

INTRODUCTION TO QUANTUM EXCLUSION PROCESSES

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ABSTRACT. The QSSEP, short for quantum symmetric simple exclusion process, is a paradigm model for stochastic quantum dynamics. Averaging over the noise, the quantum dynamics reduce to the well-studied SSEP (symmetric simple exclusion process). These notes provide an introduction to quantum exclusion processes, focusing on the example of QSSEP and its connection to free probability, with an emphasis on mathematical aspects.

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1. INTRODUCTION

The initial motivation for introducing quantum exclusion processes was to develop a quantum extension of the macroscopic fluctuation theory (MFT), which is a general framework to deal with diffusive system driven out of equilibrium introduced in [BDGJL01; BDGJL05]. Beyond this initial motivation, quantum exclusion processes are a good playground to extend the tools used in the large scale analysis of classical stochastic systems. In these notes, we mostly discuss the simplest model, the Quantum Simple Symmetric Exclusion Process (QSSEP). The QSSEP is a noisy and quantum analogue of the Symmetric Simple Exclusion Process (SSEP). The average over the noise of QSSEP reduces to its classical counterpart, the SSEP. As we will see, its analysis displays clear signs of integrability which deserve to be better understood, and is closely connected to the theory of random matrices and free probability [BJ19; Bia25; HB22]. Actually, free probability also appears in other studies of quantum systems, notably in the context of the eigenstate thermalization hypothesis [PFK22].

1.1. Towards a quantum extension of Macroscopic Fluctuation Theory. Let us elaborate on the initial motivation of finding a quantum extension to MFT [BBJ17]. Consider a stochastic particle system on a lattice, let's say a Markovian exclusion process on N sites between two reservoirs at densities $n_a \neq n_b$. At large scale, the system is often described by a density function $n(x, s)$ where the variables $x \in [0, 1]$ and $s > 0$ denote a macroscopic position x and rescaled time s : $n(x, s)$ is the average number of particles per site near position $\lfloor xN \rfloor$ at time sN^2 . Such hydrodynamic limit is generally based on the fact that locally, the system converges to a stationary measure corresponding to a certain density, which may however vary at large scale. This can be completely formalized mathematically [KL98]. If $j(x, s)$ denotes the average flux near the macroscopic position x , the conservation of particles imposes that

$$\partial_s n(x, s) + \partial_x j(x, s) = 0. \tag{1.1}$$

Macroscopic fluctuation theory allows to predict large deviation principles for various quantities related to transport in diffusive systems. The main idea is to introduce a noisy version of the Fourier-Fick's law

$$j(x, s) = -D(n(x, s))\partial_x n(x, s) + \sqrt{\frac{1}{N}}\sqrt{\sigma(n(x, s))}\xi(x, s), \quad (1.2)$$

where ξ is a space-time white noise, which induces a probability measure on density and current profiles. Here $D(n)$ is the diffusion constant and $\sigma(n)$ is called the mobility, we do not define them. Since the noise variance is of order $\frac{1}{N} \ll 1$, density and current fluctuations are small and rare. This framework allows to find a functional \mathcal{I} , called rate function, sometimes completely explicit, such that under the stationary measure of the process,

$$\mathbf{P} \left[(n(x))_{x \in [0,1]} \approx f \right] \approx e^{-N\mathcal{I}(f)},$$

to be understood using the formalism of large deviation theory — we refer to [BDGJL14] for a review on MFT. From the mathematical point of view, the validity of MFT in general is only conjectural, but its predictions have been rigorously verified on many specific models (including SSEP).

A quantum extension of MFT for diffusive quantum systems should describe:

- (1) transport and their fluctuations;
- (2) quantum effects (interferences, coherences) and their fluctuations.

We refer to the review [Ber21] for further physical background. The first task is to define quantum extensions of interacting particle systems. In these notes we focus on exclusion type systems, and often restrict to the guiding example of the symmetric simple exclusion process. Before defining it in Section 2, we need to explain what is the state space in the quantum setting, and how dynamics are defined.

1.2. The state space for quantum systems. The particle configuration state space \mathcal{C} of the classical system becomes a $*$ -algebra \mathcal{A} . The reader unfamiliar with this formalism may think of \mathcal{A} as a set of linear operators, acting on some Hilbert space, and their adjoints. A probability measure on \mathcal{C} becomes a *state* on \mathcal{A} , that is a positive unital $*$ -homomorphism $\nu : \mathcal{A} \rightarrow \mathbb{C}$, i.e. an algebra homomorphism satisfying $\nu(a^*) = \overline{\nu(a)}$, $\nu(1) = 1$, $\nu(a^*a) > 0$. Typically, the algebra \mathcal{A} is the Von Neumann algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of bounded linear operators on some Hilbert space \mathcal{H} . When $\dim \mathcal{H}$ is finite, states may be encoded by a density matrix ϱ (positive definite Hermitian) such that

$$\nu(a) = \text{Tr}(\varrho a).$$

Example 1.1. For the SSEP on N sites, we denote configurations by $\mathbf{n} = (n_1, \dots, n_N) \in \mathcal{C} = \{0, 1\}^N$. Let \mathcal{H} be a Hilbert space with basis $(|\mathbf{n}\rangle)_{\mathbf{n} \in \mathcal{C}}$ indexed by classical configurations. Here we use Dirac's notation: a so-called ket $|u\rangle$ denotes a vector in some Hilbert space. A bra $\langle v|$ is a complex number, the scalar product of $|u\rangle$ and $|v\rangle$. Hence we may view the bra $\langle v|$ as a linear form $\mathcal{H} \rightarrow \mathbb{C}$. When \mathcal{H} is finite dimensional, one should think of $\langle v|$ as being the Hermitian transpose of the conjugate of $|v\rangle$, i.e. a row vector. Finally, $|u\rangle\langle v|$ is a rank one linear operator on the Hilbert space. Since $\dim(\mathcal{H}) = 2^N$, we can focus on the $2^N \times 2^N$ density matrix ϱ . Note that if ϱ is diagonal in the basis $(|\mathbf{n}\rangle)_{\mathbf{n} \in \mathcal{C}}$, it can be written as

$$\varrho = \sum_{\mathbf{n} \in \mathcal{C}} \mathbf{P}(\mathbf{n}) |\mathbf{n}\rangle\langle \mathbf{n}|,$$

where $\mathbf{P}(\mathbf{n})$ is a probability distribution. Indeed, the condition $\nu(1) = 1$ is equivalent to $\text{Tr}(\varrho) = 1$, that is $\sum_{\mathbf{n} \in \mathcal{C}} \mathbf{P}(\mathbf{n}) = 1$, and positivity implies $0 \leq \mathbf{P}(\mathbf{n}) \leq 1$.

1.3. Stochastic quantum dynamics. We also need to define dynamics on states, instead of Markovian dynamics on probability measures. In general, quantum dynamics on quantum states can either be unitary flows for closed systems, or flows of so-called trace preserving completely positive maps (also called CP-maps or quantum channels) for open systems, since then the dynamics are in general not unitary to take into account additional dissipative processes related to the contacts with external reservoirs. Here, these dynamics will be themselves random, hence we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We wish to define a random state ν_t^ω for each $\omega \in \Omega$ (or a random density matrix ϱ_t^ω). The dynamics are constructed using random unitary flows for closed systems, and flows of random CP-maps for open systems.

We moreover impose that our quantum stochastic dynamics should be related to the classical one, in the sense that the average quantum dynamics

$$t \rightarrow \bar{\varrho}_t := \mathbb{E} [\varrho_t]$$

should contain, as a marginal, the classical dynamics.

We emphasize that there are two levels of randomness when dealing with quantum stochastic dynamics:

- ▶ Quantum randomness, encoded by ν (or ϱ). The quantum expectation value is $\nu(a)$.
- ▶ Classical randomness, coming from the fact that ν depends on $\omega \in \Omega$, associated to the probability measure \mathbb{P} and expectation \mathbb{E} – note that we use a different notation than for the probability measure \mathbf{P} and expectation \mathbf{E} associated to the classical SSEP.

Thus we should consider quantities such as

$$\mathbb{E} [\nu_t(a)], \quad \mathbb{E} [\nu_t(a_1)\nu_t(a_2)], \dots, \quad \mathbb{E} [\nu_t(a_1) \dots \nu_t(a_n)].$$

Furthermore, it may also be interesting to see how the dynamics may be influenced by a sequence of quantum measurements, but we will not discuss this here.

Remark 1.2. From a physics perspective, the origin of randomness may be understood from two different perspectives:

- ▶ Either as a way to describe generic systems;
- ▶ Or as arising from interaction with hidden degrees of freedom.

The later case is particularly adapted to describe diffusive systems. Indeed a physical particle system behaves diffusively only above a certain scale, the so-called *mean free path* which is the distance separating two collisions of the system's particle on obstacles (think about the passage from random walks to the Brownian motion). Thus, if we are aiming at describing diffusive systems, there should be low space-time scale degrees of freedom inducing these many collisions. Furthermore, after averaging over those degrees of freedom, the systems is expected to behave classically – in the physics literature, this phenomena of destructive interference induced by collisions is called decoherence. This echoes in the above condition that the average dynamics should reduce to the classical one.

Remark 1.3. For the quantum SSEP, the density matrix ϱ_t is a $2^N \times 2^N$ matrix. As we have mentioned, when it is diagonal, it describes a probability measure on configurations, but in the quantum setting we may also look at combinations of classical configurations such as

$$|010011100\rangle + |010001100\rangle,$$

and study interferences. Hence the off-diagonal nature of ϱ_t is important and codes for quantum effects.

1.4. Quantum exclusion processes. In the case of quantum exclusion processes on N sites with periodic boundary conditions, the system is closed and the dynamics are unitary, in the sense that

$$\varrho_t = U_t \varrho_0 U_t^*,$$

where $U_t = U_t(\omega)$, $\omega \in \Omega$, is a unitary matrix described by a stochastic differential equation. We consider Itô type SDEs of the form

$$\varrho_{t+dt} = e^{-idH_t} \varrho_t e^{idH_t},$$

for some hermitian matrix Hamiltonian increments dH_t . By definition, the average over the noise $\overline{\varrho}_t$ is

$$\overline{\varrho}_t = \int_{\Omega} d\mathbb{P}(\omega) U_t(\omega) \varrho_0 U_t(\omega)^* =: \overline{\Phi}_t(\varrho_0).$$

This is a convex combination of unitary transformations, but it is not unitary in general. The map $\varrho \mapsto \overline{\Phi}_t(\varrho)$ is an example of a (special class of) trace preserving completely positive map, i.e. $\text{Tr}(\overline{\Phi}(\varrho)) = 1$ for $\text{Tr}(\varrho) = 1$, and $\overline{\Phi}(\varrho) \geq 0$ for $\varrho \geq 0$. Technically, the definition of CP-maps involves a stronger notion than the simple positivity, but we don't need to enter into these details – see [Att, chapter 6] for the more complete definition.

In the case of open quantum exclusion processes, the above unitary flows are replaced by flows of CP-maps $\varrho \rightarrow \varrho_t = \Phi_t(\varrho_0)$, with $\Phi_t = \Phi_t(\omega)$, $\omega \in \Omega$. The convex combination of CP-maps is again a CP-map so that the average flow is also a CP-map.

As already mentioned, we require that the semi group $\overline{\Phi}_t$ encodes the classical dynamics. This means that $\overline{\Phi}_t$ should preserve the class of diagonal matrices (in the basis $(|\mathbf{n}\rangle)_{\mathbf{n} \in \mathcal{C}}$) and the associated flow on diagonal matrices should be the same as the evolution of probability distributions $\mathbf{P}_t(\mathbf{n})$ in the classical exclusion process.

