# Direct cavity detection of Majorana pairs (supplemental material) 

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## DEFINITION OF THE CHARGE SUSCEPTIBILITY IN THIS WORK

In this work, we define the charge susceptibility as $\chi(\omega)=\int_{-\infty}^{+\infty} d t \chi(t) e^{i \omega t}$ with

$$
\begin{equation*}
\chi(t)=-i \theta(t)\left\langle\left[\hat{h}_{\mathcal{C}}(t), \hat{h}_{\mathcal{C}}(t=0)\right]\right\rangle_{0} \tag{1}
\end{equation*}
$$

and $\left\rangle_{0}\right.$ the statistical average when $\hat{h}_{\mathcal{C}}$ is disregarded from $\hat{h}_{t o t}$. The operator $\hat{h}_{\mathcal{C}}$ only involves charge occupation numbers. Note that this definition is slightly different from the definition used in Ref.[4], where the lever arm between $\left(\hat{a}+\hat{a}^{\dagger}\right)$ and the circuit orbital energy is not included in $\chi(t)$. Here, we prefer to use Eq. (1) because it enables one to have a unique definition of $\chi(t)$ for all the cases considered. This is why the photonic lever arms $N_{\alpha \beta}, g$ or $\beta$ are included in $\chi(t)$ for the three models considered by us.

## ANDREEV DOTS AND QUANTUM DOTS IN THE SPIN-DEGENERATE CASE

The purpose of this section is to show that the impossibility to induce microwave transitions between pairs of electron-hole conjugated states is not contradictory with experiments on superconducting atomic point contacts (also called Andreev dots)[1] and predictions for quantum dots with superconducting reservoirs[2]. In these cases, transitions between energy-symmetric electron and hole states occur due to the spin degeneracy in the system. Let us use the model mesoscopic QED Hamiltonian

$$
\begin{equation*}
H_{A}=E_{0}\left(\hat{c}_{\uparrow}^{\dagger} \hat{c}_{\downarrow}^{\dagger}+\hat{c}_{\downarrow} \hat{c}_{\uparrow}\right)+g_{0}\left(\hat{c}_{\uparrow}^{\dagger} \hat{c}_{\uparrow}+\hat{c}_{\downarrow}^{\dagger} \hat{c}_{\downarrow}\right)\left(\hat{a}+\hat{a}^{\dagger}\right)+\omega_{0} \hat{a}^{\dagger} \hat{a} \tag{2}
\end{equation*}
$$

Above, the term in $E_{0}$ describes Andreev reflection processes in the Andreev or quantum dot, and $g_{0}$ describes the coupling of the dot to the cavity with frequency $\omega_{0}$. The Bogoliubov operators can be defined as

$$
\begin{align*}
& \hat{\gamma}_{\uparrow}=\left(\hat{c}_{\uparrow}+\hat{c}_{\downarrow}^{\dagger}\right) / \sqrt{2}  \tag{3}\\
& \hat{\gamma}_{\downarrow}=\left(\hat{c}_{\downarrow}-\hat{c}_{\uparrow}^{\dagger}\right) / \sqrt{2} \tag{4}
\end{align*}
$$

Hence, one finds

$$
\begin{equation*}
H_{A}=E_{0}\left(\hat{\gamma}_{\uparrow}^{\dagger} \hat{\gamma}_{\uparrow}+\hat{\gamma}_{\downarrow}^{\dagger} \hat{\gamma}_{\downarrow}-1\right)+g_{0}\left(1-\hat{\gamma}_{\uparrow}^{\dagger} \hat{\gamma}_{\downarrow}^{\dagger}-\hat{\gamma}_{\downarrow} \hat{\gamma}_{\uparrow}\right)\left(\hat{a}+\hat{a}^{\dagger}\right)+\omega_{0} \hat{a}^{\dagger} \hat{a} \tag{5}
\end{equation*}
$$

In this case, applying the general expression of $\chi\left(\omega_{0}\right)$ given in the main text for discrete systems, we obtain

$$
\begin{equation*}
\chi^{*}\left(\omega_{0}\right) \simeq g_{0}^{2}\left(\omega-2 E_{0}+i 0^{+}\right)^{-1} \tag{6}
\end{equation*}
$$

The presence of a resonance at $\omega=2 E_{0}$ is clearly due to the creation of two degenerate quasiparticles by the operators $\hat{\gamma}_{\uparrow}^{\dagger}$ and $\hat{\gamma}_{\downarrow}^{\dagger}$.

