad (16)$$

and γ large :

$$S \sim \sqrt{N} \left(\sqrt{\frac{2}{3}} \pi - \frac{\sqrt{6}}{\pi} e^{-\gamma\pi/\sqrt{6}} \right) \quad (17)$$

Note that in any case, as $N \rightarrow \infty$, $\varepsilon \rightarrow 0$ (i.e., $z \rightarrow 1$).

3. Calculation in the grand canonical ensemble.

Now, in thermodynamic language, this calculation has been made in the canonical ensemble — that is with N fixed. We think it is worth showing the corresponding calculation in the grand canonical ensemble, for the two following reasons : first, it is simpler, and second it does give the same result in the thermodynamic limit, that is provided N and L go to infinity.

The generating function $G_L(z)$ is the analogy of a partition function and $\log z$ is a chemical potential for N . For a given z the mean value of N is

$$\langle N \rangle = \frac{\partial \log G_L(z)}{\partial \log z} = L + \sum_{n=1}^L \frac{nz^n}{1 - z^n}. \quad (18)$$

As there is no internal energy, the entropy S is obtained by a Legendre transform of $\log G_L(z)$:

$$S = \log G_L(z) - N \log z. \quad (19)$$

This entropy is also given by $S = \log \Omega(N, L)$. Then

$$\log \Omega(N, L) = - \log z \sum_{n=1}^L \frac{nz^n}{1 - z^n} - \sum_{n=1}^L \log(1 - z^n). \quad (20)$$

Formulae (18) and (20) are essentially the same as (8) and (9). (18) gives the correspondence between N and z , and we recover the same discussion as above. To understand the fact that the results do not depend on the ensemble (canonical or grand canonical) we can see if the relative

fluctuations of N vanish in the large N and L limit. Thus we compute

$$\langle N^2 \rangle - \langle N \rangle^2 = \frac{\partial^2 \log G_L(z)}{\partial (\log z)^2}$$

which is precisely the quantity B (relation (10)). This quantity, as calculated above, appears to be of order $\frac{\langle N \rangle^2}{L}$, in case i), which gives

$$\left(\frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} \right)^{1/2} \sim \frac{1}{\sqrt{L}},$$

and of order $(N - L)^{3/2}$ in case ii), which gives

$$\left(\frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} \right)^{1/2} \sim N^{-1/4}.$$

4. Mean length for a given number of sites.

One can also compute the mean value of L for a given N . This can be done by looking at the generating functions $H_N(y)$ and $T(y, z)$:

$$H_N(y) = \sum_{L=0}^{\infty} y^L \Omega(N, L) \quad (21)$$

$$T(y, z) = \sum_{L \geq 0} y^L G_L(z) = 1 + \sum_{N=1}^{\infty} z^N H_N(y) \quad (22)$$

first one obtains the expression of $T(y, z)$: from formula (5), by multiplying both sides by y^L , and by summing over L , one obtains

$$T(y, z) - 1 = zyT(y, z) + z^2 yT(yz, z) + \dots \quad (23)$$

and writing the same equation for $T(yz, z)$ and subtracting, one gets

$$T(y, z) - T(yz, z) = zyT(y, z)$$

that is

$$T(y, z) = \frac{1}{1 - zy} T(yz, z)$$

which gives

$$T(y, z) = \prod_{n=1}^{\infty} \frac{1}{1 - yz^n}. \quad (24)$$

Then as above there are two possibilities : either one can use a Cauchy formula to obtain H_N from T , and then use the saddle point method ; or one can work with the grand canonical ensemble, where $\log y$ and $\log z$ are chemical potentials for L and N . Again the two methods give the same

result in the large L, N limit. We just give the result, using the second formulation :

$$\log H_N = \log T(y, z) - N \log z$$

$$\langle L \rangle = \frac{\partial \log H_N}{\partial \log y}$$

and this gives, with $-\varepsilon = \log z$, $-\phi = \log y$

$$\log H_N = N\varepsilon - \sum_{n=1}^{\infty} \log(1 - e^{-n\varepsilon - \phi}).$$

In the large N limit one has to consider the limit $\phi \rightarrow 0$, $\varepsilon \rightarrow 0$ which gives

$$\langle L \rangle = \left(\frac{\sqrt{6}}{\pi} \sqrt{N} + \frac{3}{\pi^2} \right) \left(\log \left(\frac{\sqrt{6}}{\pi} \sqrt{N} \right) - \Gamma'(1) \right) + \frac{3}{\pi^2} \quad (25)$$

where $\Gamma'(1)$ is the derivative of the gamma function at 1 :

$$\Gamma'(1) = -0.5772157.$$

5. Conclusion.

We have derived for a model of directed compact animals asymptotic formulae for the number of animals with N sites and length L , and for the mean length at fixed N . Moreover we have shown that for this problem, where the analogue of a free energy is not extensive, the canonical and grand canonical ensemble are still equivalent in the large L and N limit.

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