For instance, in the special case of the SSEP, it implies that

$$\mathbf{E}_{\text{ssep}} \left[e^{\sum_{j=1}^N h_j n_j(t)} \right] = \mathbb{E} \left[\text{Tr} \left(\varrho_t e^{\sum_{j=1}^N h_j \hat{n}_j} \right) \right], \quad (1.3)$$

where \hat{n}_j denotes the number operator, i.e. the diagonal matrix with diagonal coefficients 0 or 1 (in the basis $(|\mathbf{n}\rangle)_{\mathbf{n} \in \mathcal{C}}$) depending on the state of the j th site.

1.5. A curiosity about the classical SSEP. As t goes to infinity, the average density matrix $\overline{\varrho}_t$ must converge to the unique stationary measure of the SSEP. Under this stationary measure, it is well-known that,

$$\mathbf{E}_{\text{ssep}} \left[e^{\sum_{j=1}^N h_j n_j} \right] \asymp_{N \rightarrow \infty} e^{N \mathcal{F}_{\text{ssep}}(h)},$$

in the sense that the logarithm of both sides are asymptotically equivalent, where the function h is a continuum limit of h_j in a suitable sense (say $h_j = h(j/N)$ for some smooth function h). The statement was first derived in [DLS01a], and later with various levels of mathematical rigor in [DLS01b; BDGJL03; BDGJL07; BD22]. Via (1.3), the analysis of QSSEP led [BBBH22] to a new expression for the SSEP large deviation rate function (here we consider the system with boundary densities $n_a = 0, n_b = 1$):

$$\mathcal{F}_{\text{ssep}}(h) = \sup_{a,b} \left\{ \int_0^1 dx \left[\log \left(1 + b(x)(e^{h(x)} - 1) - a(x)b(x) \right) \right] + \tilde{\mathcal{F}}_0(a) \right\}, \quad (1.4)$$

where

$$\tilde{\mathcal{F}}_0(a) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \kappa_k \left(\int_x^1 dy a(y) \right),$$

and κ_k denotes the k th free cumulant defined in Section 3.1. The map $x \mapsto \int_x^1 dy a(y)$ should be viewed as a random variable on the space $[0, 1]$ equipped with the Lebesgue measure. The variational problem in (1.4) does not involve any boundary condition on a, b and should be understood in a sense explained below in Section 4.

Remark 1.4. Further, the analysis of QSSEP suggests that, for a generic quantum diffusive system (at large time), the large deviation fluctuations of quantum observables related to density or transport are those of its classical system counterpart.

- In particular, it has been conjectured that, under mild hypotheses, we have the \mathbb{P} -almost sure convergence

$$\frac{1}{N} \log \text{Tr} \left(\varrho_{\text{eq}} e^{\sum_{j=1}^N h_j \hat{n}_j} \right) \xrightarrow{N \rightarrow \infty} \mathcal{F}_Q(h),$$

where ϱ_{eq} denotes a random density matrix whose probability distribution is left invariant by the stochastic quantum dynamics and the functional \mathcal{F}_Q is deterministic, the same as for the classical system i.e. $\mathcal{F}_Q = \mathcal{F}_{\text{classical}}$. However, sub-leading corrections (in $1/\sqrt{N}$) are expected to be fluctuating and different in the classical and quantum processes. A similar conjecture is expected to hold for the large deviation functions for transport quantities (e.g. current) [HB22; MDGV23].

- Furthermore, it has also been conjectured in [HB22] that, if the macroscopic fluctuation theory applies to density fluctuations, then the coherent off-diagonal matrix elements of ϱ_{eq} should obey a large deviation principle (with free probability structures).

2. QUANTUM SYMMETRIC SIMPLE EXCLUSION PROCESS

As its classical counterpart, QSSEP can be defined on any graph. Here we will focus on finite graphs:

- (1) A chain of N sites indexed $1, \dots, N$ with periodic boundary conditions;
- (2) A closed chain of N sites without imposing periodic boundary conditions;
- (3) A chain of N sites connected to boundary reservoirs, which we call open boundary conditions.

As we will see, the analysis of the first two cases is very similar, while the third case is much richer. Since we have N sites in all cases, the Hilbert space $\mathcal{H} \cong (\mathbb{C}^2)^{\otimes N}$, and the system is described by a $2^N \times 2^N$ density matrix ϱ_t .

2.1. Periodic boundary conditions. In that case, the system is not in contact with external reservoirs and form a closed system. The evolution is a stochastic unitary dynamics.

2.1.1. Definition of the model. The time evolution of the density matrix $\varrho_t = \varrho_t^\omega$ is defined by the matrix stochastic differential equation

$$d\varrho_{t+dt} = e^{-idH_t} \varrho_t e^{idH_t}, \quad (2.1)$$

with Hamiltonian increment

$$dH_t = \sum_{j=1}^N c_{j+1}^\dagger c_j dW_t^{(j)} + c_j^\dagger c_{j+1} d\bar{W}_t^{(j)},$$

where $(W_t^{(j)})_{1 \leq j \leq N}$ are independent complex Brownian motions and $\bar{W}_t^{(j)}$ denote their complex conjugate, so that we have

$$\langle W_t^{(j)}, \bar{W}_t^{(k)} \rangle_t = \delta_{jk} t, \quad \langle W_t^{(j)}, W_t^{(k)} \rangle_t = 0, \quad \langle \bar{W}_t^{(j)}, \bar{W}_t^{(k)} \rangle_t = 0, \quad (2.2)$$

where δ_{jk} is the Kronecker delta symbol, equal to 1 when $j = k$ and 0 otherwise, and $\langle X, Y \rangle_t$ denotes the quadratic (co)variation of stochastic processes X_t and Y_t . There is one complex Brownian motion per edge.

The c_j^\dagger and c_k are respectively fermionic creation and annihilation operators, satisfying anti-commutation relations

$$\{c_j^\dagger, c_k\} = \delta_{jk}, \quad \{c_j^\dagger, c_k^\dagger\} = \{c_j, c_k\} = 0.$$

Here $\{a, b\}$ is the anti-commutator $ab + ba$. When $N = 1$, c and c^\dagger are given by the 2×2 matrices

$$c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad c^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so that $c^\dagger|0\rangle = |1\rangle$ and $c|1\rangle = |0\rangle$ if $|0\rangle = (1, 0)^T$ and $|1\rangle = (0, 1)^T$. We remark that the symbol \dagger denotes, in fact, the Hermitian conjugation, which we could denote by $*$. We keep using the symbol \dagger as this is the most common when dealing with creation and annihilation operators. When $N > 1$, one should define the action of operators c_j and c_k^\dagger on $(\mathbb{C}^2)^{\otimes N}$. Concretely, they have a block structure acting independently on each component of the tensor product:

$$c_k = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{k-1 \text{ times}} \otimes c \otimes \underbrace{I \otimes \cdots \otimes I}_{N-k \text{ times}},$$

and c_k^\dagger is the conjugate – see e.g. [Ste20, Appendix A]. We will see another equivalent definition in Section 2.1.4 with a more transparent algebraic interpretation.

By Itô's formula, (2.1) can be rewritten as

$$d\rho_t = -\mathbf{i}[dH_t, \rho_t] - \underbrace{\frac{1}{2}[dH_t, [dH_t, \rho_t]]}_{\text{Itô contraction}},$$

where the double commutator is computed by applying the rules (2.2), i.e. that all quadratic terms in $dW_t^{(j)}$ or $d\bar{W}_t^{(j)}$ are replaced by the differential of their quadratic (co)variation (these are the rules of Itô's calculus). This description shows that ρ_t solves a matrix valued linear stochastic differential equation, hence it admits a unique strong solution.

2.1.2. Average dynamics. We can now easily describe the average dynamics $\bar{\rho}_t = \mathbb{E}[\rho_t]$. We have

$$d\bar{\rho}_t = -\frac{1}{2}\mathbb{E}[dH_t, [dH_t, \bar{\rho}_t]] =: \bar{\mathcal{L}}(\bar{\rho}_t) dt, \quad (2.3)$$

where, again, the double commutator is simplified using (2.2). The linear operator $\bar{\mathcal{L}}$ is called a *Lindbladian*. This is the generator associated to the semi-group $\bar{\Phi}_t$ of trace preserving completely positive maps,

$$\bar{\rho}_t = e^{t\bar{\mathcal{L}}}\rho_0 = \bar{\Phi}_t(\rho_0).$$

Explicitly, the Lindbladian is given by $\bar{\mathcal{L}} = \vec{\mathcal{L}} + \overleftarrow{\mathcal{L}}$ with

$$\vec{\mathcal{L}}\rho = \sum_{j=1}^N \ell_j \rho \ell_j^\dagger - \frac{1}{2} \left(\ell_j^\dagger \ell_j \rho + \rho \ell_j^\dagger \ell_j \right),$$

where $\ell_j = c_{j+1}^\dagger c_j$. The operator $\overleftarrow{\mathcal{L}}$ is defined similarly, simply exchanging the roles of ℓ_j and ℓ_j^\dagger . Both $\vec{\mathcal{L}}$ and $\overleftarrow{\mathcal{L}}$ preserve density matrices ρ_{diag} diagonal in the particle number basis $|\mathbf{n}\rangle$, $\mathbf{n} \in \mathcal{C}$. As mentioned above, such density matrix specifies a probability measure on classical configurations since it can be written as $\rho_{\text{diag}} = \sum_{\mathbf{n} \in \mathcal{C}} \mathbf{P}(\mathbf{n}) |\mathbf{n}\rangle \langle \mathbf{n}|$. To see how

$\overline{\mathcal{L}}$ acts on ϱ_{diag} , it is enough to check how it acts on any edge configuration (since $\overline{\mathcal{L}}$ is the sum of terms acting on edges). We have:

$$\begin{aligned}\overline{\mathcal{L}}(|00\rangle\langle 00|) &= 0, \\ \overline{\mathcal{L}}(|01\rangle\langle 01|) &= |10\rangle\langle 10| - |01\rangle\langle 01|, \\ \overline{\mathcal{L}}(|10\rangle\langle 10|) &= |01\rangle\langle 01| - |10\rangle\langle 10|, \\ \overline{\mathcal{L}}(|11\rangle\langle 11|) &= 0.\end{aligned}$$

Thus $\overline{\mathcal{L}}$ acts as the Markov generator of SSEP on diagonal density matrices.

Remark 2.1. Since ϱ_t is a random variable, one may also look at the evolution of its higher moments. Those are going to evolve linearly in time, as for the average (2.3), but with a Lindblad operator depending on the order of the moment.

2.1.3. *An important property.* The QSSEP dynamics possesses a $U(1)^N$ invariance. This means that we can change c_j into $c_j e^{i\theta_j}$, $\theta_j \in \mathbb{R}$, (so that c_j^\dagger is changed into $c_j^\dagger e^{-i\theta_j}$) without changing the law of the states ϱ_t because such transformation can be absorbed in a redefinition of the Brownian motions without changing their law.