## QUANTUM DOT CONTACTED TO A SUPERCONDUCTOR AND SUBJECT TO A ZEEMAN FIELD

It is important to discuss the signatures of Andreev bound states with a trivial lifting of spin-degeneracy due to a Zeeman field. The minimal model for this situation is a single-orbital quantum dot tunnel contacted to a superconductor and subject to a Zeeman field $E_{z}$. This system is described by the Hamiltonian

$$
\begin{equation*}
H_{B}=E_{0}\left(\hat{c}_{\uparrow}^{\dagger} \hat{c}_{\downarrow}^{\dagger}+\hat{c}_{\downarrow} \hat{c}_{\uparrow}\right)+E_{z}\left(\hat{c}_{\uparrow}^{\dagger} \hat{c}_{\uparrow}-\hat{c}_{\downarrow}^{\dagger} \hat{c}_{\downarrow}\right)+\varepsilon_{\text {orb }}\left(\hat{c}_{\uparrow}^{\dagger} \hat{c}_{\uparrow}+\hat{c}_{\downarrow}^{\dagger} \hat{c}_{\downarrow}\right) \tag{7}
\end{equation*}
$$



FIG. 1: Schemes of the density of states in the quantum dot (left panel) and microwave cavity response (right panel) in the presence of Andreev bound states with a trivial spin-splitting by a Zeeman field $E_{z}$

For the sake of generality, we have included above the dot orbital energy $\varepsilon_{\text {orb }}$ which was assumed to be $\varepsilon_{\text {orb }}=0$ in the previous section. A Bogoliubov-De Gennes transformation gives

$$
\begin{equation*}
H_{B}=\varepsilon_{A} \hat{d}_{A}^{\dagger} \hat{d}_{A}+\varepsilon_{B} \hat{d}_{B}^{\dagger} \hat{d}_{B}+\varepsilon_{C} \hat{d}_{C}^{\dagger} \hat{d}_{C}+\varepsilon_{D} \hat{d}_{D}^{\dagger} \hat{d}_{D} \tag{8}
\end{equation*}
$$

The quasiparticle modes in the above expression belong to two independent subspaces. The subspace combining electrons with spin up and holes with spin down gives the modes $\hat{d}_{A}^{\dagger}$ and $\hat{d}_{A}^{\dagger}$ with energies $\varepsilon_{A}$ and $\varepsilon_{B}$ defined by

$$
\begin{aligned}
& \varepsilon_{A}=E_{z}+\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}, \hat{d}_{A}^{\dagger}=\sqrt{\frac{1}{2}\left(1+\frac{\varepsilon_{o r b}}{\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}}\right)} \hat{c}_{\uparrow}^{\dagger}+\sqrt{\frac{1}{2}\left(1-\frac{\varepsilon_{o r b}}{\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}}\right)} \hat{c}_{\downarrow} \\
& \varepsilon_{B}=E_{z}-\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}, \hat{d}_{B}^{\dagger}=-\sqrt{\frac{1}{2}\left(1-\frac{\varepsilon_{o r b}}{\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}}\right)} \hat{c}_{\uparrow}^{\dagger}+\sqrt{\frac{1}{2}\left(1+\frac{\varepsilon_{o r b}}{\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}}\right)} \hat{c}_{\downarrow}
\end{aligned}
$$

The subspace combining electrons with spin down and holes with spin up gives the modes $\hat{d}_{C}^{\dagger}$ and $\hat{d}_{D}^{\dagger}$ with energies $\varepsilon_{C}$ and $\varepsilon_{D}$ defined by

$$
\begin{aligned}
& \varepsilon_{C}=-E_{z}+\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}, \hat{d}_{C}^{\dagger}=-\sqrt{\frac{1}{2}\left(1-\frac{\varepsilon_{o r b}}{\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}}\right)} \hat{c}_{\uparrow}+\sqrt{\frac{1}{2}\left(1+\frac{\varepsilon_{o r b}}{\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}}\right)} \hat{c}_{\downarrow}^{\dagger} \\
& \varepsilon_{D}=-E_{z}-\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}, \hat{d}_{D}^{\dagger}=\sqrt{\frac{1}{2}\left(1+\frac{\varepsilon_{o r b}}{\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}}\right)} \hat{c}_{\uparrow}+\sqrt{\frac{1}{2}\left(1-\frac{\varepsilon_{o r b}}{\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}}\right)} \hat{c}_{\downarrow}^{\dagger}
\end{aligned}
$$

