Growth processes and Integrability (Gregory Schehr)

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lecture notes from a series of lectures given in the SFT school in Paris, September 2019. The notes were taken by Naftali Smith.

Contents

0.1 Outline

There is a deep link between the following systems:
KPZ equation.
Random matrix theory (RMT).
Non-intersecting paths.
Fermions.

1 KPZ story:

1.1 Introduction

The interface is between two phases of matter: one stable and one unstable. For example, Ising model + magnetic field (drawing). We will focus on 1+1 dimension. The interface is described by a height field $h(x,t)$. And it is described by the stochastic PDE (the KPZ equation):

$$\partial_t h(x,t) = \nu \triangle h + \frac{\lambda}{2} (\nabla h)^2 + \sqrt{2D}\eta(x,t) \tag{1}$$

where $\eta(x,t)$ is Gaussian white noise $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$ and we will assume $\lambda > 0$.

The simpler case $\lambda = 0$ gives the Edwards-Wilkinson model which is much easier to solve. Let's say we fix some initial condition $h(x,0)$. Then $h(x,t)$ is a Gaussian at any x and t. Fluctuations are of order $h(x=0,t) \propto t^{1/4}$. Suppose a finite substrate of length L, so $x \in [0,L]$ with periodic boundary conditions. Then for long times $t \gg L^z$ where $z = 2$ is the dynamical exponent, the system reaches a stationary state. The stationary distribution is given by Boltzmann:

$$P_{st}[\{h(x)\}] \sim \exp\left[-\frac{\nu}{D}\int_0^L dx\ (\nabla h)^2\right]. \tag{2}$$

Notice that this is exactly the measure of a Brownian $h(x)$. (Strictly speaking, we should prevent the interface from running away to infinity. This can be done, e.g. by fixing the zero mode, so multiply P_{st} by the factor $\delta\left(\int_0^L h(x)\,dx\right)$.

Note: there is a problem with the KPZ equation because the $(\nabla h)^2$ term is very singular. So the equation must (in principle) be regularized. A rigorous treatment of KPZ was achieved by M. Hairer (renormalization).

Let us return to the KPZ equation ($\lambda \neq 0$). We will assume from now on

$$\nu = \frac{1}{2}, \quad \lambda = D = 1$$

(corresponding to rescaling x, t, h). There is a very nice property that only holds in 1 dimension. On a finite system with periodic B.C., the stationary measure is exactly the same as for $\lambda = 0$. So the stationary state is well understood. How does the system reach the stationary state, in the so-called "growth regime" $t \ll L^z$ (where here $z \neq 2$)? The standard way to charactarize this is through the roughness. This measures the fluctuations and is defined as follows:

$$w^2(L,t) = \left\langle \frac{1}{L}\int_0^L dx\,[h(x,t) - \langle h(x,t)\rangle]^2 \right\rangle. \tag{3}$$

The scaling form of the roughness at $t \gg 1$ and $L \gg 1$ with t/L^z fixed is

$$w_L(t) \sim t^\beta F\left(\frac{t}{L^z}\right) \to \begin{cases} t^\beta & t \ll L^z \\ L^\alpha & t \gg L^z \end{cases} \tag{4}$$

The first line in the previous equation corresponds to the limit $F(u \to 0) \to$ const. The second line implies the connection $z = \alpha/\beta$ between the exponents. For KPZ in 1+1 dimension the exponents are

$$\beta = \frac{1}{3}, \quad z = \frac{3}{2}, \quad \alpha = \frac{1}{2}$$

(for EW they are $\beta = \frac{1}{4}, \quad z = 2, \quad \alpha = \frac{1}{2}$). Between 1986 and 1999 these properties were referred to as the "KPZ universality class" [Review by Halpin-Healy and Zhang, Phys. Rep. 1995]. Then in 1999 there was a breakthrough by mathematicians. It was realized that the full distribution of $h(x,t)$ is universal.

1.2 KPZ universality: beyond the exponents

[Baik, Deift, Johansson '99, Johanson 2000, Spohn, Prahofer 2000]. Exact solutions of some specific models in the KPZ universality class. This uncovered a connection to random matrices. They also noticed a strong dependence on the initial condition, even at (relatively) long times $\tau_{\text{microscopic}} \ll t \ll L^z$ where $\tau_{\text{microscopic}}$ is some microscopic timescale. So some initial conditions (ICs) were identified:

* Flat: $h(x, t = 0) = 0$.
* Droplet

(* "stationary").

Note: the polynuclear growth (PNG) model is in the KPZ universality class.

Let us start with the flat IC. What are the fluctuations of $h(x = 0, t)$? It turns out that at long times

$$h(x = 0, t) \approx c_{\text{flat}} t + (\Gamma_{\text{flat}} t)^{1/3} \chi_1 \tag{5}$$

where χ_1 is a Tracy-Widom random variable with $\beta = 1$, and it is related to the Gaussian Orthogonal Ensemble (GOE) from RMT. For the droplet,

$$h(x = 0, t) \approx c_{\text{drop}} t + (\Gamma_{\text{drop}} t)^{1/3} \chi_2 \tag{6}$$

where χ_1 is a Tracy-Widom random variable with $\beta = 2$, and it is related to the Gaussian Unitary Ensemble (GUE) from RMT. The Tracy-Widom dsitribution ('94-'96) are defined as follows:

Let X be an $N \times N$ random matrix, with entries which are independent Gaussian random variables. $\beta = 1$ corresponds to a real symmetric matrix (GOE) and $\beta = 2$ corresponds to complex Hermitian (GUE). So

$$P(X)\,dX = \frac{1}{Z}\exp\left(-\frac{\beta N}{2}\mathrm{Tr}X^2\right)dX. \tag{7}$$