2.1.4. *Fock space formalism and further remarks.* It may be convenient to work with the Fock space representation of the Hilbert space \mathcal{H} spanned by $|\mathbf{n}\rangle$, $\mathbf{n} \in \mathcal{C}$, considered in the previous section. It is isomorphic to the exterior algebra $\bigwedge V$ where $V = \mathbb{C}^N$ is the one-particle Hilbert space, i.e.

$$\mathcal{H} \cong \bigwedge V = \mathbb{C} \oplus V \oplus V \wedge V \oplus V \wedge V \wedge V \oplus \dots$$

We then view \mathbb{C} as spanned by zero particle configurations, V spanned by one particle configurations, $V \wedge V$ by two particle configurations, etc. In this representation, the creation and annihilation fermion operator act via wedge products or contractions on elements of the exterior algebra $\bigwedge V$. More precisely, if we denote by $(e_i)_{1 \leq i \leq N}$ an orthonormal basis of V , we define $c_i^\dagger = c^+(e_i)$ and $c_i = c^-(e_i)$ where the linear operators $c^+(u)$ and $c^-(u)$ are defined for $u \in V$ as

$$\begin{aligned}c^+(u)(u_1 \wedge \dots \wedge u_n) &= u \wedge u_1 \wedge \dots \wedge u_n, \\ c^-(u)(u_1 \wedge \dots \wedge u_n) &= \sum_{i=1}^n (-1)^i \langle u, u_i \rangle u_1 \wedge \dots \wedge \widehat{u}_i \wedge \dots \wedge u_n,\end{aligned}$$

where \widehat{u}_i indicates that the contribution of u_i is absent in the wedge product. The local number operators \hat{n}_j can be written as $\hat{n}_j = c_j^\dagger c_j$. They count the number of particles at site j . Again, we refer to [Mey95] for a more complete presentation.

Remark 2.2. By the conservation of particles, for each $1 \leq k \leq N$, the subspace $V^{\wedge k}$ is stable under the dynamics, hence U_t is a block matrix. Restricting to the sub-block corresponding to V , i.e. the one particle dynamics, the Hamiltonian increment is given by

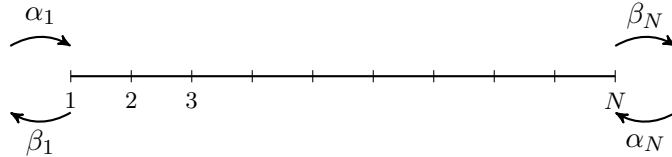
$$dh_t := dH_t|_V = \begin{pmatrix} 0 & dW_t^{(1)} & 0 & \dots & 0 & d\overline{W}_t^{(N)} \\ d\overline{W}_t^{(1)} & 0 & dW_t^{(2)} & 0 & \dots & 0 \\ 0 & d\overline{W}_t^{(2)} & 0 & dW_t^{(3)} & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & 0 & d\overline{W}_t^{(N-1)} \\ dW_t^{(N)} & 0 & \dots & 0 & d\overline{W}_t^{(N-1)} & 0 \end{pmatrix}. \quad (2.4)$$

Remark 2.3. If we replace the fermionic Fock space and creation/annihilation operators by bosonic analogues, QSSEP becomes the quantum symmetric simple inclusion process (QSSIP). As the name suggests, when averaging over the noise, it reduces to the symmetric simple inclusion process [BJSW25].

Remark 2.4. Given the amount of interest in the asymmetric simple exclusion process (ASEP) and in particular in its the totally asymmetric version (TASEP), it would be natural to define a quantum analogue. It may be checked that that $\overleftarrow{\mathcal{L}}_{\text{ASEP}} = p\overrightarrow{\mathcal{L}} + q\overleftarrow{\mathcal{L}}$ acts as the ASEP Markov matrix on diagonal density matrices, which implies that one needs to consider a family of independent *non-commutative* Brownian motions, i.e. Brownian motions such that Itô contractions are computed using quadratic variations $\langle W_t^{(j)}, \overline{W}_t^{(j)} \rangle_t = pt$ and $\langle \overline{W}_t^{(j)}, W_t^{(j)} \rangle_t = qt$, see [JKB20; RE21].

Remark 2.5. One may wonder whether the QSSEP admit a natural discrete analogue. Of course, one may consider discrete dynamics $\varrho_{t+1} = e^{-i(H_{t+1}-H_t)}\varrho_t e^{i(H_{t+1}-H_t)}$ where H_t is the same as before, or replacing Brownian motions by random walks. However, the averaged dynamics $\overline{\varrho}_t$ would no longer be described as a discrete particle system with local moves. Alternatively, one can consider random unitary circuits (locally conserving the number of particles) whose average dynamics are expected to be described by the macroscopic fluctuation theory of SSEP at large time [GH19; MDGV23]. More generally, random quantum circuits are examples of discrete time stochastic quantum dynamics – see the survey [FKNV23].

2.2. Open boundary conditions. Let us now consider the system in contact with reservoirs at its boundary, which we call open boundary conditions. In the classical case, particles may enter or exit the system at sites 1 and N at certain exponential rates, as depicted below:



The dynamics of the open quantum SSEP is defined in a similar fashion as in the closed case, except at the boundary where the gain/loss process are defined using other CP maps [BJ19]. The dynamics are defined by

$$d\varrho_t = -\mathbf{i}[dH_t, \varrho_t] - \frac{1}{2}[dH_t, [dH_t, \varrho_t]] + \mathcal{L}_{\text{bdry}}\varrho_t dt,$$

where now

$$dH_t = \sum_{j=1}^{N-1} c_{j+1}^\dagger c_j dW_t^{(j)} + c_j^\dagger c_{j+1} d\overline{W}_t^{(j)},$$

with again one complex Brownian motion per edge, and $\mathcal{L}_{\text{bdry}}$ is a boundary Lindbladian term

$$\mathcal{L}_{\text{bdry}} = \alpha_1 \mathcal{L}_1^+ + \beta_1 \mathcal{L}_1^- + \alpha_N \mathcal{L}_N^+ + \beta_N \mathcal{L}_N^-,$$

where (for $i = 1, N$ the boundary sites)

$$\mathcal{L}_i^+ \varrho = c_i^\dagger \varrho c_i - \frac{1}{2} \left(c_i c_i^\dagger \varrho + \varrho c_i c_i^\dagger \right),$$

and \mathcal{L}_i^- is defined similarly exchanging c_i and c_i^\dagger . The operator \mathcal{L}_i^+ (resp. \mathcal{L}_i^-) injects (resp. extracts) particle at the boundary site i at unit rate, since $\mathcal{L}_i^+ |0\rangle \langle 0| = |1\rangle \langle 1| - |0\rangle \langle 0|$ and $\mathcal{L}_i^+ |1\rangle \langle 1| = 0$ (and similarly for \mathcal{L}_i^-).

Remark 2.6. The restriction of the Hamiltonian increment to the one-particle sector is slightly different in the open case, compared to the closed case, to take the boundary conditions into account:

$$dh_t := dH_t|_V = \begin{pmatrix} 0 & dW_t^{(1)} & 0 & \dots & 0 & 0 \\ d\bar{W}_t^{(1)} & 0 & dW_t^{(2)} & 0 & \dots & 0 \\ 0 & d\bar{W}_t^{(2)} & 0 & dW_t^{(3)} & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & 0 & dW_t^{(N-1)} \\ 0 & 0 & \dots & 0 & d\bar{W}_t^{(N-1)} & 0 \end{pmatrix}. \quad (2.5)$$

2.3. Another approach via Wick's theorem. Let us assume that the initial state has the form

$$\varrho_0 = \frac{1}{Z} e^{-\sum_{i,j=1}^N c_i^\dagger M_{i,j} c_j}, \quad Z = \text{Tr} \left(e^{-\sum_{i,j=1}^N c_i^\dagger M_{i,j} c_j} \right). \quad (2.6)$$

for some Hermitian matrix M , possibly random. This is a reasonable assumption because this class is preserved by the dynamics. Further, we believe that the stationary measure of the system is unique and does belong to that class.

For any t , the quantum probability that there are particles at locations j_1, \dots, j_k is $\text{Tr}(\varrho_t \hat{n}_{j_1} \dots \hat{n}_{j_k})$, with number operators $\hat{n}_j = c_j^\dagger c_j$ (note that they are projectors, $\hat{n}_j^2 = \hat{n}_j$). By Wick's theorem (for distinct indices j_k)

$$\text{Tr}(\varrho_t \hat{n}_{j_1} \dots \hat{n}_{j_k}) = \det \left(\text{Tr}(\varrho_t c_n^\dagger c_m) \right)_{n,m=1}^k. \quad (2.7)$$

This means that \mathbb{P} -almost surely, the quantum SSEP configuration at any time is a determinantal point process (DPP) on $\{1, \dots, n\}$ with a random correlation kernel

$$G_t(i, j) := \text{Tr}(\varrho_t c_i^\dagger c_j).$$

It is remarkable that the classical SSEP particle locations do not form a determinantal process but they may be realized as an explicit average of random determinantal processes, given by particles locations in QSSEP.

For more general observables of the form $O = e^{\sum_{i,j=1}^N c_i^\dagger A_{i,j} c_j}$, we have

$$\text{Tr}(\varrho_t O) = \det[I + G_t(e^A - I)], \quad (2.8)$$

so that ϱ_t is determined (at least its action on observables of the above form) by the random $N \times N$ matrix G . We note that (2.8) implies (2.7), by considering an expansion for small A . We thus have reduced the problem of understanding the $2^N \times 2^N$ density matrix ϱ_t to the control of the $N \times N$ two-point correlation matrix G_t .

For closed or periodic boundary conditions, the evolution of G_t is unitary,

$$G_{t+dt} = e^{-i dh_t} G_t e^{i dh_t}, \quad (2.9)$$

where dh_t is the reduction of the Hamiltonian increment dH_t to the one-particle sector,

$$dh_t = \sum_{j=1}^N E_{j,j+1} dW_t^{(j)} + E_{j+1,j} d\bar{W}_t^{(j)}, \quad (2.10)$$

where $E_{i,j}$ is the $N \times N$ matrix with zero entries except a 1 in position (i, j) modulo N . In other terms, dh_t is given by (2.4) (periodic boundary conditions) or (2.5) (closed chain without reservoirs). Again, by Itô's formula,

$$dG_t = -\mathbf{i}[dh_t, G_t] - \frac{1}{2} \underbrace{[dh_t, [dh_t, G_t]]}_{\text{Itô contraction}}. \quad (2.11)$$

In the open boundary case, one needs to adapt the one-particle Hamiltonian increment so that dh_t is then given by (2.5) and to add boundary terms. Each coefficient of G_t obeys a stochastic differential equation of the form (see Supplementary Materials in the Arxiv version of [BJ19])

$$dG_t(i, i) = \text{unitary dynamics} + \sum_{p \in \{1, N\}} \delta_{pi} (\alpha_p (1 - G_t(i, i)) - \beta_p G_t(i, i)) dt, \quad (2.12a)$$

$$dG_t(i, j) = \text{unitary dynamics} - \frac{1}{2} \sum_{p \in \{1, N\}} (\alpha_p + \beta_p) (\delta_{pi} + \delta_{pj}) G_t(i, j) dt. \quad (2.12b)$$

Example 2.7. Take $N = 2$ and $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ (closed chain). There is only one edge and hence only one complex Brownian motion W_t . Then, G_t is a 2×2 Hermitian matrix (if G_0 is Hermitian) solving the system of SDE,

$$\begin{aligned} dG_t(1, 1) &= -iG_t(2, 1)dW_t - iG_t(1, 2)d\bar{W}_t + (G_t(2, 2) - G_t(1, 1))dt, \\ dG_t(1, 2) &= -i(G_t(2, 2) - G_t(1, 1))dW_t - G_t(1, 2)dt, \end{aligned}$$

under the constraints $G_t(2, 1) = \overline{G_t(1, 2)}$ and $G_t(2, 2) = n_1 + n_2 - G_t(1, 1)$. The term $n_1 + n_2$, equal to the total number of particles, comes from the fact that $\text{Tr}(G_t) = \text{Tr}((c_1^\dagger c_1 + c_2^\dagger c_2)\varrho_t) = n_1 + n_2$. Let us assume that the total number of particles is 1 (it is conserved because the boundary parameters are all zero). Letting $D_t = G_t(1, 1) - G_t(2, 2)$, $R_t = \Re[G_t(1, 2)]$, $I_t = \Im[G_t(1, 2)]$ and let $W_t = \frac{B_1(t) + iB_2(t)}{\sqrt{2}}$, we arrive at

$$\begin{cases} dD_t &= -2D_t dt + 2\sqrt{2}(R_t dB_2 - I_t dB_1), \\ dR_t &= \frac{-1}{\sqrt{2}}D_t dB_2 - R_t dt, \\ dI_t &= \frac{1}{\sqrt{2}}D_t dB_1 - I_t dt. \end{cases} \quad (2.13)$$

One may check that $D_t^2 + 4(R_t^2 + I_t^2)$ is conserved, which of course we already knew since it corresponds to the conservation of the determinant under the unitary dynamics.