The modes B and C cross at zero energy for $E_{z}=\sqrt{E_{0}^{2}+\varepsilon_{o r b}^{2}}$, which enables a zero energy crossing in the density of states $\nu(\omega)$ of the dot (see Fig. 1, left panel). Each of the levels B and C can lead to photo-assisted tunneling to the fermionic reservoirs, which should give signals similar to feature 1 of Fig. 2 of the main text (see Fig. 1, right panel, pink line). The coupling term between transition B-C and the cavity is strictly zero because these two states belong to two different Nambu-spin subspaces (see above equations). This implies that direct microwave transitions between $B$ and $C$ are not possible. Hence, one could worry that the trivial zero energy crossing between states B and C could mimic the first zero energy level crossing in Fig.1,b (case B) and its microwave response. However, these two cases can be discriminated by considering the analogue of feature 2 which reveals a specific signature of the trivial degeneracy lifting provided by the magnetic field. Indeed, transitions between the states A and B or C and D can occur at a frequency $\omega_{2}=\sqrt{E_{0}^{2}+\varepsilon_{\text {orb }}^{2}} / \pi \hbar$ which is independent of $E_{z}$, with a matrix element $-\sqrt{E_{0}^{2} /\left(E_{0}^{2}+\varepsilon_{o r b}^{2}\right)}$ which is also independent of $E_{z}$. At low temperatures, this transition should be visible for low values of $E_{z}\left(E_{z}<\sqrt{E_{0}^{2}+\varepsilon_{\text {orb }}^{2}}\right)$ such that states C and B are populated and empty respectively $\left(\varepsilon_{C}<0\right.$ and $\left.\varepsilon_{B}>0\right)$, as shown by the see red line
in Figure 1, right panel. The independence of $\omega_{2}$ from $E_{z}$ is a specific property of the trivial case, because in the Majorana case, the frequency of feature 2 strongly depends on $E_{z}$ due to the topological phase transition (see the strongest resonance in Figure 3a). One should keep in mind that this distinction problem raises only if it is not possible to observe several consecutive zero energy crossings in $\nu(\omega)$, because a trivial pair of Andreev states should give only one zero energy crossing when $E_{z}$ is increased from 0 .

## DEFINITION OF THE RESERVOIRS CONTRIBUTION $\hat{h}_{\mathcal{R}}$ TO OUR CHAIN HAMILTONIAN

We describe explicitly the $S$ and $N_{L(R)}$ reservoirs with the Hamiltonian

$$
\begin{equation*}
\hat{h}_{\mathcal{R}}=\sum_{O, k, \sigma} \varepsilon_{k O} \hat{c}_{O k \sigma}^{\dagger} \hat{c}_{O k \sigma}+\Delta \sum_{k}\left(\hat{c}_{S k \uparrow}^{\dagger} \hat{c}_{S-k \downarrow}^{\dagger}+h . c .\right)+\sum_{n, O, k, \sigma}\left(t_{O k n} \hat{d}_{n \sigma}^{\dagger} \hat{c}_{O k \sigma}+h . c .\right)+h_{d i s s}^{S} \tag{9}
\end{equation*}
$$

where $\hat{c}_{O k \sigma}^{\dagger}$ is the creation operator for an electron with momentum $k$ and spin $\sigma$ in the reservoir $O \in\left\{S, N_{L}, N_{R}\right\}$. Above, $t_{O k n}$ is the tunneling constant between site $n$ and reservoir $O$. In the following we use energy-independent tunnel constants $t_{S k n}=t_{S}$ and $t_{N k n}=t_{N}\left(\delta_{n, 1}+\delta_{n, \mathbf{N}}\right)$, and a broad band approximation for the reservoirs. The tunnel rates are defined as $\Gamma_{O}=2 \pi\left|t_{O, n}\right|^{2} \rho_{O}$ with $\rho_{O}$ the density of states per spin direction in reservoir $O \in\left\{S, N_{L}, N_{R}\right\}$. In the main text, we use $\Gamma_{N_{L(R)}}=\Gamma_{N}$. The term $h_{\text {diss }}^{S}$ accounts for the broadening of the BCS peaks in the DOS of $S$ and the finite subgap DOS (see details in Ref.[4]). These effects depend on the DOS broadening parameter $\Gamma_{b}$.