The statistics of the eigenvalues of these models are as follows: first of all, there are N eigenvalues which are all real $\lambda_1, \ldots, \lambda_N$. Their joint PDF is

$$P(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N}\prod_{i<j}|\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{2}\sum_i \lambda_i^2}. \tag{8}$$

The density (here the averaging $\langle\ldots\rangle_\beta$ denotes averaging over this joint PDF) converges, in the $N\to\infty$ limit, to the Wigner semi-circle:

$$\rho_N(\lambda)=\left\langle\frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\right\rangle_\beta\to\frac{1}{\pi}\sqrt{2-\lambda^2},\qquad\lambda\in\left[-\sqrt{2},+\sqrt{2}\right]\tag{9}$$

In order to study the edge of the semi-circle, we can look at $\lambda_{\max}=\max_{1\le i\le N}\lambda_i$. Then it behaves, in the $N\to\infty$ limit, as

$$\lambda_{\max}\to\sqrt{2}+\frac{1}{\sqrt{2}}N^{-2/3}\chi_\beta\tag{10}$$

where χ_β is the Tracy-Widom distribution.

(end of first lecture)

What is the Tracy-Widom distribution? We'll focus on the case $\beta=2$ because it is a little easier. It is given in terms of a Fredholm determinant

$$\mathcal{F}_2(s)=\mathbb{P}(\chi_2\le s)=\mathrm{Det}\left(I-P_sK_{\mathrm{Ai}}P_s\right)\tag{11}$$

where for $\tilde{K}=\tilde{K}(x,y)$, the Fredholm determinant is

$$\mathrm{Det}\left(I-\tilde{K}\right)=\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\tilde{K}^n\right)\tag{12}$$

where the Trace is defined by the integral

$$\mathrm{Tr}\tilde{K}=\int dx\tilde{K}(x,x)\tag{13}$$

and powers are given by

$$\tilde{K}^2(x,y)=\int dz\tilde{K}(x,z)\tilde{K}(z,y)\tag{14}$$

and the Airy kernel is

$$K_{\mathrm{Ai}}(x,y)=\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}=\int_0^\infty d\lambda\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)\tag{15}$$

where Ai is the Airy function, $\mathrm{Ai}''(x)=x\mathrm{Ai}(x)$, and P_s is the projector on $[s,+\infty[$. So, for instance

$$\mathrm{Tr}P_sK_{\mathrm{Ai}}P_s=\int_s^{+\infty}K_{\mathrm{Ai}}(x,x)\,dx.\tag{16}$$

The achievement of Tracy and Widom was the connection to Painlevé equations. The P.II equation is

$$q''(x)=xq(x)+2q^3(x),\qquad q(x\to+\infty)\sim\mathrm{Ai}(x)\tag{17}$$

(at large x, the q^3 term in the equation becomes negligible). So TW found [CMP '94] that

$$\mathcal{F}_2(s)=\exp\left[-\int_s^{+\infty}dx\,(x-s)\,q^2(x)\right].\tag{18}$$

(They also found a similar result for $\beta=1$). The Tracy-Widom distribution is highly non-Gaussian. $\mathcal{F}_2(s)$ is the cumulative distribution. The tails of the PDF are

$$\mathcal{F}_2'(s)\sim\begin{cases}\exp\left(-\frac{2\beta}{3}s^{3/2}\right), & s\to+\infty,\\ \exp\left(-\frac{\beta}{24}|s|^3\right), & s\to-\infty.\end{cases}\tag{19}$$

(the $s\to+\infty$ asymptote can be found fairly easily from the Fredholm representation, but the $s\to-\infty$ was found using Painlevé). The Tracy-Widom distributions were observed in experiments using liquid crystals [Takeuchi et. al. 2011].

2 Exact solution of KPZ at finite time t

[Le Doussal, Calabrese, Rosso, Dotsenko, Sasamoto, Spohn, Schehr, Majumdar, Borodin, Amir, Corwin, Quastel,...]

The KPZ equation in rescaled units is

$$\partial_t h=\frac{1}{2}\partial_x^2 h+\frac{1}{2}\left(\partial_x h\right)^2+\xi(x,t)\tag{20}$$

where $\langle\xi(x,t)\xi(x',t')\rangle=\delta(x-x')\delta(t-t')$. There is a problem with the KPZ equation, and it can be circumvented using the Cole-Hopf transform:

$$Z(x,t)=e^{h(x,t)}\tag{21}$$

and then one finds that Z satisfies the stochastic heat equation (SHE)

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \underbrace{Z(x,t)\,\xi(x,t)}_{\text{Ito}} \tag{22}$$

so the equation is linear, but the price we have to pay is that the noise is now multiplicative (as opposed to additive as in KPZ). The initial condition is of course $Z(x,0) = e^{h(x,0)}$. One can sure that (almost surely) $Z(x,t)$ is always positive (which is expected from its definition).

Looking at the SHE, we recognize that it is a Schrödinger equation in imaginary time where $\xi(x,t)$ is a (random) potential. This leads to a connection to polymers in random media as we now show.

$Z(x,t)$ is related to the partition function of a directed polymer in a random medium. The Green's function can be found in the form of a path integral:

$$\mathcal{Z}(x,x_0,t) = \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}x(\tau)\, e^{-\int_0^\tau \left[\frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \xi(x,\tau)\right]} \tag{23}$$

with initial condition

$$\mathcal{Z}(x,x_0,t=0) = \delta(x-x_0). \tag{24}$$

This corresponds to the partition function of a directed polymer in a random potential. The term $\left(\frac{\partial x}{\partial \tau}\right)^2$ is interpreted as elastic energy, and $\xi(x,\tau)$ is the random potential. Then $Z(x,t)$ is found by integrating the Green's function together with the initial condition:

$$Z(x,t) = \int_{-\infty}^{\infty} dx_0 \mathcal{Z}(x,x_0,t)\, e^{h(x_0,0)}. \tag{25}$$

What are the initial conditions?