Remark 2.8. Here is another perspective from the point of view of representation theory. The Lie algebra $\mathfrak{gl}_N(\mathbb{C})$ acts on $\wedge V$ by mapping $E_{i,j}$ to $c_i^\dagger c_j$. By exponentiation, this yields a representation of $GL_N(\mathbb{C})$ on $\wedge V$. Let us denote by $\Gamma : GL_N(\mathbb{C}) \rightarrow \text{End}(\wedge V)$ this representation map and denote by γ the Lie algebra representation map from $\mathfrak{gl}_N(\mathbb{C})$ to $\text{End}(\wedge V)$. Elements in $\Gamma(GL_N(\mathbb{C}))$ are of the form

$$\Gamma(e^A) = e^{\sum_{i,j=1}^N c_i^\dagger A_{i,j} c_j},$$

as considered above, and the character formula is $\text{Tr}(\Gamma(e^A)) = \det(I + e^A)$. We thus let the $*$ -algebra \mathcal{A} be generated by those elements and we restrict to density matrices $\varrho = \frac{\Gamma(R)}{\text{Tr}(\Gamma(R))}$ with $R = R^* \in GL_N(\mathbb{C})$, as in (2.6) above. Using the group law $\Gamma(R)\Gamma(O) = \Gamma(RO)$, we have,

$$\text{Tr}(\varrho\Gamma(O)) = \frac{\det(I + RO)}{\det(I + R)} = \det(I + G(O - I)),$$

as in (2.8), with $G = \frac{R}{I+R}$. Taking $O = e^{\varepsilon X}$ for $X \in \mathfrak{gl}_N(\mathbb{C})$ and letting ε going to zero allows to identify G as the $N \times N$ matrix

$$G_{i,j} = \text{Tr}(\varrho\gamma(E_{i,j})) = \text{Tr}(\varrho c_i^\dagger c_j),$$

which thus fully determines the density matrix.

We henceforth use the symbol ‘tr’ to denote the trace over the N -dimensional space (i.e. the one-particle sector), as opposed to the trace ‘Tr’ over the 2^N -dimensional Fock space.

2.4. Stationary measure of periodic/closed QSSEP and random matrix theory.

In the periodic or closed boundary case, dynamics are unitary, so that the spectrum of G_t is preserved. Clearly, the evolution defined by (2.9) and (2.10) is unitary, so that the spectrum of G_t is constant. This implies that G_t is supported on the orbit of G_0 under the adjoint action of the unitary group.

Theorem 2.9 ([BBJ18, Proposition 1]). *Let G_0 be a $N \times N$ Hermitian matrix. The Haar distribution on the orbit of G_0 is the unique stationary measure of the stochastic differential equation on Hermitian matrices given by (2.11) where dh_t is given by (2.10) with initial condition G_0 . The same result holds when dh_t is given by (2.5).*

Sketch of proof. First of all, the dynamics preserves the unitary orbit of G_0 . This is clear from (2.1) and it can be checked formally using the definition of the dynamics (2.11), by verifying that $\text{tr}(G_t^k)$ remains constant for all k , so that the spectrum is preserved.

Thus, we are considering a Markovian diffusion on the unitary orbit of G_0 . Since this space is compact, there exists at least one stationary probability measure.

Then, the essential input is the fact that the set of matrices $E_{j,j+1}$ and $E_{j+1,j}$ for $1 \leq j \leq N-1$ that appear in (2.10) forms a system of simple root generators of the Lie algebra $\mathfrak{su}(N)$. This implies that any stationary measure is invariant by unitary conjugation, hence it is unique and Haar distributed.

To prove this $U(N)$ invariance, assume that G is distributed according to a stationary measure, and let

$$Z(A) = \mathbb{E} \left[e^{\text{tr}(AG)} \right],$$

where the expectation is taken over the stationary measure. Stationarity implies that

$$Z(A) = \mathbb{E} \left[Z[e^{\text{id}h_t} A e^{-\text{id}h_t}] \right]$$

where the expectation is now taken with respect Brownian motions in dh_t . Applying Itô's lemma, this implies that

$$\left(\sum_{j=1}^{N-1} D_j^+ D_j^- + D_j^- D_j^+ \right) Z(A) = 0, \quad (2.14)$$

where $D_j^+ = D(E_{j,j+1})$, $D_j^- = D(E_{j+1,j})$ and for $\mathbf{i}X \in \mathfrak{su}(N)$, $D(X)$ is the operator

$$D(X)F(A) = \partial_s F(e^{\mathbf{i}sX} A e^{-\mathbf{i}sX}) \Big|_{s=0}.$$

From (2.14), one shows that $D_j^\pm Z(A)$ for all $1 \leq j \leq N-1$. Because the $E_{j,j+1}$ and $E_{j+1,j}$ for a system of simple roots, one then concludes that $D(X)Z(A) = 0$ for all $\mathbf{i}X \in \mathfrak{su}(N)$ which eventually implies that for any $U \in SU(N)$,

$$Z(A) = Z(U^* A U).$$

Details of the argument can be found in [BBJ18, Appendix B]. \square

We stress that the Markov process describing the evolution of G_t does not really have a unique stationary measure, in the sense that it depends on the initial condition G_0 . As a consequence of Claim 2.9, we expect that as t goes to infinity, the law of G_t weakly converges to the law of $U^* G_0 U$ where U is a random Haar distributed unitary matrix. We may then compute the Laplace transform of the probability measure of G_t in the $t \rightarrow \infty$ limit as the Harish-Chandra–Itzykson–Zuber integral

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[e^{N \text{tr}(AG_t)} \right] = \int_{U(N)} dU e^{N \text{tr}(AU^* G_0 U)} = Z(NA), \quad (2.15)$$

where dU is Haar measure. A first contact with free probability emerges in the large N limit: if G_0 admits a limiting spectral measure μ_0 , and A has finite rank (and further

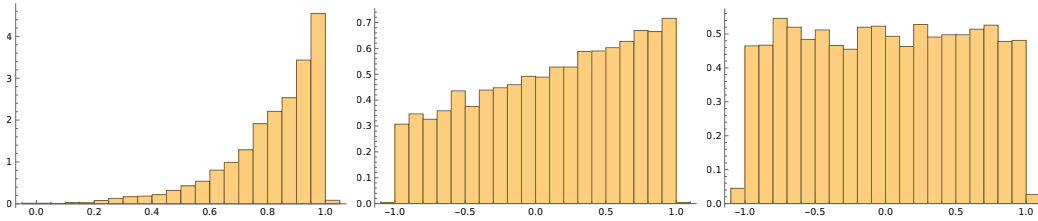


FIGURE 1. Probability distribution of D_t for $t = 0.1, t = 1$ and $t = 10$. Initially the distribution is concentrated at 1, and it converges to the uniform distribution on $[-1, 1]$. The few values outside $[-1, 1]$ are numerical errors due to the simulated dynamics being not perfectly unitary.

assumptions), this integral over the unitary group becomes in the $N \rightarrow \infty$ limit the formal series [IZ80; Col02]

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log Z(zNA) = \sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr}(A^n) \kappa_n(\mu_0),$$

where κ_n denote the free cumulants of μ_0 which are defined in Section 3.1.

Example 2.10. We reconsider the special case of Example 2.7. Let us fix the initial condition, for instance

$$G_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For a Haar distributed unitary matrix U , it is well-known that $|U_{11}|^2$ is uniformly distributed in $[0, 1]$. Hence, by Claim 2.9, $G(1, 1)$ should be uniformly distributed in $[0, 1]$ for large t . Equivalently, D_t in Example 2.7 should be uniform in $[-1, 1]$. To check that, it is not clear that the system of SDE (2.13) can be solved explicitly, but we do find on numerical simulations (see Figure 1) that the probability distribution of D_t approaches the uniform one.

Remark 2.11. The solution to the SDE $U_{t+dt} = e^{-idht} U_t$ is different from the standard notion of Brownian motion on $U(N)$ or $SU(N)$. This stochastic process, as well as the process $G_{t+dt} = e^{-idht} G_t e^{idht}$ from (2.9), does not seem to have been considered in the mathematics literature, and it seems interesting to study it further. In particular, one could compare the mixing time of these diffusions to the mixing time of the Brownian motion on $U(N)$ [Sal04; M el24]. In the context of random walks on the symmetric group, the analogous problem is well-known (and solved): the mixing time is different when the walk is generated by all transpositions [DS81] or only by adjacent transpositions [Lac16] — the latter case being related to the classical SSEP (with closed boundary conditions).

3. OPEN QSSEP AND FREE PROBABILITY

3.1. Preliminaries about classical and free cumulants. For random variables X_1, \dots, X_n , and any $(k_1, \dots, k_n) \in \mathbb{Z}_{\geq 1}^n$, the classical cumulants $\mathbf{K}_{k_1, \dots, k_n}(X_1, \dots, X_n)$ are generally defined through the coefficients in the power series

$$\log \mathbf{E} \left[e^{t_1 X_1 + \dots + t_n X_n} \right] = \sum_{k_1, \dots, k_n \geq 1} \mathbf{K}_{k_1, \dots, k_n}(X_1, \dots, X_n) \prod_{i=1}^n \frac{t_i^{k_i}}{k_i!}.$$

The cumulants $\mathbf{K}(X_1, \dots, X_n) := \mathbf{K}_{1, \dots, 1}(X_1, \dots, X_n)$ are related to the moments via

$$\mathbf{E}[X_1 \cdots X_n] = \sum_{\pi \in \mathcal{P}_n} \prod_{B \in \pi} \mathbf{K}(\{X_i\}_{i \in B}).$$

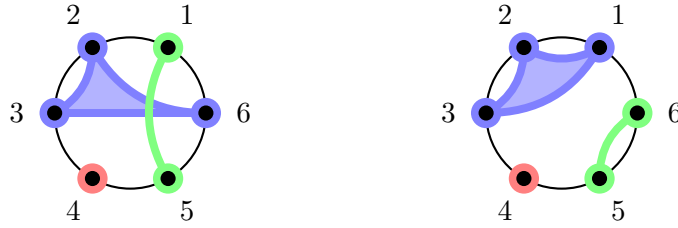


FIGURE 2. Left: A partition of $\{1, 2, \dots, 6\}$ into 3 blocks (green, blue and red). We see that the segment joining the green points and segments joining blue points do cross. Right: A non crossing partition.

where the sum runs over \mathcal{P}_n , the set of partitions π of $\{1, \dots, n\}$ into disjoint subsets denoted B . Equivalently, cumulants are expressed in terms of moments as

$$\mathbf{K}(X_1, \dots, X_n) = \sum_{\pi \in \mathcal{P}_n} (|\pi| - 1)! (-1)^{|\pi|-1} \prod_{B \in \pi} \mathbf{E} \left[\prod_{i \in B} X_i \right], \quad (3.1)$$

where $|\pi|$ denotes the number of blocks in the partition. In particular, the first cumulant is the expectation $\mathbf{K}(X) = \mathbf{E}[X]$, the second cumulant is the covariance

$$\mathbf{K}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y], \quad (3.2)$$

and the third cumulant is given by

$$\mathbf{K}(X, Y, Z) = \mathbf{E}[XYZ] - \mathbf{E}[X]\mathbf{E}[YZ] - \mathbf{E}[Y]\mathbf{E}[ZX] - \mathbf{E}[Z]\mathbf{E}[XY] + 2\mathbf{E}[X]\mathbf{E}[Y]\mathbf{E}[Z]. \quad (3.3)$$

A well known property of classical cumulants is that when the variables X_i are independent, $\mathbf{K}_k(X_1 + \dots + X_n) = \mathbf{K}_k(X_1) + \dots + \mathbf{K}_k(X_n)$ for all $k \geq 1$.