## DETAILS OF OUR KELDYSH APPROACH

The retarded Green's function $\mathcal{G}^{r}$ of the chain has site-indexed elements with the following structure in the Nambu $\otimes$ spin subspace:

$$
\mathcal{G}_{n, n^{\prime}}^{r}=\left[\begin{array}{llll}
\mathcal{G}_{\hat{d}_{n, \uparrow}, \hat{d}_{n, \uparrow}^{\dagger}}^{r} & \mathcal{G}_{\hat{d}_{n, \uparrow}, \hat{d}_{n, \downarrow}}^{r} & \mathcal{G}_{\hat{d}_{n, \uparrow}, \hat{d}_{n, \downarrow}^{\dagger}}^{r} & \mathcal{G}_{\hat{d}_{n, \uparrow}, \hat{d}_{n, \uparrow}}^{r}  \tag{10}\\
\mathcal{G}_{\hat{d}_{n, \downarrow}^{\dagger}}^{\dagger}, \hat{d}_{n, \uparrow}^{\dagger} & \mathcal{G}_{\hat{d}_{n, \downarrow}^{\dagger}, \hat{d}_{n, \downarrow}}^{r} & \mathcal{G}_{\hat{d}_{n, \downarrow}^{\dagger}, \hat{d}_{n, \downarrow}^{\dagger}}^{r} & \mathcal{G}_{\hat{d}_{n, \downarrow}^{\dagger}}^{r} \hat{d}_{n, \uparrow} \\
\mathcal{G}_{\hat{d}_{n, \downarrow}, \hat{d}_{n, \uparrow}^{\dagger}}^{r} & \mathcal{G}_{\hat{d}_{n, \downarrow}, \hat{d}_{n, \downarrow}}^{r} & \mathcal{G}_{\hat{d}_{n, \downarrow}, \hat{d}_{n, \downarrow}^{\dagger}}^{r} & \mathcal{G}_{\hat{d}_{n, \downarrow}, \hat{d}_{n, \uparrow}}^{r} \\
\mathcal{G}_{\hat{d}_{n, \uparrow}^{\dagger}, \hat{d}_{n, \uparrow}^{\dagger}}^{r} & \mathcal{G}_{\hat{d}_{n, \uparrow}^{\dagger}, \hat{d}_{n, \downarrow}}^{r} & \mathcal{G}_{\hat{d}_{n, \uparrow}^{\dagger}, \hat{d}_{n, \downarrow}^{\dagger}}^{r} & \mathcal{G}_{\hat{d}_{n, \uparrow}^{\dagger}, \hat{d}_{n, \uparrow}}^{r}
\end{array}\right]
$$

For any operators $A$ and $B$, we use $\mathcal{G}_{A, B}^{r}(t)=-i \theta(t)\langle\{A(t), B(t=0)\}\rangle$. Using Hamiltonian $\hat{h}_{\mathcal{W}}+\hat{h}_{\mathcal{R}}$, we obtain:

$$
\begin{equation*}
\mathcal{G}_{d d}^{r}=\left[\omega \mathbf{1}+\Omega-\Sigma_{N}^{r}-\Sigma_{S}^{r}\right]^{-1} \tag{11}
\end{equation*}
$$

Above, the matrices $\Omega, \Sigma_{N}^{r} \Sigma_{S}^{r}$ are given by:

$$
\left.\begin{array}{rl}
\Omega_{n, n^{\prime}} & =\left[\begin{array}{llll}
\mu-E_{z} & 0 & 0 & 0 \\
0 & -\mu-E_{z} & 0 & 0 \\
0 & 0 & \mu+E_{z} & 0 \\
0 & 0 & 0 & -\mu+E_{z}
\end{array}\right] \\
& +\left[\begin{array}{lll}
t & 0 & 0 \\
0 \\
0 & -t & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0
\end{array}\right]\left(\delta_{n-1, n}+\delta_{n+1, n}\right)+\left[\begin{array}{lll}
0 & 0 & -\Lambda_{x} \\
0 & 0 & 0 \\
\Lambda_{x} & 0 & 0 \\
0 & \Lambda_{x} & 0 \\
0
\end{array}\right] \delta_{n, n^{\prime}} \\
\left.\Sigma_{N, n, n^{\prime}}^{r}=-i \frac{\Gamma_{N, n}}{2}\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right] \delta_{n+1, n}-\delta_{n-1, n}\right) \tag{13}
\end{array}\right] \delta_{n, n^{\prime}} .
$$