* Droplet - Z$(x,0) = \delta(x)$.

* Flat - $Z(x,0) = 1$.

So the Droplet corresponds to the point-to-point partition function. The flat case corresponds to point-to-line. In the following, we will consider the droplet case because it is the simplest. We want to compute $\mathcal{Z}(x,0,t)$. For simplicity we will focus on $x=0$. Eventually we would like to calculate the full distribution of $\mathcal{Z}$. We will begin by calculating the moments $\langle \mathcal{Z}(0,0,t)^n \rangle$. Then n corresponds to the number of replicas, and we proceed as follows:

$$\langle \mathcal{Z}(0,0,t)^n \rangle = \left\langle \int \mathcal{D}x_1 \ldots \mathcal{D}x_n \exp\left[-\frac{1}{2}\int_0^t d\tau \sum_{a=1}^n \left(\frac{\partial x_a}{\partial \tau}\right)^2 - \sum_{a=1}^n \int_0^t d\tau \xi(x_a(\tau),\tau)\right]\right\rangle. \tag{26}$$

Reminder on Gaussian:

$$\frac{1}{A_N}\int dx_1 \ldots dx_N \exp\left(-\frac{1}{2}\sum_{i,j} x_i C_{ij}^{-1} x_j - \sum_{i=1}^N k_i x_i\right) = \exp\left(+\frac{1}{2}\sum_{i,j} k_i \underbrace{C_{ij}}_{=\langle x_i x_j\rangle} k_j\right). \tag{27}$$

So taking $i = a,\tau, k_i = 1$, we get

$$\begin{aligned}\langle \mathcal{Z}(0,0,t)^n \rangle &= \left\langle \int \mathcal{D}x_1 \ldots \mathcal{D}x_n \exp\left[-\frac{1}{2}\int_0^t d\tau \sum_{a=1}^n \left(\frac{\partial x_a}{\partial \tau}\right)^2 + \frac{1}{2}\sum_{a,b}\int_0^t d\tau d\tau' \underbrace{\langle \xi(x_a(\tau),\tau)\,\xi(x_b(\tau'),\tau')\rangle}_{=\delta(\tau-\tau')\delta(x_a(\tau)-x_b(\tau'))}\right]\right\rangle = \\ &= \left\langle \int \mathcal{D}x_1 \ldots \mathcal{D}x_n \exp\left[-\frac{1}{2}\int_0^t d\tau \sum_{a=1}^n \left(\frac{\partial x_a}{\partial \tau}\right)^2 + \frac{1}{2}\sum_{a,b}\int_0^t d\tau \delta(x_a(\tau) - x_b(\tau'))\right]\right\rangle \end{aligned} \tag{28}$$

(Note: the invariance under the transformation $x_a(\tau) \to x_a(\tau) + f(t)$ is a manifestation of the statistical tilt symmetry of the KPZ equation).

Now $\mathcal{Z}(0,0,t)$ satisfies

$$\partial_t \mathcal{Z} = -H_{\text{rep}}\mathcal{Z} \tag{29}$$

where

$$H_{\text{rep}} = -\frac{1}{2}\sum_{a=1}^n \left(\frac{\partial x_a}{\partial \tau}\right)^2 - \frac{1}{2}\sum_{a,b}\delta(x_a - x_b) \tag{30}$$

which is called the attractive (because of the sign of the second term) Lieb-Lieniger. This can be solved exactly using Bethe ansatz. We won't go into the details, but we will give the result.

It turns out that the good object to compute is a sort of generating function

$$g_t(s) = \left\langle \exp\left(-\exp\left[t^{1/3}\left(\tilde{h}(0,t) - s\right)\right]\right)\right\rangle \tag{31}$$

where $\tilde{h}(0,t)$ is a shifted and rescaled version of h:

$$\tilde{h}(0,t) = \frac{h(0,t) + \frac{t}{12}}{t^{1/3}}. \tag{32}$$

So it turns out that this can be written as a Fredholm determinant

$$g_t(s) = \mathrm{Det}\left(I - P_s K_t P_s\right) \tag{33}$$

where P_s is the projector as we saw before and

$$K_t(x,y) = \int_{-\infty}^{\infty} \frac{\mathrm{Ai}(x+u)\,\mathrm{Ai}(y+u)}{1 + e^{-t^{1/3}u}} du \tag{34}$$

is a generalization of the Airy kernel. This result is exact and valid at all times. Now, in the limit $t \to +\infty$, this becomes the Airy kernel:

$$K_t(x,y) \to \int_0^{\infty} \mathrm{Ai}(x+u)\,\mathrm{Ai}(y+u)\,du = K_{\mathrm{Ai}}(x,y). \tag{35}$$

Moreover, in the large-t limit

$$g_t(s) \to \left\langle \theta\left(s - \tilde{h}(0,t)\right)\right\rangle = \mathbb{P}\left(\tilde{h}(0,t) \le s\right) \tag{36}$$

which recovers the Tracy-Widom result from before. Note that the factor $\frac{1}{1+e^{-t^{1/3}u}}$ can be interpreted as a Fermi factor, and indeed we will show in a later lecture that this problem is closely connected to noninteracting Fermions at finite temperature.