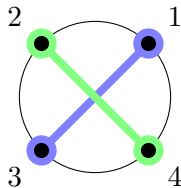
There exist analogous properties in free probability theory. In this context, random variables become elements of a unital associative $*$ -algebra \mathcal{A} over \mathbb{C} equipped with a linear map (non-commutative expectation) $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ such that $\varphi(1) = 1$ and $\varphi(a^*a) > 0$ for $a \in \mathcal{A}$. The free cumulants $\kappa(a_1, \dots, a_n)$ for $a_i \in \mathcal{A}$ are then defined implicitly via

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in \text{NC}(\mathcal{P}_n)} \prod_{B \in \pi} \kappa((a_i)_{i \in B}), \quad (3.4)$$

where $\text{NC}(\mathcal{P}_n)$ is the subset of \mathcal{P}_n consisting of non-crossing partitions (the notion of non-crossing partition can be understood from the examples in Figure 2). As in (3.1), the relation (3.4) can be inverted to express free cumulants in terms of free moments. One has $\kappa(a) = \varphi(a)$ and

$$\begin{aligned} \kappa(a, b) &= \varphi(ab) - \varphi(a)\varphi(b), \\ \kappa(a, b, c) &= \varphi(abc) - \varphi(a)\varphi(bc) - \varphi(b)\varphi(ca) - \varphi(c)\varphi(ab) + 2\varphi(a)\varphi(b)\varphi(c), \end{aligned}$$

in a way similar to the above classical moment-cumulant formula. However, at order 4, the free cumulant $\kappa(a_1, a_2, a_3, a_4)$ no longer has the same expression as the classical cumulant $\mathbf{K}(X_1, X_2, X_3, X_4)$, because there exists a crossing partition of order 4:



Adapting the notion of independent variables to the non-commutative setting, elements $a_1, a_2 \in \mathcal{A}$ are said to be respectively free iff all their mixed free cumulants vanish. We refer to [NS06; Spe19] for more details on free probability.

3.2. First moments of the stationary measure. Let us go back to quantum exclusion processes and consider the case of open QSSEP with boundary parameters chosen so as to impose densities $n_a < n_b$ at both ends (the relation is $n_a = \frac{\alpha_1}{\alpha_1 + \beta_1}$ and $n_b = \frac{\alpha_N}{\alpha_N + \beta_N}$ as for the SSEP). We observe that in (2.11) and (2.12), all terms are linear in the entries of the matrix G , so that one can compute the expectation under the stationary measure $\mathbb{E}[G(i, j)]$ by imposing that the expectation of all such coefficients remain constant over time, leading to solving a system of linear equations. In this section, all expectations are with respect to the stationary measure of the process G_t . One obtains that $\mathbb{E}[G(i, j)] = 0$ for $i \neq j$ and for $i = \lfloor xN \rfloor$ [BJ19]

$$\mathbb{E}[G(i, i)] = n_a + x(n_b - n_a) + O\left(\frac{1}{N}\right).$$

At second order, again, one can compute all the terms of the form $\mathbb{E}[G(i, j)G(k, \ell)]$ through a linear system. For $i = \lfloor xN \rfloor$, $j = \lfloor yN \rfloor$, the dominant terms have the form $\mathbb{E}[G(i, j)G(j, i)]$ and one finds [BJ19], letting $\Delta n := n_b - n_a$,

$$\mathbb{K}[G(i, j), G(j, i)] = \frac{(\Delta n)^2}{N}x(1 - y) + O\left(\frac{1}{N^2}\right), \text{ for } x < y, \quad (3.5)$$

and

$$\begin{aligned} \mathbb{K}[G(i, i), G(i, i)] &= \frac{(\Delta n)^2}{N}x(1 - x) + O\left(\frac{1}{N^2}\right), \\ \mathbb{K}[G(i, i), G(j, j)] &= -\frac{(\Delta n)^2}{N^2}x(1 - y) + O\left(\frac{1}{N^3}\right), \text{ for } x < y, \end{aligned}$$

where $\mathbb{K}(X_1(\omega), \dots, X_n(\omega))$ denotes the classical cumulants of random variables X_1, \dots, X_n with respect to the measure and expectation \mathbb{P}, \mathbb{E} . For example, (3.5) is the covariance between random variables $G(i, j)$ and $G(j, i)$. It is useful to note that the polynomial $x(1 - y)$ may also be written as

$$x \wedge y - xy \quad (3.6)$$

to remove the constraint that $x < y$. Here $x \wedge y = \min(x, y)$. At third order, dominant terms have the form

$$\mathbb{K}[G(i, j), G(j, k), G(k, i)] = \frac{(\Delta n)^3}{N^2}x(1 - 2y)(1 - z) + O\left(\frac{1}{N^3}\right), \text{ for } x < y < z,$$

with $i = \lfloor xN \rfloor$, $j = \lfloor yN \rfloor$ and $k = \lfloor zN \rfloor$. Again, this can be rewritten in a more symmetric way as

$$x \wedge y \wedge z - x(y \wedge z) - y(z \wedge x) - z(x \wedge y) + 2xyz. \quad (3.7)$$

The formulas (3.6) and (3.7) are reminiscent of expressions (3.2) and (3.3) for classical cumulants in terms of moments. But as we will see, things become slightly different at the fourth order and higher where free probability comes into play.

3.3. General case and free probability. Let us turn to the general case. It is convenient to associate to a product of the form $G(i_1, j_1)G(i_2, j_2) \dots G(i_r, j_r)$ an oriented graph \mathcal{G} with edges (i_k, j_k) . It turns out that very few graphs have a nontrivial average. We have $\mathbb{E}[\mathcal{G}] = 0$ unless \mathcal{G} is an Eulerian graph (at each vertex, the number of ingoing and outgoing arrows are the same). This follows from $U(1)^N$ invariance, i.e. we can change c_j into $c_j e^{i\theta_j}$ so that c_j^\dagger is changed into $c_j^\dagger e^{-i\theta_j}$ without changing the probability distribution of $G(i, j)$. Taking the θ_j as independent uniformly random phases, one can see that the expectation vanishes if the graph is not Eulerian. Again, all expectations in this section are with respect to the stationary measure for the process G_t .

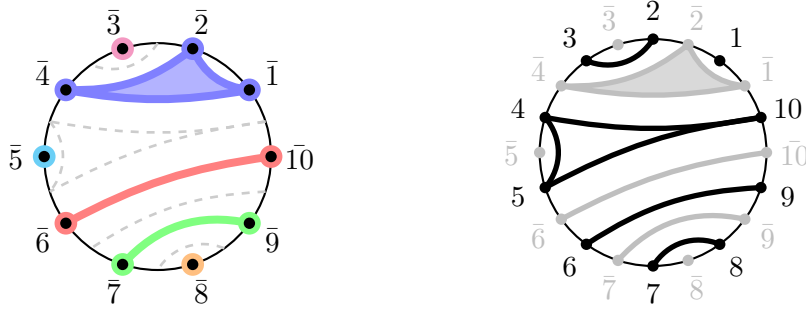


FIGURE 3. Left: a non crossing partition in \mathcal{P}_{10} . Right: Its Kreweras complement is depicted in black. In applications to cyclic loop expectations in QSSEP or random matrix theory, the partition π is partitioning the edges and its dual the points joining the edges into a loop (this is why the elements on the left are denoted $\bar{1}, \bar{2}, \dots$ instead of $1, 2, \dots$).

Claim 3.1 was first obtained in [Bia25], based on the combinatorial analysis of a recursive characterization of the functions g_p , established in [BJ21]. Finding a full proof of this characterization remains an open mathematical problem, however, the deduction of (3.9) from the recursive characterization is a mathematical theorem. We will explain in Section 3.4 why free cumulants arise. We will not follow the combinatorial analysis of [Bia25] but rather present a more direct derivation from [HB22] based on free probability.

Remark 3.2. The scaling (3.8) of the g_p 's with $1/N$ suggests a large deviation behaviour

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \left[e^{N \text{tr}(AG)} \right] = \mathcal{F}(A),$$

where the functional \mathcal{F} is related to the polynomials g_p [BH23].

3.4. Elements of proof. Let us explain now how free cumulants arise. First, we recall that the cyclic polynomials g_p are defined in (3.8) as classical cumulants. Using (3.1), we have that

$$\mathbb{E} [G(i_1, i_2) \dots G(i_p, i_1)] = \sum_{\pi \in \mathcal{P}_p} \prod_{B \in \pi} \mathbb{K}[(G(i_j, i_{j+1}))_{j \in B}].$$

We first argue that not all cumulants contribute in this sum, and that those which contribute are associated to non-crossing partitions. When $p = 2$,

$$\mathbb{E} \left[\text{loop}(i, j) \right] = \underbrace{\mathbb{K} [G(i, j), G(j, i)]}_{N^{-1} g_2(x, y)} + \underbrace{\mathbb{K} [G(i, j)] \mathbb{K} [G(j, i)]}_{\delta_{i, j} g_1(x) g_1(y)}.$$

At large scale, the indicator function δ_{ij} should be viewed as a discrete approximation of $N^{-1} \delta(x - y)$ so that as $N \rightarrow \infty$,

$$\mathbb{E} \left[\text{loop}(x, y) \right] \approx N^{-1} (g_2(x, y) + \delta(x - y) g_1(x) g_1(y)),$$

to be understood in a weak sense. More generally let us define the expectations (not the cumulants) of cyclic loops

$$g^s(x) := \lim_{N \rightarrow \infty} N^{p-1} \mathbb{E} [G(i_1, i_2) \dots G(i_p, i_1)],$$

where $i_j = \lfloor Nx_j \rfloor$ as before and $x = (x_1, \dots, x_p)$. $U(1)^N$ invariance and scalings with $1/N$ imply the following

Claim 3.3 ([HB22]). *We have*

$$g^s(x) = \sum_{\pi \in \text{NC}(\mathcal{P}_p)} g_\pi(x) \delta_{\pi^*}(x), \quad (3.10)$$

where the sum is over non-crossing partition π and g_π is the product of the cyclic cumulants associated to each parts of the partition π ,

$$g_\pi(x_1, \dots, x_p) = \prod_{B \in \pi} g_{|B|}(x_B), \quad x_B = (x_j)_{j \in B}, \quad (3.11)$$

and the partition π^* denotes the Kreweras complement of the non crossing partition π (see Figure 3) and $\delta_{\pi^*}(x) = \prod_{B \in \pi^*} \delta_{|B|}(x_B)$ where $\delta_n(x_1, \dots, x_n) := \delta(x_1 - x_2) \delta(x_2 - x_3) \dots \delta(x_{n-1} - x_n)$.