and

$$
\Sigma_{S}^{r}=-i \Gamma_{S} \tilde{G} / 2
$$

with

$$
\begin{align*}
\tilde{G}_{n, n^{\prime}} & =\left[\begin{array}{llll}
G_{\omega} & F_{\omega} & 0 & 0 \\
F_{\omega} & G_{\omega} & 0 & 0 \\
0 & 0 & G_{\omega} & -F_{\omega} \\
0 & 0 & -F_{\omega} & G_{\omega}
\end{array}\right] \delta_{n, n^{\prime}}  \tag{14}\\
G_{\omega} & =-i\left(\omega+i \frac{\Gamma_{b}}{2}\right) / D_{\omega}, F_{\omega}=i \Delta / D_{\omega} \tag{15}
\end{align*}
$$

and $D_{\omega}=\sqrt{\Delta^{2}-\left(\omega+i\left(\Gamma_{b} / 2\right)\right)^{2}}$.
The lesser self energy $\Sigma^{<}(\omega)$ can be expressed as:

$$
\Sigma_{n, n^{\prime}}^{<}(\omega)=i f(\omega) \Gamma_{S n} \operatorname{Re}[\tilde{G}] \delta_{n, n^{\prime}}+i \Gamma_{N n} f\left(\omega-e V_{b}\right)\left[\begin{array}{llll}
1 & 0 & 0 & 0  \tag{16}\\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] \delta_{n, n^{\prime}}+i \Gamma_{N n} f\left(\omega+e V_{b}\right)\left[\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right] \delta_{n, n^{\prime}}
$$

The above expressions can be used to calculate the charge susceptibility $\chi(\omega)$ from Eq. (2) of the main text. Note that this last Eq. is divided by a factor 2 in comparison with Ref.[4] because we have introduced a redundancy in the spin sector to describe properly non-collinearities induced by the spin-orbit coupling.

## ABSENCE OF MICROWAVE TRANSITIONS INSIDE A MAJORANA DOUBLET IN THE PRESENCE OF DISSIPATION

In the main text, we have derived analytically the absence of microwave transitions inside a Majorana doublet by considering a nanocircuit with a discrete spectrum. We have also found with a Keldysh numerical approach that this result persists in the presence of fermionic continua of states from metallic reservoirs, at least in the regime of parameters explored by us. It is instructive to reconsider this second case analytically with a simplified approach.

We assume that a pair of Majorana fermions $\left(\hat{m}_{L}, \hat{m}_{R}\right)$ is coupled to a fermionic continuum of states described by an energy-dependent fermion creator operator $f^{\dagger}(E)$. The total Hamiltonian of the system is

$$
\begin{equation*}
\hat{H}=\hat{H}_{0}+\hat{V} \tag{17}
\end{equation*}
$$

with

$$
\begin{gather*}
\hat{H}_{0}=\varepsilon \hat{\gamma}_{1}^{\dagger} \hat{\gamma}_{1}+\int_{0}^{+\infty} d E E \hat{f}^{\dagger}(E) \hat{f}(E)  \tag{18}\\
\hat{V}=\int_{0}^{+\infty} d E \lambda(E)\left(\hat{\gamma}_{1}^{\dagger} \hat{f}(E)+\hat{f}^{\dagger}(E) \hat{\gamma}_{1}\right)+\int_{0}^{+\infty} d E \mu(E)\left(\hat{\gamma}_{1}^{\dagger} \hat{f}^{\dagger}(E)+\hat{f}(E) \hat{\gamma}_{1}\right) \tag{19}
\end{gather*}
$$

and $\hat{\gamma}_{1}^{\dagger}=\left(\hat{m}_{L}-i \hat{m}_{R}\right) / \sqrt{2}$ the ordinary fermion operator associated to the Majorana pair. The coupling term $\hat{V}$ between the operators $\left(\hat{\gamma}_{1}^{\dagger}, \hat{\gamma}_{1}\right)$ and $\left(\hat{f}^{\dagger}, \hat{f}\right)$ contains a particle conserving term in $\lambda$ as well as a non-conserving term in $\mu$ due to the presence of superconductivity in the circuit. One can diagonalize $\hat{H}$, i.e.

$$
\begin{equation*}
\hat{H}=\int_{0}^{+\infty} d E E \hat{d}^{\dagger}(E) \hat{d}(E) \tag{20}
\end{equation*}
$$

with $E>0$ by using:

$$
\begin{equation*}
\hat{d}^{\dagger}(E)=\hat{f}^{\dagger}(E)+u(E) \hat{\gamma}_{1}^{\dagger}+v(E) \hat{\gamma}_{1}+\int_{0}^{+\infty} d E^{\prime} \quad\left(x\left(E, E^{\prime}\right) \hat{f}^{\dagger}\left(E^{\prime}\right)+y\left(E, E^{\prime}\right) \hat{f}\left(E^{\prime}\right)\right) \tag{21}
\end{equation*}
$$