(end of second lecture)

3 Random Growth, DPRM and RMT

(third lecture begins with a reminder of the KPZ equation and the connection to directed polymers in a random medium (DPRM))

Two discrete models:

1) Toy model of Hammerseley

2) Johansson's model

Let us begin by the simplest model, defined on $\mathbb{Z}^2$. Let us think of a square lattice (a square lattice is drawn on the board). Think of paths on the lattice beginning at the point $(0,0)$ and ending at (N,N), where in each step exactly one of the coordinates increases by one. Define the energy of the path $\mathcal{P}$ by

$$E(\mathcal{P}) = \sum_{(i,j)\in\mathcal{P}} \xi_{i,j} \tag{37}$$

and the partition function

$$Z(\beta) = \sum_{\mathcal{P}} e^{-\beta E(\mathcal{P})}. \tag{38}$$

(This model is supposed to be a discrete version of the polymer model). Let's say we are interested in calculating the energy of the ground state

$$h(0,t) \approx E_{\min} = \min_{\mathcal{P}} E(\mathcal{P}) \tag{39}$$

(or equivalently $E_{\max}$).

<u>Toy model:</u>

Take a square, and throw N points randomly into a unit square (an example is drawn on the board with $N = 8$ points). Now consider all of the (continuous) directed paths that begin at the bottom-left corner and end at the upper-right corner. By "directed" we mean that the path only goes up and right. Now, the points are "pinning centers", and the system gains energy if the path goes through the points. So to each path we assign the energy

$$-\epsilon_0 \times (\#\text{ points touched}). \tag{40}$$

So, for example, we can ask what the ground state energy is?

Let us order the points (independently) according on the x and y axis. Then there is a permutation that is defined by the function x-axis-label -> y-axis-label. Notice that all of the allowed (directed) configurations of the polymer correspond to increasing subsequences of this permutation. (An example is drawn on the board, for which the permutation is $\begin{matrix}12345678\\37451268\end{matrix}$ and the polymer's configuration corresponds to the subsequence 126). So the energy of the ground state is

$$E_{\min} = (-\epsilon_0) \times \text{length of the longest increasing subsequence of the permutation.} \tag{41}$$

One can show that throwing the points randomly on the square leads to the uniform measure on the space of permutations, where each permutation occurs with probability $1/N!$. This problem is known as the "Ulam problem" (1961). It turns out that the PNG model (discussed in the previous lecture) can also be mapped to Ulam's problem.

Ulam's problem:

Assign a uniform weight to each permutation $1/N!$. Define

$$\ell_N \equiv \text{length of the Longest Increasing Subsequence (LIS).}$$

What is the distribution of ℓ_N? Ulam showed nuemrically that $\langle \ell_N \rangle \sim c\sqrt{N}$, and this was proven by Hammerseley in '72, and in '77 Vershik-Kerov showed that $c = 2$. Baik-Deift-Johansson ('99) showed that, in the limit $N \to +\infty$,

$$p_{n,N} = \mathbb{P}(\ell_N \le n) \to \mathcal{F}_2\left(\frac{n - 2\sqrt{N}}{N^{1/6}}\right) \tag{42}$$

where $\mathcal{F}_2$ is the Tracy-Widom distribution with $\beta = 2$. They used fairly complicated methods.

Connection to matrix models: instead of taking N to be constant, it turns out to be easier to let it fluctuate (in the same way as when one moves from the canonical to the grand-canonical ensemble). This is done by considering the generating function

$$\phi_n(\lambda) = \sum_{N=0}^{\infty} p_{n,N} e^{-\lambda^2} \frac{\lambda^{2N}}{N!} \tag{43}$$

(so the number of points N is a random variable whose distribution is $\text{Poisson}(\lambda^2)$). Then it turns out that it can be written as an integral over all matrices in the unitary group:

$$\phi_n(\lambda) = e^{-\lambda^2} \int_{U \in \mathbb{U}(n)} \mathcal{D}U \; e^{\lambda \mathrm{Tr}(U + U^\dagger)} \tag{44}$$

This is a well-known model in lattice QCD [Wilson, Gross& Wilten, Wadia,...]. This model is known to exhibit a phase transition as a function of λ between the so-called "strong-coupling" and "weak-coupling" phases. Since U is unitary, its eigenvalues are of modulus 1 so we can write them as $e^{i\theta_1}, \dots e^{i\theta_n}$, so we can write

$$\phi_n(\lambda) = e^{-\lambda^2} \int_0^{2\pi} d\theta_1 \dots d\theta_n \prod_{i<j} \left| e^{i\theta_i} - e^{i\theta_j} \right|^2 \exp\left[\sum_{i=1}^{n} 2\lambda \cos(\theta_i)\right]. \tag{45}$$

And, if λ is correctly scaled with n, we get a phase transition

$$-\lim_{n\to\infty} \frac{1}{n^2} \ln \phi_n(\lambda = bn) = \begin{cases} 0, & 0 < b < \frac{1}{2}, \\ b^2 - 2b + \frac{3}{4} + \frac{1}{2}\ln 2b, & b > \frac{1}{2}. \end{cases} \tag{46}$$

The asymptotic behavior at $b \to \frac{1}{2}^+$ is $\propto \left|\frac{1}{2} - b\right|^3$ so the phase transition is of the third order. A third-order transition is rather unusual (most of the known phase transitions are of order 1 or 2) and, as we will see, it is a fingerprint of the Tracy-Widom distribution. How can we understand the transition? Draw a phase diagram in $\left(b, \frac{1}{n}\right)$ plane. Then taking the limit $n \to \infty$ means that we are on the b axis. What do the phases look like? At $b > \frac{1}{2}$ (the "weak-coupling phase") there is a **gap** in the distribution of the eigenvalues $e^{i\theta}$ on the unit circle, and at $b = 1/2$ the gap closes so that there is no gap at $b < \frac{1}{2}$ (the "strong-coupling phase"). The transition is of course smoothened to a crossover at finite n. Finite-n effects were also studied. It turns out that it is interesting to study the "double-scaling limit" where

$$b = \frac{1}{2} - \frac{s}{2^{4/3} n^{2/3}}. \tag{47}$$

So the interesting thing to look at is the equivalent of the specific heat, and it turns out that