It is useful to understand $\pi \in \text{NC}(\mathcal{P}_p)$ as a partition of the edges and its dual π^* as a partition of the points. For a loop the number of edges is equal to the number of points. The identity (3.10) should be understood in a weak sense, i.e. that for any continuous function $\psi : [0, 1]^p \rightarrow \mathbb{R}$,

$$\int_{[0,1]^p} dx_1 \dots dx_p \psi(x) g^s(x) = \sum_{\pi \in \text{NC}(\mathcal{P}_p)} \int_{[0,1]^p} dx_1 \dots dx_p \psi(x) g_\pi(x) \delta_{\pi^*}(x).$$

Alternatively, one can consider $\lim_{N \rightarrow \infty} N^{-1} \mathbb{E}[\text{tr}(G \Delta_1 \dots G \Delta_p)]$, with $\Delta_1, \dots, \Delta_p$ diagonal matrices, see Claim 4.5.

Let us now explain the origin of non-crossing partitions and how (3.10) is obtained for $p = 4$. The fourth order cyclic moment can be expressed, by (3.1), as a sum over partitions of products of cumulants:

where the symbol \odot denote similar terms obtained by rotations of the preceding term. Each term in the expansion corresponds to a different partition of $\{1, 2, 3, 4\}$. Since the random variables $G(i, j)$ are attached to the edges of the graph, we partition the edge set, this is why the pictures are somewhat different between Figure 2 and Figure 3. Then, each term in the expansion should be understood as a product of cumulants, each cumulant corresponding to a block in the partition (i.e. a colour). To enforce $U(1)^N$ invariance, these cumulants are multiplied by (a product of) delta functions $N^{-1} \delta(x_i - x_j)$ whenever there is a dashed line joining x_i and x_j . However, since cyclic cumulants of order p are $O(N^{1-p})$ and delta terms are of order N^{-1} , we see that all terms in the expansion above

are of order N^{-3} except for the term

$$\mathbb{K} \left[\begin{array}{c} x_1 \quad x_2 \\ \text{---} \\ x_4 \quad x_3 \end{array} \right] \sim N^{-4} g(x_1, x_2) g(x_3, x_4) \delta(x_2 - x_3) \delta(x_1 - x_4),$$

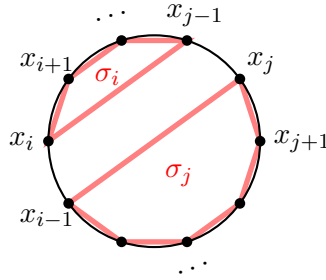
which is subdominant. More generally, terms corresponding to partitions with crossings are always subdominant. This explains why (3.10) holds.

The rest of the argument of the Claim 3.1 goes as follows [HB22]:

- (1) First one looks at the dynamical equations satisfied by expectation values of cyclic loops, $g^s(x, \tau) := \lim_{N \rightarrow \infty} N^{p-1} \mathbb{E} [G_{N^2\tau}(i_1, i_2) \dots G_{N^2\tau}(i_p, i_1)]$, in the large N scaling limit, with $i_j = \lfloor Nx_j \rfloor$, $t = N^2\tau$. Here, the expectation is not taken with respect to the stationary measure, but with respect to the Brownian motions involved in the dynamics. One can show that

$$(\partial_\tau - \Delta) g^s(x, \tau) = 2 \sum_{i < j} \partial_{x_i} \partial_{x_j} (\delta(x_i - x_j) g^s(x, \tau | \sigma_i) g^s(x, \tau | \sigma_j)), \quad (3.12)$$

where $\Delta := \sum_{j=1}^p \partial_{x_j}^2$ and $g^s(x, \tau | \sigma_i)$ (resp. $g^s(x, \tau | \sigma_j)$) are the expectation values of the smaller loops σ_i , containing x_i but not x_j , (resp. σ_j , containing x_j but not x_i) obtained by splitting the initial loop as follows



The evolution for these g^s has therefore a triangular structure.

- (2) Given the expectation values $g^s(x, \tau)$, one defines the cyclic cumulants $g_p(x, \tau)$ using (3.10) and $g_\pi(x, \tau) = \prod_{B \in \pi} g_{|B|}(x, \tau)$ for any non-crossing partition. Then, mimicking the moment-cumulant formula (3.4) in the free probability context, one defines

$$\varphi(x, \tau) := \sum_{\pi \in \text{NC}(\mathcal{P}_p)} g_\pi(x, \tau),$$

Compare to (3.10), this formula does not contain the Dirac delta-function associated to the Kreweras of π . Then, one shows using the type of reasoning above and (combinatorial) properties of non-crossing partitions that we have

$$(\partial_t - \Delta) \varphi(x, \tau) = 2 \sum_{i < j} \delta(x_i - x_j) (\partial_{x_i} \varphi(x, \tau | \sigma_i)) (\partial_{x_j} \varphi(x, \tau | \sigma_j)). \quad (3.13)$$

where $\varphi(x, \tau | \sigma_i)$ (resp. $\varphi(x, \tau | \sigma_j)$) is defined as above by breaking the initial loop in two pieces σ_i and σ_j and restricting $\varphi(x, \tau)$ to the smaller loop σ_i (resp. σ_j).

- (3) Finally, one checks that $\varphi(x) = \min\{x_1, \dots, x_p\}$ is the stationary solution of (3.13) with appropriate boundary conditions.

This implies that $g_\pi(x)$ in the stationary measure are the free cumulants associated to moments given by the function $\varphi(x)$, that is the free cumulants of the variables I_x as in Claim 3.1.

Remark 3.4. The proof of the Claim 3.1 in [Bia25] is different from the argument above. It starts from the stationarity conditions for the cumulants of cyclic loops [BJ19]. To

write them we need an alternative notation. Until now we ordered the points on the loops. Instead we could order them on the interval $[0, 1]$, so that $0 \leq x_1 < x_2 < \dots < x_p \leq 1$ and consider cumulants (with respect to the stationary measure for the process G_t)

$$g_\sigma(x) = \mathbb{K} \left[\begin{array}{c} x_{\sigma^1(1)} x_{\sigma^2(1)} \\ \bullet \\ x_1 \bullet x_{\sigma^3(1)} \\ \bullet \\ \vdots \bullet x_{\sigma^4(1)} \\ \bullet \\ \vdots \\ \bullet \\ \vdots \end{array} \right],$$

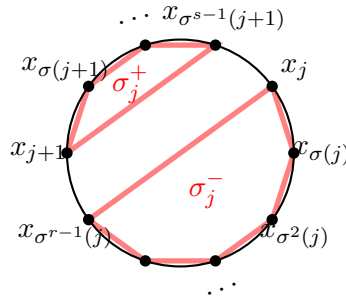
for any cyclic permutation $\sigma \in \mathcal{S}_p$. Those cyclic cumulants are polynomials [BJ19]. Given $\sigma \in \mathcal{S}_p$, and for any pair of adjacent variables x_j and x_{j+1} , $g_\sigma(x)$ can be decomposed as

$$g_\sigma(x) = A_j(\sigma) + B_j(\sigma)x_j + C_j(\sigma)x_{j+1} + D_j(\sigma)x_jx_{j+1}.$$

where the coefficients A_j, B_j, C_j, D_j may depend on all variables x_k but not on x_j and x_{j+1} . The stationarity conditions on g_σ then translate into conditions on those coefficients [BJ21]

$$\begin{aligned} A_j(\tau_j \circ \sigma) &= A_j(\sigma), \\ B_j(\tau_j \circ \sigma) &= C_j(\sigma) + \partial_{x_j} g_{\sigma_j^-}(x) \partial_{x_{j+1}} g_{\sigma_j^+}(x), \\ C_j(\tau_j \circ \sigma) &= B_j(\sigma) - \partial_{x_j} g_{\sigma_j^-}(x) \partial_{x_{j+1}} g_{\sigma_j^+}(x), \\ D_j(\tau_j \circ \sigma) &= D_j(\sigma), \end{aligned}$$

where $\tau_j \in \mathcal{S}_p$ is the transposition exchanging j and $j+1$, and $\tau_j \circ \sigma$ denotes the composition of the two permutations τ_j and σ . Similarly as in (3.12), the cyclic permutations σ_j^\pm are defined by breaking the initial cyclic permutation σ into two smaller cyclic permutations. Namely, let r, s be such that $\sigma^r(j) = j+1$ and $\sigma^s(j+1) = j$. Then, the cyclic permutations σ_j^- and σ_j^+ are defined in such a way that the cyclic order induced by σ in the r variables $x_j, x_{\sigma(j)}, \dots, x_{\sigma^{r-1}(j)}$ is encoded by σ_j^- while the cyclic order induced by σ in the s variables $x_{j+1}, x_{\sigma(j+1)}, \dots, x_{\sigma^{s-1}(j+1)}$ is encoded by σ_j^+ . Pictorially,



The above equations admit a unique solution. The proof in [Bia25] consists in solving them combinatorially, and understanding their free probability interpretation.

4. ENSEMBLES OF STRUCTURED RANDOM MATRICES

In this section, following [BH23; BH25] we describe a possible general framework for dealing with a large class of structured random matrices. In random matrix theory, the probability distribution of matrix entries is often invariant under permutations of the indices. This is the case of Wigner matrices for instance. Another example which is important for us is a Haar distributed orbit $M = UDU^*$, with D some fixed diagonal

matrix and U is unitary Haar distributed. In both cases, the law of M is $U(N)$ invariant, so it is a fortiori permutation invariant. This implies that quantities such as

$$\mathbb{E} [M_{i_1, j_1} \cdots M_{i_p, j_p}]$$

depend on the indices only through the topology of the associated graph. For example, in the case of a random matrix $M = UDU^*$, the expectation values of cyclic loops, with all indices i_1, \dots, i_p distinct, behave as

$$\mathbb{E} [M_{i_1, i_2} \cdots M_{i_p, i_1}] = \frac{1}{N^{p-1}} \kappa_p(\mu_D) + O(N^{-p}),$$

where $\kappa_p(\mu_D)$ denote the free cumulants of the limiting spectral distribution of the diagonal matrix D , as the size N goes to infinity. This expression, and in particular its scaling, resembles those we encounter in the previous Section 3, except that in QSSEP those expectation values were depending explicitly on the points $x_k = i_k/N$. We are interested in this Section in a generalization of these ensembles of random matrices for *structured* matrices whose law is not invariant under permutations. Similar random matrix ensembles are considered in the context of the Eigenstate Thermalization Hypothesis [FK19; PFK22]. The class of random matrices we consider is however different from [Han16] who also employs the phrase “structured random matrices”.