The operator $\hat{d}^{\dagger}(E)$ can be obtained by solving the Lippmann-Schwinger equation[3]

$$
\begin{equation*}
\hat{d}^{\dagger}(E)=\hat{f}^{\dagger}(E)+\frac{1}{E-a d_{\hat{H}_{0}}+i \eta} a d_{\hat{V}}\left(\hat{d}^{\dagger}(E)\right) \tag{22}
\end{equation*}
$$

with $\eta \rightarrow 0^{+}$and $a d_{\hat{b}}(\hat{a})=[\hat{b}, \hat{a}]$. This equation can be transformed into:

$$
\begin{equation*}
\left(E-a d_{\hat{H}}+i \eta\right) \hat{d}^{\dagger}(E)=i \eta \hat{f}^{\dagger}(E) \tag{23}
\end{equation*}
$$

By combining Eqs.(18), (22) and (23), we obtain the system of equations:

$$
\begin{gather*}
(E-\varepsilon+i \eta) u(E)-\int_{0}^{+\infty} d E^{\prime}\left(x\left(E, E^{\prime}\right) \lambda\left(E^{\prime}\right)+y\left(E, E^{\prime}\right) \mu\left(E^{\prime}\right)\right)=\lambda(E)  \tag{24}\\
(E+\varepsilon+i \eta) v(E)+\int_{0}^{+\infty} d E^{\prime}\left(x\left(E, E^{\prime}\right) \mu\left(E^{\prime}\right)+y\left(E, E^{\prime}\right) \lambda\left(E^{\prime}\right)\right)=-\mu(E)  \tag{25}\\
\left(E-E^{\prime}+i \eta\right) x\left(E, E^{\prime}\right)-\lambda\left(E^{\prime}\right) u(E)+\mu\left(E^{\prime}\right) v(E)=0  \tag{26}\\
\left(E+E^{\prime}+i \eta\right) y\left(E, E^{\prime}\right)-\mu\left(E^{\prime}\right) u(E)+\lambda\left(E^{\prime}\right) v(E)=0 \tag{27}
\end{gather*}
$$

From Eqs.(24-27), we obtain

$$
\left[\begin{array}{c}
u(E)  \tag{28}\\
v(E)
\end{array}\right]=D^{-1}(E)\left[\begin{array}{cc}
E+\varepsilon+i \eta-\tilde{\Sigma}_{d} & -\Sigma_{c} \\
-\Sigma_{c} & E-\varepsilon+i \eta-\Sigma_{d}
\end{array}\right]\left[\begin{array}{c}
\lambda(E) \\
-\mu(E)
\end{array}\right]
$$

with

$$
\begin{equation*}
D(E)=\left(E-\varepsilon+i \eta-\Sigma_{d}\right)\left(E+\varepsilon+i \eta-\tilde{\Sigma}_{d}\right)-\Sigma_{c}^{2} \tag{29}
\end{equation*}
$$

Above, the quantities

$$
\begin{align*}
& \Sigma_{c}=\int_{0}^{+\infty} d E^{\prime} \lambda\left(E^{\prime}\right) \mu\left(E^{\prime}\right)\left(\frac{1}{E-E^{\prime}+i \eta}+\frac{1}{E+E^{\prime}+i \eta}\right)  \tag{30}\\
& \Sigma_{d}=\int_{0}^{+\infty} d E^{\prime}\left(\frac{\lambda^{2}\left(E^{\prime}\right)}{E-E^{\prime}+i \eta}+\frac{\mu^{2}\left(E^{\prime}\right)}{E+E^{\prime}+i \eta}\right)  \tag{31}\\
& \tilde{\Sigma}_{d}=\int_{0}^{+\infty} d E^{\prime}\left(\frac{\lambda^{2}\left(E^{\prime}\right)}{E+E^{\prime}+i \eta}+\frac{\mu^{2}\left(E^{\prime}\right)}{E-E^{\prime}+i \eta}\right) \tag{32}
\end{align*}
$$

are self-energies which account for the coupling between the Majorana pair and the continuum of states.
Accordingly with the main text, we now assume that the Majorana doublet couples to the cavity through the term $\hat{h}_{\mathcal{C}}\left(\hat{a}+\hat{a}^{\dagger}\right)$ with

$$
\begin{equation*}
\hat{h}_{\mathcal{C}}=\beta \hat{\gamma}_{1}^{\dagger} \hat{\gamma}_{1} \tag{33}
\end{equation*}
$$