$$\lim_{n\to\infty} \frac{\partial^2}{\partial s^2}\left[-\frac{1}{n^2} \ln \phi_n\left(\lambda = n\left(\frac{1}{2} - \frac{s}{2^{4/3} n^{2/3}}\right)\right)\right] = -q^2(s) \tag{48}$$

where $q(s)$ is the solution of the Painlevé II equation

$$q'' = sq + 2q^3. \tag{49}$$

Remark: since

$$\mathcal{F}_2(s) = \exp\left(-\int_s^\infty dx\,(x-s)\,q^2(x)\right), \tag{50}$$

we have

$$\frac{\partial^2}{\partial s^2}\ln \mathcal{F}_2(s) = -q^2(s). \tag{51}$$

So

$$\phi_n\left(\lambda = n\left(\frac{1}{2} - \frac{s}{2^{4/3}n^{2/3}}\right)\right) \to \mathcal{F}_2(s). \tag{52}$$

(we will skip Johansson's calculation).
(end of third lecture).

4 KPZ and non-interacting Fermions

We will discuss here first the case of temperature $T = 0$ which is connected to random matrices and KPZ at long times, and then the case $T > 0$ which is connected to KPZ at finite time.

4.1 $T = 0$

The model is N spinless Fermions with no interaction

$$\hat{H}_N = \sum_{i=1}^N \hat{h}_i, \quad \hat{h}_i = \frac{p_i^2}{2m} + V(x_i) \tag{53}$$

where the potential is harmonic:

$$V(x) = \frac{1}{2}m\omega^2 x^2. \tag{54}$$

The single particle eigenstates are

$$\varphi_k(x) \propto e^{-\frac{\alpha^2}{2}x^2} H_{k-1}(\alpha x), \quad k = 1, 2, \ldots \tag{55}$$

where $\alpha = \sqrt{m\omega/\hbar}$ (in the following we will set $\alpha = 1$) and H_k are the Hermite polynomials. These can be used in order to construct the many-body wave function:

$$\begin{aligned}\Psi_0(\underline{x}) = \Psi_0(x_1,\ldots,x_N) &= \frac{1}{\sqrt{N!}} \det_{1\le k,l\le N} \varphi_k(x_l) = \\ &= \frac{1}{\sqrt{N!}} e^{-\frac{1}{2}\left(x_1^2+\cdots+x_N^2\right)} \det_{1\le k,l\le N} H_{k-1}(x_l)\end{aligned} \tag{56}$$

Let us calculate this determinant explicitly for $N = 3$. Manipulating the columns of the matrix, we find that it is proportional to a Vandermonde determinant:

$$\det_{1\le k,l\le 3} H_k(x_l) = \begin{vmatrix} 1 & 2x_1 & 4x_1^2-2 \\ 1 & 2x_2 & 4x_2^2-2 \\ 1 & 2x_3 & 4x_3^2-2 \end{vmatrix} = 8\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = 8\prod_{1\le k<l\le N}(x_l - x_k). \tag{57}$$

So the joint PDF of the locations of the Fermions is

$$|\Psi_0(\underline{x})|^2 = \frac{1}{Z_n} e^{-\sum_{i=1}^N x_i^2} \prod_{i<j}(x_i - x_j)^2 \tag{58}$$

which is similar to the joint PDF of the eigenvalues of the GUE random matrices ($\mathcal{P}(X) \propto \exp\left(-\mathrm{Tr}X^2\right)$).

The x_i's form a determinantal point process (DPP). Let us rewrite the joint PDF as

$$|\Psi_0(\underline{x})|^2 = \frac{1}{N!}\det_{k,l}\varphi_k^*(x_l)\det_{k,l}\varphi_k(x_l) = \frac{1}{N!}\det A \det B = \frac{1}{N!}\det AB \tag{59}$$

where we defined

$$A_{k,l} = \varphi_l^*(x_k), \quad B_{k,l} = \varphi_k(x_l). \tag{60}$$

Let us calculate the product of these two matrices

$$(AB)_{k,l} = \sum_{m=1}^{N} A_{k,m} B_{m,l} = \sum_{m=1}^{N} \varphi_m^* (x_l) \varphi_m (x_k) \equiv \underbrace{K_N (x_l, x_k)}_{\text{the "kernel"}} \tag{61}$$

so altogether we get

$$|\Psi_0 (\underline{x})|^2 = \frac{1}{N!} \det_{k,l} K_N (x_l, x_k) . \tag{62}$$

A useful property of this kernel is that it is reproducible, meaning that:

$$\int_{-\infty}^{\infty} K_N (x, z) K_N (z, y) dz = \sum_{m,m'=1}^{N} \varphi_m^* (x) \underbrace{\int_{-\infty}^{\infty} dz \varphi_m^* (z) \varphi_{m'} (z)}_{\delta_{m,m'}} \varphi_{m'} (y) = K_N (x, y) . \tag{63}$$

The reproducibility implies that the p-point correlation function, defined as the marginal of the joint distribution, is given by a $p \times p$ determinant with the same kernel:

$$R_p (x_1, \dots, x_p) = \frac{N!}{(N-p)!} \int dx_{p+1} \dots dx_N |\Psi_0 (x_1, \dots, x_p, x_{p+1}, \dots x_N)|^2 = \det_{1 \le k,l \le p} K_N (x_l, x_k) \tag{64}$$

(we will not prove this property here). Remark: this property can also be viewed as a consequence of Wick's theorem. In 2nd quantized formalism, the kernel is

$$K_N (x, y) = \langle \Psi_0 | \Psi^\dagger (x) \Psi (y) | \Psi_0 \rangle \tag{65}$$

where $\Psi^\dagger$ and Ψ are creation and annihilation operators while $|\Psi_0\rangle$ denotes the many-body ground state.