4.1. Cyclic cumulants and free probability. Let us describe the ensembles of random matrices introduced in [BH23].

Assumptions 1. Let \mathbb{P}_N be a sequence of probability measures on $N \times N$ matrices M_N satisfying the following properties (we will often drop the subscript N and write simply the measure \mathbb{P} and the matrix M):

- (1) ($U(1)^N$ invariance) For any diagonal matrix $u \in U(1)^N$, i.e. $u = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_N})$, $\theta_j \in \mathbb{R}$,

$$M \stackrel{(d)}{=} uMu^*.$$

- (2) (Scaling of cyclic cumulants) For all $p \geq 1$, there exists a continuous and bounded function $g_p(x_1, \dots, x_p)$ such that if $i_j = i_j(N)$ are distinct integers chosen so that $i_j(N)/N$ converges to x_j ,

$$\mathbb{E} [M_{i_1, i_2}, \dots, M_{i_p, i_1}] = \frac{1}{N^{p-1}} g_p(x_1, \dots, x_p) + O\left(\frac{1}{N^p}\right). \quad (4.1)$$

- (3) (Pinching of loops) If in the sequences (i_1, \dots, i_p) some of the indices coincide, e.g. $i_n(N) = i_m(N)$ for $n \neq m$, then

$$\lim_{N \rightarrow \infty} N^{p-1} \mathbb{K} [M_{i_1, i_2}, \dots, M_{i_p, i_1}] = g_p(x),$$

- (4) (Scaling of disconnected cycles) The cumulant associated to r disjoint cycles with a total of n matrix elements is of order $O(N^{2-r-n})$. More precisely, using the graphical notations that we have set up in Section 3,

$$\mathbb{K} \left[\begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \end{array} \quad \dots \quad \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \end{array} \right] = O(N^{2-r-n}), \quad (4.2)$$

in the sense that there exists a constant $C_{n,r}$ such that all such cumulants are bounded by $C_{n,r} N^{2-r-n}$.

The property (4) is stronger than the corresponding property in the original reference [BH23], this new version was introduced in a later addition [BH25]. In (2), because of $U(1)^N$ invariance, we can replace the expectation values by the cumulants since we assume

that all indices are distinct. Thus, using the graphical notation, the function g_p defined in (4.1) are the scaled cumulants of cyclic loops,

$$g_p(x_1, \dots, x_p) = \lim_{N \rightarrow \infty} N^{1-p} \mathbb{K} \left[\begin{array}{c} \text{Diagram of a cyclic loop with } p \text{ vertices labeled } x_1, x_2, x_3, x_4, \dots, x_p \end{array} \right]. \quad (4.3)$$

These g_p are called *local free cumulants*, since they have a natural interpretation in conditioned free probability (see Remark 4.6 below).

The general approach to structured random matrices developed below is based on the moments method. This is why properties (2), (3) and (4) in Assumptions 1 concern moments and cumulants. Consequently, a number of statements have to be interpreted as formal power series, as we stress below. In random matrix theory, it is often necessary to impose some additional assumptions on the operator norm of the involved matrices. Estimating the operator norm of matrices from minimal assumptions is typically a difficult problem [HT05] – see also [Hay20; BC22; BC23; Han25] for recent developments around the Peterson-Thom conjecture. In these notes, we do not discuss such assumptions.

Remark 4.1. Since these properties are formulated in terms of cumulants, they are preserved by shifting the matrices M by diagonal matrices, $M_{ij} \rightarrow M_{ij} - \delta_{ij} a_i$. Similarly these properties are preserved by multiplying M by a diagonal matrix, on the left or on the right, that is under transformation $M \rightarrow \Delta_L M \Delta_R$ with Δ_L, Δ_R being diagonal matrices.

Remark 4.2. As already mentioned, Assumptions 1 are fulfilled by Wigner matrices (possibly with a variance profile [Gir12; Shl96; Gui]) or Haar distributed orbits. It is believed that they are also satisfied in QSSEP, although a proof is missing (the proof of the scaling property (4) is lacking).

Remark 4.3. Defining an ensemble of structured random matrices requires the data of its local free cumulants (4.3). It is an apparently difficult problem to characterize which set of functions g_p forms a possible set of local free cumulants.

Example 4.4. The scalings (4.2) imply for example that

$$\mathbb{K} \left[\begin{array}{c} \text{Diagram of two separate cyclic loops, one with vertex } i \text{ and one with vertex } j \end{array} \right] = O(N^{2-2-2}) = O(N^{-2})$$

but

$$\mathbb{K} \left[\begin{array}{c} \text{Diagram of a single cyclic loop with vertices } i \text{ and } j \end{array} \right] = O(N^{2-1-2}) = O(N^{-1}).$$

Comparing with Section 3, we see that these scaling properties are satisfied in QSSEP.

More generally, the scalings in Assumptions 1 imply that $\text{tr}(M)$ concentrates fast around its mean: we have that (as a formal power series in z)

$$\lim_{N \rightarrow \infty} N^{-1} \log \mathbb{E} \left[e^{z \text{tr}(M)} \right] = \lim_{N \rightarrow \infty} N^{-1} z \mathbb{E} [\text{tr}(M)], \quad (4.4)$$

in the sense that all cumulants are negligible compared to the first one as N goes to infinity. Indeed, let us consider the first two cumulants. The first one is

$$\mathbb{K} [\text{tr}(M)] = \sum_{i=1}^N \mathbb{K} \left[\begin{array}{c} \text{Diagram of a single cyclic loop with vertex } i \end{array} \right] \sim N \int_0^1 dx g_1(x),$$

while the second cumulant is

$$\begin{aligned} \mathbb{K}[\mathrm{tr}(M), \mathrm{tr}(M)] &= \sum_{1 \leq i \neq j \leq N} \mathbb{K} \left[\begin{array}{c} \bigcirc \\ \bullet \\ i \end{array} \quad \begin{array}{c} \bigcirc \\ \bullet \\ j \end{array} \right] + \sum_{i=1}^N \mathbb{K} \left[\begin{array}{c} \bigcirc \\ \bullet \\ \bigcirc \\ \bullet \\ \bigcirc \end{array} \right] \\ &= N^2 \times O(N^{-2}) + N \times O(N^{-1}) = O(1). \end{aligned}$$

It is easy to check using Assumptions 1 that higher cumulants are sub-leading in $1/N$. For such matrices, non-crossing partitions play a role similar as in QSSEP. We have:

Claim 4.5. [BH23] *Let $M = M_N$ be a sequence of random matrices with law \mathbb{P}_N satisfying Assumptions 1. Let $\Delta_1, \dots, \Delta_p$ be diagonal matrices such that $(\Delta_k)_{ii} = \psi_k(i/N)$ for some continuous functions ψ_k . Then,*

$$\lim_{N \rightarrow \infty} N^{-1} \mathbb{E}[\mathrm{tr}(M \Delta_1 M \Delta_2 \cdots M \Delta_p)] = \int_{[0,1]^p} T_p(x_1, \dots, x_p) \prod_{j=1}^p dx_j \psi_j(x_j), \quad (4.5)$$

where

$$T_p(x_1, \dots, x_p) = \sum_{\pi \in \mathrm{NC}(\mathcal{P}_p)} g_\pi(x) \delta_{\pi^*}(x), \quad (4.6)$$

and $g_\pi(x)$ and $\delta_{\pi^*}(x)$ are defined as in (3.10).

This is obtained via the same argument as in Section 3 (which we have sketched only for $p = 4$).

Remark 4.6. This formula illustrates the fact that the notion of conditional probability, in a non-commutative setup, is a well-suited framework to study structured random matrices. Let $\mathcal{D} \subset \mathcal{A}$ be unital subalgebra. Following [MS17], a *conditional expectation value* is a map $E^{\mathcal{D}} : \mathcal{A} \rightarrow \mathcal{D}$ such that $E^{\mathcal{D}}[\Delta] \in \mathcal{D}$ and $E^{\mathcal{D}}[\Delta a \Delta'] = \Delta E^{\mathcal{D}}[a] \Delta'$ for all $a \in \mathcal{A}$ and $\Delta, \Delta' \in \mathcal{D}$. Furthermore, the *operator-valued distribution* of a random variable $a \in \mathcal{A}$ is given by all *operator-valued moments* $E^{\mathcal{D}}[a \Delta_1 a \cdots a \Delta_{n-1} a] \in \mathcal{D}$ where $\Delta_1, \dots, \Delta_{n-1} \in \mathcal{D}$.

In the present context, we should consider the case where \mathcal{A} is the ensemble of $N \times N$ random matrices M satisfying assumptions (1)-(3), and \mathcal{D} is the subalgebra of deterministic (bounded) diagonal matrices. In the large N limit the algebra \mathcal{D} is identified with $L^\infty[0, 1]$. Furthermore, for $M \in \mathcal{A}$ we define the conditional expectation value to be

$$E^{\mathcal{D}}[M] := \mathrm{diag}(\mathbb{E}[M_{11}], \dots, \mathbb{E}[M_{NN}]),$$

that is, one takes the usual expectation value of the matrix elements and sets all non-diagonal elements to zero. Since we are only interested in this concrete example, we will always denote elements of \mathcal{A} by M in the following definitions (instead of a).

As in the scalar case, the operator-valued free cumulants are defined via a moment-cumulant relation [MS17]: The \mathcal{D} -valued free cumulants $\kappa_n^{\mathcal{D}} : \mathcal{A}^n \rightarrow \mathcal{D}$ are defined through the \mathcal{D} -valued moments by

$$E^{\mathcal{D}}[M_1 \cdots M_n] = \sum_{\pi \in \mathrm{NC}(\mathcal{P}_n)} \kappa_\pi^{\mathcal{D}}(M_1, \dots, M_n),$$

where the $\kappa_\pi^{\mathcal{D}}$ are constructed from the family of linear functions $\kappa_n^{\mathcal{D}} := \kappa_{1_n}^{\mathcal{D}}$ respecting the nested structure of the parts appearing in π – see Figure 4.

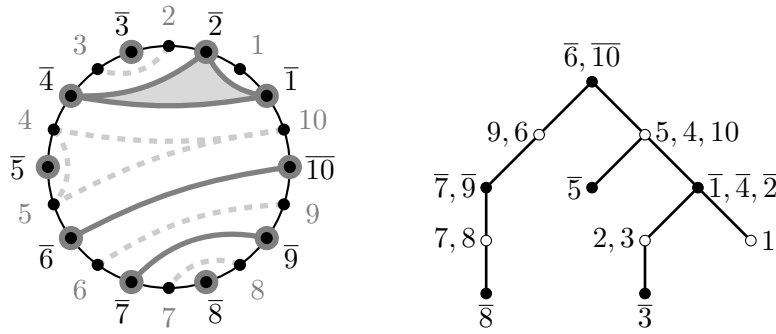


FIGURE 4. Non-crossing partitions have a nested structure: for instance in the present example, the part $\{\bar{8}\}$ is nested inside the part $\{\bar{7}, \bar{9}\}$ which is itself nested in $\{\bar{6}, \bar{10}\}$. One associates to this example the planar tree shown on the right, where the black vertices correspond to the blocks of π and the white vertices correspond to the blocks of the Kreweras complement π^* .

It turns out – see [BH23] for details – that the cumulants of cyclic loops $g_\pi(x)$ appearing in (4.6) can be identified with the operator-valued free cumulants, that is:

$$\begin{aligned} \kappa_\pi^{\mathcal{D}}(M\Delta_1, \dots, M\Delta_1, M\Delta_n)(x) \\ = \int \left(\prod_{k=1}^{n-1} dx_k \psi_k(x_k) \right) \psi_n(x) g_\pi(x_1, \dots, x_{n-1}, x) \delta_{\pi^*}(x_1, \dots, x_{n-1}, x), \end{aligned}$$

for Δ_k diagonal matrices with entries $(\Delta_k)_{ii} = \psi_k(i/N)$ with Δ_k bounded functions on $[0, 1]$.