We also assume that the system is in the ground state of Hamiltonian (20), i.e. all quasiparticle states are filled up to $\varepsilon=0$. For our circuit, this corresponds to a filling of the energy levels up the Fermi energy, in an equilibrium situation $(V=0, T=0)$. In these conditions, using the equations

$$
\begin{equation*}
\hat{\gamma}_{1}^{\dagger}=\int_{0}^{+\infty} d E\left(u^{*}(E) \hat{d}^{\dagger}(E)+v(E) \hat{d}(E)\right) \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\gamma}_{1}=\int_{0}^{+\infty} d E\left(v^{*}(E) \hat{d}^{\dagger}(E)+u(E) \hat{d}(E)\right) \tag{35}
\end{equation*}
$$

which result from Eq.(21), one finds that the only term of $\hat{h}_{\mathcal{C}}$ which contributes to the charge susceptibility of the circuit is:

$$
\begin{equation*}
\hat{O}_{\mathcal{C}}^{a d d} \simeq \int_{0}^{+\infty} \int_{0}^{+\infty} d E d E^{\prime} N\left(E, E^{\prime}\right) \hat{d}^{\dagger}(E) \hat{d}^{\dagger}\left(E^{\prime}\right) \tag{36}
\end{equation*}
$$

with

$$
\begin{equation*}
N\left(E, E^{\prime}\right)=\beta\left(u^{*}(E) v^{*}\left(E^{\prime}\right)-u^{*}\left(E^{\prime}\right) v^{*}(E)\right) / 2 \tag{37}
\end{equation*}
$$



FIG. 2: (a): $\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]$ and $\operatorname{Im}\left[\tilde{\chi}\left(\omega_{0}\right)\right]$ versus $\omega_{0}$. (b): $D^{-1}(E)$ and $D^{-1}\left(\omega_{0}-E\right)$ versus $E$ for different values of $\omega_{0}$, which correspond to the empty circles in panel (a).

Indeed, by generalizing the expression $\chi^{*}\left(\omega_{0}\right) \simeq \sum_{\alpha \beta}\left|N_{\alpha \beta}\right|^{2}\left(\omega_{0}-E_{\alpha}-E_{\beta}+i 0^{+}\right)^{-1} / 2$ of the main text, we obtain

$$
\begin{equation*}
\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]=\frac{\pi}{2} \int_{0}^{+\infty} \int_{0}^{+\infty} d E d E^{\prime}\left|N\left(E, E^{\prime}\right)\right|^{2} \delta\left(\omega_{0}-E-E^{\prime}\right) \tag{38}
\end{equation*}
$$

which gives

$$
\begin{equation*}
\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]=\frac{\pi}{2} \int_{0}^{\omega_{0}} d E\left|N\left(E, \omega_{0}-E\right)\right|^{2} \tag{39}
\end{equation*}
$$

Equations (36) and (38) are particularly instructive, since they show how dissipation could possibly induce a microwave resonance in the Majorana subspace for $\omega_{0}=2 \varepsilon$. One has $N(E, E)=0$, similarly to the fact that $N_{\alpha \alpha}=0$ in the main text. However, due to the broadening of the Majorana states, terms in $\hat{d}^{\dagger}(E) \hat{d}^{\dagger}\left(E^{\prime}\right)$, with $E \simeq E^{\prime} \simeq \varepsilon$ but $E \neq E^{\prime}$, could modify the cavity response. Therefore, the effect of $\hat{O}_{\mathcal{C}}^{\text {add }}$ must be considered carefully.

To evaluate $\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]$, we now make the wide band approximation on the continuum of states, like in the main text. We furthermore assume that $\lambda(E)$ and $\mu(E)$ are independent from energy, which is relevant for $\varepsilon \ll \Delta$ if the $S$ contact dominates low energy quasiparticle tunneling to/from the chain sites. We finally perform the usual simplification $|\lambda|=|\mu|$ (see Refs.[5-9]). This gives $\Sigma_{c}=\Sigma_{d}=\Sigma_{d}=-i \Gamma$ with $\Gamma=2 \pi \lambda^{2}=2 \pi \mu^{2}$ the dissipation rate associated to the continuum of states. With the above assumptions we find

$$
\begin{equation*}
N\left(E, E^{\prime}\right)=\beta \frac{\varepsilon \Gamma}{\pi} \frac{\left(E-E^{\prime}\right)}{D(E) D\left(E^{\prime}\right)} \tag{40}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]=\frac{(\beta \varepsilon \Gamma)^{2}}{2 \pi} \int_{0}^{\omega_{0}} d E \frac{\left(2 E-\omega_{0}\right)^{2}}{\left|D(E) D\left(\omega_{0}-E\right)\right|^{2}} \tag{41}
\end{equation*}
$$

with

$$
\begin{equation*}
D(E)=(E-\varepsilon+i \Gamma)(E+\varepsilon+i \Gamma)+\Gamma^{2} \tag{42}
\end{equation*}
$$