Using this formalism, one can make many useful calculations. For example, the hole probability: What is the statistics of the number of particles N_J of particles within some set $J \subset \mathbb{R}$ (for example, J can be an interval). The generating function of this probability can be written as a Fredholm determinant:

$$\sum_{n=0}^{\infty} \mathbb{P} (N_j = n) e^{-pn} = \text{Det} \left(\mathbf{1} - \left(1 - e^{-p}\right) P_J K_N P_J \right), \qquad P_J (x) = \begin{cases} 1, & x \in J, \\ 0, & x \notin J. \end{cases} \tag{66}$$

In particular, taking the limit $p \to +\infty$ in the last equation, we will get the hole probability:

$$\mathbb{P} (N_J = 0) = \text{Det} (\mathbf{1} - P_J K_N P_J) . \tag{67}$$

A nice application of this hole probability is to calculate the distribution of the location of the rightmost Fermion:

$$x_{\max} = \max_{1 \le i \le N} x_i . \tag{68}$$

The cumulative distribution of $x_{\max}$ is the probability of a hole on a semi-infinite interval, as follows:

$$\mathbb{P} (x_{\max} \le M) = \mathbb{P} \left(N_{[M,+\infty]} = 0 \right) = \text{Det} \left(\mathbf{1} - P_{[M,+\infty]} K_N P_{[M,+\infty]} \right) \tag{69}$$

which makes sense because the distribution of $x_{\max}$ is closely related to the distribution of the largest eigenvalue of a GUE random matrix (which, as we already saw, is a Fredholm determinant).

Let us consider the particle density (which we choose to normalize to unity, consistent with the usual RMT convention). Then, in the large-N limit, it will converge to the Wigner semi-circle law:

$$\rho_N (x) = \frac{1}{N} \sum_{k=1}^{N} \delta (x - x_k) \to \frac{1}{\sqrt{N}} f \left(\frac{x}{\sqrt{N}} \right), \qquad f (z) = \frac{1}{\pi} \sqrt{2 - z^2}. \tag{70}$$

This result is valid in the bulk regime. On the other hand, one can consider the "edge regime" where $x \simeq x_{\text{edge}} \equiv \sqrt{2N}$.

What are the typical length scales in each regime? In the bulk regime, the interparticle spacing is

$$\ell_{\text{bulk}} \sim \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}. \tag{71}$$

What about the edge? the interparticle spacing there can be estimated as follows: require that there will be of order 1 particles in the edge regime, so

$$\int_{x_{\text{edge}} - \ell_{\text{edge}}}^{x_{\text{edge}}} N \rho_N (x) dx \sim 1 \implies \ell_{\text{edge}} \sim N^{-1/6}. \tag{72}$$

However, if we want to study correlations etc we need to consider the kernel (and not just the density). The behavior of the kernel in the two regimes is

$$K_N(x,y) \to \frac{1}{\ell_{\text{bulk}}} K_{\text{sine}}\left(\frac{x-y}{\ell_{\text{bulk}}}\right), \qquad N \to \infty \tag{73}$$

where

$$K_{\text{sine}}(z) = \frac{\sin(2z)}{\pi z} \tag{74}$$

is the well-known sine kernel. Near the edge,

$$K_N(x,y) \to \frac{1}{\ell_{\text{edge}}} K_{\text{Ai}}\left(\frac{x_{\text{edge}}-x}{\ell_{\text{edge}}}, \frac{x_{\text{edge}}-y}{\ell_{\text{edge}}}\right) \tag{75}$$

with the Airy kernel which we have already seen:

$$K_{\text{Ai}}(z,z') = \int_0^\infty du\, \text{Ai}(z+u)\,\text{Ai}(z'+u). \tag{76}$$

Reminder: $x_{\max}$ in the Fermion problem is equivalent to $\lambda_{\max}$ in the RMT problem. In turn (as we saw in the previous lecture), this corresponds to the height in KPZ in the droplet geometry, at very large time.

How does one obtain these results? One way is to use various asymptotic behaviors and formulas of the Hermite polynomials. Another way is a semi-classical analysis: consider the single-particle high energy levels,

$$-\frac{1}{2}\varphi_k''(x) + V(x)\varphi_k(x) = \epsilon_k \varphi_k(x), \qquad \epsilon_k = k - \frac{1}{2}. \tag{77}$$

The classical turning point is

$$V(x_{\text{edge}}) \sim \epsilon_N. \tag{78}$$

Linearizing the potential in the Schrödinger equation around this point, we find

$$-\frac{1}{2}\varphi_{N-\delta}''(x) + \left[V(x_{\text{edge}}) + (x - x_{\text{edge}})V'(x_{\text{edge}})\right]\varphi_{N-\delta}(x) = \epsilon_{N-k}\varphi_{N-\delta}(x) \tag{79}$$

so locally the potential is linear, so the wave functions are Airy functions. As a result, the kernel

$$K_N(x,y) = \sum_{k=1}^{N} \varphi_k^*(x)\varphi_k(y) \tag{80}$$

becomes the Airy kernel. This method shows us that the Airy kernel is quite universal, and will describe the edge regime for any smooth potential (e.g., any potential that behaves as $V(x) \sim x^p$).

In the next lecture, we will consider a system at $T > 0$ in the canonical ensemble. We will find that the determinantal structure is lost, but that in the grand-canonical ensemble the structure is determinantal. Moreover, we will find that the location of the rightmost Fermion is connected to KPZ at finite time t, and we will find that $1/t$ is related to the temperature.

(end of fourth lecture)

4.2 $T > 0$

(fifth lecture begins with a reminder of some of the results from the previous lectures).