4.2. Stability under non-linear transformation. The class of matrices satisfying Assumptions 1 is stable under certain non-linear transformations.

Claim 4.7 ([BH23; BH25]). *Let $M = M_N$ satisfy Assumptions 1. Then for all polynomial P , the sequence of random matrices $P(M)$ also satisfies Assumptions 1.*

The matrix $P(M)$ has new explicit local free cumulants, which can be expressed in terms of those of M . As consequence of this stability property, we also obtain that $\text{tr}(P(M))$ concentrates to its expectation in the sense of (4.4), namely

$$\lim_{N \rightarrow \infty} N^{-1} \log \mathbb{E} \left[e^{z \text{tr} P(M)} \right] = \lim_{N \rightarrow \infty} z N^{-1} \mathbb{E} [\text{tr} P(M)], \quad (4.7)$$

for any polynomial P .

Claim 4.8 ([BH23; BH25]). *Let $M = M_N$ satisfy Assumptions 1. Assume that they are centred, $\mathbb{E}[M_{ii}] = 0$. Define a matrix Y with coefficients*

$$Y_{i,j} = M_{i,j} f_{i,j}^{(N)}(N|M_{i,j}|^2)$$

where the $f_{i,j}^{(N)}$ are polynomials

$$f_{i,j}^{(N)}(u) = \sum_{k=0}^d a_k \left(\frac{i}{N}, \frac{j}{N} \right) u^k$$

where the functions $a_k(x, y)$ are continuous in (x, y) . Then the sequences of matrices Y satisfies Assumptions 1.

The property (4) in Assumptions 1 was reinforced in [BH25], and arguments for its stability under the above non-linear transformations were given there but, as the authors recognized, these are not enough to complete the proof of these claims. The type of

entrywise non-linear transformations considered in Claim 4.8 is a type of transformations considered in machine learning [Spe23; SW24].

Let us now mention briefly two applications of the previous claims, one in random matrix theory and another in quantum stochastic processes.

4.3. Application: Spectrum of sub-matrices. Let $M = M_N$ be (a sequence of) random matrices satisfying Assumptions 1 and consider M_{red} a $\ell N \times \ell N$ sub-matrix of M (we take the same subset of rows and columns, this is sometimes called a principal sub-matrix). What can be said about the law of M_{red} ? If M is a Wigner matrix, it is clear that M_{red} is still a Wigner matrix. Let us consider the slightly more involved case of a Haar-distributed orbit $M = UDU^*$. How are sub-matrices distributed? Asymptotically, they are distributed as a Haar orbit with spectral measure $\mu_D^{(\ell)}$, the free compression of μ_D obtained by scaling all the free cumulants of μ_D by $1/\ell$ (see more details in Example 4.11). Obtaining this answer using the present framework is not more difficult than computing the spectrum of the full matrix, thanks to the stability of Assumptions 1 under left/right multiplication by diagonal matrices. More generally, we have the following.

Claim 4.9 ([BH23]). *Let $M = M_N$ satisfy Assumptions 1. Let $H = H_N$ be a sequence of diagonal matrices such that $H_{ii} = h(i/N)$ where h is a continuous function, and set $M_h = H^{1/2}MH^{1/2}$. Then, the spectrum of M_h is characterized by the formal power series in $1/z$,*

$$\mathcal{F}(h; z) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\text{tr}(\log(1 - z^{-1}M_h)) \right],$$

which is given explicitly by

$$\mathcal{F}(h; z) = \int_0^1 dx \left[\log \left(1 - z^{-1}h(x)b(x) \right) + a(x)b(x) \right] - \mathcal{F}_0(a), \quad (4.8)$$

where $a(x)$ and $b(x)$ solve

$$a(x) = \frac{h(x)}{z - h(x)b(x)}, \quad b(x) = \frac{\delta \mathcal{F}_0(a)}{\delta a(x)}, \quad (4.9)$$

and

$$\mathcal{F}_0(a) := \sum_{p \geq 1} \frac{1}{p} \int_{[0,1]^p} g_p(x_1, \dots, x_p) \prod_{j=1}^p dx_j a(x_j).$$

The relations (4.9) are the extremization condition for the l.h.s. of (4.8) viewed as functional of a and b .

The functional derivative in (4.9) is given by

$$\frac{\delta \mathcal{F}_0(a)}{\delta a(x)} = \sum_{p \geq 1} \frac{1}{p} \sum_{j=1}^p \int_{[0,1]^{p-1}} g_p(x_1, \dots, x_p) \Big|_{x_j=x} \prod_{i \neq j} dx_i a(x_i).$$

Different proofs of this claim were presented in [BH23]. The most conceptual proof is probably the one using conditional non-commutative probability and the relation between the cyclic loop cumulants and operator valued free cumulants. The relations (4.9) are then equivalent to the relation between the operator valued Cauchy and R -transforms in free probability [MS17, Chap. 9, Theorem 11]. All these proofs rely on the tree structure underlying non-crossing probability, which is induced by the nested structure of non-crossing partitions as in Figure 3. In the case of Wigner matrices with variance profile, the equation for the eigenvalue density in [CG93; Shl96] are equivalent to (4.9) with only g_1 and g_2 non vanishing.

Remark 4.10. The large N limit of the spectrum of M_h is determined by the limit of its Stieltjes transform $G(z) := \lim_{N \rightarrow \infty} N^{-1} \text{tr} \left(\frac{1}{z - M_h} \right)$. Under Assumptions 1, using (4.7), the Stieltjes transform is equal to its mean in the large N limit. By Claim 4.9, the later is simply obtained by taking a derivative with respect to z in the solution (4.8), i.e.

$$G(z) = \int_0^1 \frac{dx}{z - h(x)b(x)}$$

where b is given by (4.9). In order to get information on the sub-matrix of M made by selecting the row/columns corresponding to an interval $I \subset [0, 1]$, it suffices to apply the formula above with $h(x) = \mathbf{1}_{x \in I}$.

Example 4.11. As an illustration let us consider the spectrum of sub-matrices of Haar distributed orbits. let $M = U^*DU$ with D diagonal and U Haar distributed unitary. In that case $g_p = \kappa_p(\mu_D)$, the free cumulants of the spectral measure of D (they are x -independent since Haar orbits are unstructured). Then, b_ℓ is x -independent and $a_\ell(x) = 0$ for $x \notin I$ (of course both a and b depend on ℓ). Equation (4.9) then becomes

$$A_\ell = \frac{\ell}{z - b_\ell}, \quad b_\ell = \sum_{k \geq 1} A_\ell^{k-1} \kappa_k(\mu_D), \quad (4.10)$$

with $A_\ell = \int_I dx a_\ell(x)$. Recall now that, given a measure μ , the generating function of its free cumulants,

$$K(z) := \frac{1}{z} + \sum_{k \geq 1} \kappa_k(\mu) z^{k-1}$$

is the inverse of its Cauchy (or Stieltjes) transform $G(z)$, i.e. $K(G(z)) = z$. Let K_1 be the free cumulant generating function of μ^D , i.e. $K_1(z) = \frac{1}{z} + \sum_{k \geq 1} \kappa_k(\mu^D) z^{k-1}$ and let K_ℓ be that of its free compression of μ_D (obtained by replacing $\kappa_k(\mu^D)$ by $\kappa_k(\mu^D)/\ell$). Then $\ell K_\ell(z) = K_1(z) - (1 - \ell)/z$. Similarly let G_1 (resp. G_ℓ) be the Cauchy transform of μ^D (resp. of its free dilatation).

The equations (4.10) can be rewritten, using the definition of free compression, as

$$K_\ell(A_\ell) = z/\ell \text{ that is equivalent to } A_\ell = G_\ell(z/\ell).$$

Since, from Claim 4.9, the Cauchy transform of the subblock is $\frac{\ell}{z - b_\ell} = A_\ell$, this shows that the Cauchy transform of the sub-matrix is $G_\ell(z/\ell)$, so that the spectrum of the sub-matrix is the free compression of μ^D .

4.4. Application: Almost sure classicality of quantum transport. As we have mentioned in (2.7), the multi-point correlation functions of QSSEP are given by the determinant of the random matrix G . This allows to compute the Laplace transform of the density profile for any realization of the noise as a determinant (see (2.8))

$$\text{Tr} \left(\varrho_t e^{\sum_{j=1}^N h_j \hat{n}_j} \right) = \det \left[I + G(e^H - I) \right],$$

where H is the diagonal matrix with diagonal elements h_i . By taking the logarithm, we define

$$\mathcal{F}_Q^N(h) = \frac{1}{N} \log \text{Tr} \left(\varrho_t e^{\sum_{j=1}^N h_j \hat{n}_j} \right) = \frac{1}{N} \text{tr} \log \left(I + G(e^H - I) \right). \quad (4.11)$$

The matrix $\log \left(I + G(e^H - I) \right)$ should be viewed as a formal power series in G (mathematically, one should impose some hypotheses on the norm of G to make all series converge). The statement (4.7) suggests that \mathcal{F}_Q^N is asymptotically deterministic, hence equal to its mean.

Furthermore, thanks to the correspondence (1.3) between the classical SSEP and averaged QSSEP, we have

$$\mathcal{F}_{\text{ssep}}(h) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \left[\text{Tr} \left(\varrho_t e^{\sum_{j=1}^N h_j \hat{n}_j} \right) \right].$$

Hence using (4.7) with $P(G) = \log(I + G(e^H - I))$ – extrapolating the result from polynomials to power series, which would require a mathematical argument – we obtain

$$\mathcal{F}_{\text{ssep}}(h) = \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathcal{F}_Q^N(h) \right].$$

This limit can be computed via (4.11) using Claim 4.9, reducing it to a variational problem. (The similarity between the variational problem in Claim 4.9 and that in (1.4) should be striking).

Using again that $\mathcal{F}_Q^N(h)$ is asymptotically deterministic, we learn

$$\lim_{N \rightarrow \infty} \mathcal{F}_Q^N(h) = \mathcal{F}_{\text{ssep}}(h).$$

This is one form of the *classicality* of quantum transport in large diffusive systems, as mentioned in Remark 1.4. This argument is not yet fully rigorous: the steps described above need to be better justified. But the alternative combinatorial approach below yields strong hints of the validity of (4.4), hence indirectly that the matrix G of stationary QSSEP satisfies property (4) of Assumptions 1.

4.5. Back to classical SSEP. Since the occupation numbers in SSEP are Bernoulli variables, $n_j^2 = n_j$, their cumulant generating function

$$\log \mathbf{E}_{\text{ssep}} \left[e^{\sum_{j=1}^N h_j n_j} \right]$$

is fully determined by the multiple cumulants at non-coinciding points, that is, by the set of cumulants $\mathbf{K}(n_{j_1}, \dots, n_{j_p})$ with all indices j_k distincts. Expressing this generating function in terms of those cumulants is an intricate combinatorial problem, solved in [BBBH22]. If the multiple cumulants at non-coinciding points scale appropriately with N , its solution takes a variational form in the large N limit. Furthermore, the correspondence (1.3) between the classical SSEP and QSSEP, in the form (2.7), allows to express these cumulants in terms of the QSSEP local free cumulants. This leads to the variational problem (1.4) connecting $\mathcal{F}_{\text{ssep}}$ to free probability as announced in the Introduction.

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