Figure 1a shows with a black full line $\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]$ versus $\omega_{0}$, calculated from Eqs. (41) and (42). First, we recover the general result $\operatorname{Im}\left[\chi\left(\omega_{0}=0\right)\right]=0$ of the adiabatic $\operatorname{limit}[4]$. Then, $\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]$ shows a maximum for $\omega_{0}=\varepsilon$, whereas no particular feature is visible at $\omega_{0}=2 \varepsilon$. To understand how this behavior arises from Eq.(41), it is useful to study the variations of $|D(E)|^{-1}$ and $\left|D\left(\omega_{0}-E\right)\right|^{-1}$ with $E$ (see Figure 1 b , blue and red full lines respectively). The integration window $E \in\left[0, \omega_{0}\right]$ in Eq.(41) is indicated with gray rectangles. The function $|D(E)|^{-1}$ shows two peaks at $E= \pm \varepsilon$, whereas $\left|D\left(\omega_{0}-E\right)\right|^{-1}$ shows two peaks at $E=\omega_{0} \pm \varepsilon$. From Fig.1b, the maximum of $\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]$ at $\omega_{0}=\varepsilon$ occurs because a peak from $|D(E)|^{-1}$ and a peak from $\left|D\left(\omega_{0}-E\right)\right|^{-1}$ enter the integration window. At first
glance, it may seem surprising that no resonance occurs in $\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]$ at $\omega_{0}=2 \varepsilon$ since the two peaks from $|D(E)|^{-1}$ and $\left|D\left(\omega_{0}-E\right)\right|^{-1}$ coincide and are in the middle of the integration window in this case. To understand this fact, we show with pink dashed lines the pseudo-susceptibility

$$
\begin{equation*}
\operatorname{Im}\left[\tilde{\chi}\left(\omega_{0}\right)\right]=2(\beta \varepsilon)^{2} \Gamma^{4} \int_{0}^{\omega_{0}} d E \frac{1}{\left|D(E) D\left(\omega_{0}-E\right)\right|^{2}} \tag{43}
\end{equation*}
$$

obtained by replacing the numerator $\left(2 E-\omega_{0}\right)^{2}$ in the integrand of Eq.(41) by an arbitrary constant $4 \pi \Gamma^{2}$. The pseudo-susceptibility $\operatorname{Im}\left[\tilde{\chi}\left(\omega_{0}\right)\right]$ shows a step at $\omega_{0}=\varepsilon$ but also a resonance at $\omega_{0}=2 \varepsilon$. Hence, the absence of resonance in $\chi\left(\omega_{0}\right)$ is clearly due to the $\left(2 E-\omega_{0}\right)^{2}$ factor. Importantly, this factor is directly due to the Pauli exclusion principle since it stems from the factor $\left(E-E^{\prime}\right)$ in Eq. 40 , which ensures $N(E, E)=0$. We recall that the Pauli exclusion principle is at work in this situation due to the fact that a single Majorana pair is coupled to cavity photons. We conclude that the transition between the two MBSs at $\omega_{0}=2 \varepsilon$ is still inhibited by the self adjoint character of MBSs when these MBSs are broadened by the coupling to a fermionic continuum of states. Finally, $\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]$ decreases for $\omega_{0}>\varepsilon$, and vanishes for $\omega_{0} \gg \varepsilon$. This is because in the limit $\omega_{0} \gg \varepsilon$, photon absorption and emission processes between the Majorana doublet and the continuum of states compensate each other, regardless of the details of the fermionic continuum in the area $E \sim 0$. Note that in this supplementary material, we have studied the absence of peak in $\operatorname{Im}\left[\chi\left(\omega_{0}\right)\right]$ at $\omega_{0}=2 \varepsilon$ for the case $T=0$. However, from the numerical Keldysh approach of the main text, we find that this transition remains absent for finite temperatures.
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