The joint PDF in the canonical ensemble is given by

$$P_{\text{joint}}(x_1,\dots,x_N) = \langle x_1,\dots,x_N|\hat{\rho}|x_1,\dots,x_N\rangle, \quad \hat{\rho} = \frac{1}{Z_N(\beta)}e^{-\beta \hat{H}_N} \tag{81}$$

$$P_{\text{joint}}(x_1,\dots,x_N) = \frac{1}{Z_N(\beta)} \sum_{k_1<\cdots<k_N} \frac{1}{N!}\left|\det_{1\le i,j\le N} \varphi_{k_i}(x_j)\right|^2 e^{-\beta\left(\epsilon_{k_1}+\cdots+\epsilon_{k_N}\right)} \tag{82}$$

where $Z_N(\beta)$ is the partition function. It turns out that this can be written as a single determinant, using the Cauchy-Binet formula: for integrable functions f_i, g_i, w:

$$\int dx_1 \dots dx_N \left[\det_{1\le i,j\le N} f_i(x_j) \det_{1\le i,j\le N} g_i(x_j)\right] \prod_{i=1}^{N} w(x_i) = N! \det_{1\le i,j\le N} \int dx f_i(x)\, g_j(x)\, w(x) \tag{83}$$

(the way to show this is through a brute-force decomposition of the matrices). This formula turns out to be correct also if the x_i's are discrete. Using this discrete version of the formula we get

$$P_{\text{joint}}\left(x_1,\ldots,x_N\right)=\frac{1}{Z_N\left(\beta\right)}\det_{1\le i,j\le N}\underbrace{\sum_{k=1}^{\infty}\varphi_k^*\left(x_i\right)\varphi_k^*\left(x_j\right)e^{-\beta\epsilon_k}}_{G\left(x_i,x_j,\beta\right)} \tag{84}$$

where we notice that $G\left(x_i,x_j,\beta\right)$ is nothing but the propagator in imaginary time. As a result, it is not reproducible, because

$$\int dzG\left(x,z,\beta\right)G\left(z,y,\beta\right)=G\left(x,y,2\beta\right). \tag{85}$$

So the process is not determinantal in the canonical emsemble.

The trick is that in the grand-canonical ensemble, we will see that the process is determinantal, and then we will use the equivalence between the canonical and grand-canonical ensembles in the large-N limit. We will not prove that the process is determinantal but only mention that this is done using the independence of the occupation numbers of the different energy levels. The kernel is

$$K_\mu\left(x,y\right)=\sum_{k=1}^{\infty}\frac{\varphi_k^*\left(x\right)\varphi_k\left(y\right)}{e^{\beta\left(\epsilon_k-\mu\right)}+1}. \tag{86}$$

It is not reproducible, but nevertheless the process is determinantal.

How does the temperature affect the density profile? Qualitative arguments: We saw that at $T=0$ there are two length scales $\ell_{\text{bulk}}\sim N^{-1/2}$, $\ell_{\text{edge}}\sim N^{-1/6}$. At $T>0$ there is an additional length scale which controls the crossover from quantum to classical behavior: the thermal De-Broglie wave length

$$\lambda_T=\sqrt{\frac{\hbar^2}{2\pi mk_BT}}. \tag{87}$$

So the crossover between quantum and classical behavior will happen at different temperatures in the bulk and at the edge: in the bulk we will get

$$\lambda_T\sim\ell_{\text{bulk}}\implies T_{\text{bulk}}\sim N \tag{88}$$

which could be expected, because this is just the Fermi energy. However, at the edge:

$$\lambda_T\sim\ell_{\text{edge}}\implies T_{\text{edge}}\sim N^{1/3}. \tag{89}$$

The edge is much more sensitive to fluctuations, because the density is much lower. So for such temperatures, the bulk will be unaffected (still be described by the Wigner semi-circle), while the edge will be modified. So let us set $T=\frac{1}{b}N^{1/3}$ and take the large-N limit. Analysing Eq. (86), we find that the wave functions near the edge are Airy functions, and we will get (in the limit $N\to\infty$)

$$K_\mu\left(x,y\right)\to\frac{1}{\ell_{\text{edge}}}K_{\text{KPZ}}\left(\frac{x-\sqrt{2N}}{\ell_{\text{edge}}},\frac{y-\sqrt{2N}}{\ell_{\text{edge}}}\right) \tag{90}$$

with

$$K_{\text{KPZ}}\left(z,z'\right)=\int_{-\infty}^{\infty}d\lambda\frac{\text{Ai}\left(z+\lambda\right)\text{Ai}\left(z'+\lambda\right)}{e^{-b\lambda}+1} \tag{91}$$

Reminder: we already saw K_{KPZ} in a previous lecture in the context of KPZ with droplet geometry at finite time t:

$$\left\langle e^{-\exp\left(t^{1/3}\left(\tilde{h}(0,t)-s\right)\right)}\right\rangle=\text{Det}\left(\mathbf{1}-P_sK_{\text{KPZ}}P_s\right). \tag{92}$$

So the connection is $b=t^{1/3}$. What is the interpretation of the Fredholm determinant in the Fermion language?

$$\mathbb{P}\left(x_{\max}\left(T\right)\le\sqrt{2N}+\frac{N^{-1/6}}{\sqrt{2}}s\right)=\text{Det}\left(\mathbf{1}-P_sK_{\text{KPZ}}P_s\right). \tag{93}$$

[Le Doussal, Dean, Majumdar, Schehr]. Is the connection between Fermions and KPZ a pure coincidence or is there something else behind it?

(Remark: it appears to be difficult to rigorously prove mathematically that these results are valid in the canonical ensemble, although for physicists it is rather natural to expect that the predictions from the canonical and the grand-canonical ensembles (for local observables) indeed coincide in thermodynamic limit).

5 KPZ and non-Intersecting line ensemble

The motivation is to understand correlations. Let us consider a KPZ interface. So far we only discussed the distribution of height at one point. We can study spatial correlations, i.e. the correlation between $h(x,t_1)$ and $h(x',t_1)$, or temporal correlations $h(x'',t_2)$. (this is drawn on the blackboard). We will focus on spatial correlations which are much easier, and on the droplet geometry. Reminder: in the large-time limit, the mean interface height is $O(t)$ and the fluctuations are of order $O\left(t^{1/3}\right)$. Remember also that locally, h as a function of x is a Brownian motion, so from the scaling of fluctuations in Brownian motion together with the scale $t^{1/3}$ of fluctuations, we find that fluctuations in the x-direction are $O\left(t^{2/3}\right)$.

5.1 Connection to line ensembles (multilayer PNG) [Prähofer,Spohn]

Definition: a Brownian bridge on $[0,T]$ is a Brownian motion which is conditioned on returning to the origin at time T: $B(t=0)=B(t=T)=0$. Remark: one way to construct such a bridge is $B(t)=X(t)-\frac{t}{T}X(T)$ where $X(t)$ is a standard Brownian motion $X(0)=0$, $\dot{X}(t)=\xi(t)$.

Now let us consider N non-intersecting Brownian bridges, $x_1(t),\dots,x_N(t)$ with $x_1(t)<\dots<x_N(t)$, and set $T=1$ for convenience (this is drawn on the board). This occurs rather naturally in the multilayer PNG model. Then, in the large-N limit we will have $x_N(t)\sim\sqrt{N}$. Consider $x_N(t)$ at some fixed time, say $t=1/2$. Then it turns out that the fluctuations (in the x direction) are of order $\sim N^{-1/6}$, and the correlation time (in the t direction) is $\sim N^{-1/3}$.

Statement (for the PNG model): there is a connection to the KPZ height $h(x,t)$:

$$\lim_{t\to\infty}\frac{h\left(ut^{\frac{2}{3}},t\right)-2t}{t^{1/3}}\overset{\text{'law'}}{=}\lim_{N\to\infty}\frac{x_N\left(\frac{1}{2}+\frac{u}{2N^{\frac{1}{3}}}\right)-\sqrt{N}}{N^{-\frac{1}{6}}}=\mathcal{A}_2(u)-u^2 \tag{94}$$

where the converegence is in law, and $\mathcal{A}_2(u)$ is the Airy$_2$ process. This process is believed to describe correlations in KPZ. Note that it is stationary (independent of u), and

$$\mathbb{P}(\mathcal{A}_2(u)\le s)=\mathbb{P}(\mathcal{A}_2(0)\le s)=\mathcal{F}_2(s) \tag{95}$$

where $\mathcal{F}_2(s)$ is the Tracy-Widom distribution with $\beta=2$.

Remark: for the flat initial condition the correlations are descrbied by the Airy$_1$ process, but there is no connection to non-intersecting lines there.

5.2 Connection to RMT and Dyson's Brownian motion

Now, back to the non-intersecting Brownian bridges (drawn on the board). There is a subtlety here: it is impossible to require $x_1(0)=\dots x_N(0)=0$ but for the x_i's to not intersect at $t>0$. This must be regularised, e.g. by requiring $x_j(0)=j\epsilon$ and then taking the limit $\epsilon\to0$. Consider a given time τ and ask about the joint PDF of $x_1(\tau),\dots,x_N(\tau)$: $P_{\text{joint}}(x_1,\dots,x_N;\tau)$. Relying on the Karlin-McGregor formula '59 (which allows one to calculate the propagator of the Brownian motions, and has a very nice determinantal structure), it has been shown that

$$P_{\text{joint}}(x_1,\dots,x_N;\tau)=\frac{1}{Z_N(\tau)}\prod_{i<j}(x_i-x_j)^2\exp\left[-\frac{1}{\sigma^2(\tau)}\sum_{i=1}^{N}x_i^2\right] \tag{96}$$

so

$$\frac{x_i}{\sigma(\tau)}\overset{\text{law}}{=}\lambda_i=\text{eigenvalues of GUE.} \tag{97}$$

So the density of the x_i's is the Wigner semi-circle, and the top Brownian bridge will behave like the largest eigenvalue of a GUE matrix

$$\frac{x_N}{\sigma(\tau)}\overset{\text{law}}{=}\lambda_{\max}\text{ of GUE}=\sqrt{2N}+\frac{1}{\sqrt{2}N^{1/6}}\underbrace{\chi_2}_{\text{Tracy-Widom}}. \tag{98}$$

How can one generate these non-intersecting Brownian bridges?

It turns out that $x_1(t),\dots,x_N(t)$ can be generated by looking at the eigenvalues of a random matrix whose elements are time dependent - a version of Dyson's Brownian motion. Let $H(t)$ be a $N\times N$ Hermitian matrix whose elements are Brownian bridges:

$$H_{mn}(t)=\begin{cases}\frac{1}{\sqrt{2}}\left[B_{mn}(t)+i\tilde{B}_{mn}(t)\right], & m<n,\\ B_{mm}(t), & m=n,\\ \frac{1}{\sqrt{2}}\left[B_{nm}(t)-i\tilde{B}_{nm}(t)\right], & m>n,\end{cases} \tag{99}$$

where the B_{nm}'s and $\tilde{B}_{nm}$'s are independent Brownian bridges. Then the claim is that the eigenvalues $\lambda_1(t) < \cdots < \lambda_N(t)$ behave as non-intersecting Brownian motions (the way to show this is to write the Fokker-Planck equations of the two processes and show that they coincide). So nowadays it is common to charactarize the Airy process directly from Dyson's Brownian motion.

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