Subradiant split Cooper pairs (Supplemental Material)

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A. HAMILTONIAN OF THE CPS

Inside the left and right dots $i \in \{L, R\}$, an electron with spin $\sigma \in \{\uparrow, \downarrow\}$ can be in the orbital $\tau \in \{K, K'\}$, which is reminiscent from the K/K' degeneracy of graphene. We use a double dot effective hamiltonian which includes the superconducting proximity effect due to the superconducting contact, i.e.

$$H_{ddot}^{eff} = \sum_{i,\tau,\sigma} (\varepsilon + \Delta_{so}\tau\sigma) n_{i\tau\sigma} + H_{int} + \Delta_{K\leftrightarrow K'} \sum_{i,\sigma} (d_{iK\sigma}^\dagger d_{iK'\sigma} + d_{iK'\sigma}^\dagger d_{iK\sigma}) \quad (1)$$
$$+ t_{ee} \sum_{\tau,\sigma} (d_{L\tau\sigma}^\dagger d_{R\tau\sigma} + d_{R\tau\sigma}^\dagger d_{L\tau\sigma}) + t_{eh} \sum_{\tau} \left\{ \left(d_{L\tau\uparrow}^\dagger d_{R\overline{\tau}\downarrow}^\dagger - d_{L\overline{\tau}\downarrow}^\dagger d_{R\tau\uparrow}^\dagger \right) + h.c. \right\}$$

with $\overline{K} = K'$ and $\overline{K'} = K$. The term in t_{eh} accounts for coherent injection of singlet Cooper pairs inside the double dot [1]. Taking into account the superconducting contacts with the term in t_{eh} is valid provided quasiparticle transport between the superconducting contact and the double dot can be disregarded (see part C of the Supplemental Material). For simplicity, we assume that the orbital energies in dots L and R are both equal to ε, which can be obtained by tuning properly the dots' gate voltages. The term H_{int} accounts for Coulomb charging effects. We assume that there cannot be more than one electron in each dot, due to a strong intra-dot Coulomb charging energy. The constant Δ_{so} corresponds to an effective spin-orbit coupling [2]. The term $\Delta_{K\leftrightarrow K'}$ describes a coupling between the K and K' orbitals of dot i, due to disorder at the level of the carbon nanotube atomic structure [2–4]. The hamiltonian H_{ddot}^{eff} must be supplemented by the normal leads hamiltonian $H_{leads} = \sum_{k_\tau,i,\sigma} \varepsilon_{ik_\tau} c_{ik_\tau\sigma}^\dagger c_{ik_\tau\sigma} + h.c.$ and the tunnel coupling between the dots and normal leads $H_t = \sum_{k_\tau,\tau,i,\sigma} t c_{ik_\tau\sigma}^\dagger d_{i\tau\sigma} + h.c.$, with $c_{ik_\tau\sigma}$ the annihilation operator for an electron with spin σ in orbital k_τ of the normal lead $i \in \{L, R\}$.

B. EXPRESSION OF THE CPS EIGENSTATES

We now discuss the eigenstates and eigenvectors of hamiltonian (1) in the general case. The diagonalization of hamiltonian (1) can be performed by block in five different subpaces, the two subspaces of states occupied with a single spin $\sigma \in \{\uparrow, \downarrow\}$, the two subspaces of states occupied with two equal spins $\sigma \in \{\uparrow, \downarrow\}$, and the subspace comprising the empty state and states occupied with two opposite spins.

B.1. Subspace of singly occupied states with spin σ

One can treat separately the subspace of singly occupied states with spin $\uparrow$ and the subspace of singly occupied states with spin $\downarrow$. For a given spin direction $\sigma \in \{\uparrow, \downarrow\}$, the eigenenergies and corresponding eigenvectors of H_{ddot}^{eff}

are:

eigenenergy	eigenvector
$\varepsilon_{1\sigma} = \varepsilon - t_{ee} - \Delta_r$	$\|s_{1\sigma}\rangle = \frac{1}{2}\sqrt{1-\sigma\frac{\Delta_{so}}{\Delta_r}}\left(\|K\sigma,0\rangle - \|0,K\sigma\rangle\right) + \frac{\Delta_{K/K'}}{2\Delta_r\sqrt{1-\sigma\frac{\Delta_{so}}{\Delta_r}}}\left(\|0,K'\sigma\rangle - \|K'\sigma,0\rangle\right)$
$\varepsilon_{2\sigma} = \varepsilon + t_{ee} - \Delta_r$	$\|s_{2\sigma}\rangle = -\frac{1}{2}\sqrt{1-\sigma\frac{\Delta_{so}}{\Delta_r}}\left(\|K\sigma,0\rangle + \|0,K\sigma\rangle\right) + \frac{\Delta_{K/K'}}{2\Delta_r\sqrt{1-\sigma\frac{\Delta_{so}}{\Delta_r}}}\left(\|0,K'\sigma\rangle + \|K'\sigma,0\rangle\right)$
$\varepsilon_{3\sigma} = \varepsilon - t_{ee} + \Delta_r$	$\|s_{3\sigma}\rangle = -\frac{1}{2}\sqrt{1+\sigma\frac{\Delta_{so}}{\Delta_r}}\left(\|K\sigma,0\rangle - \|0,K\sigma\rangle\right) + \frac{\Delta_{K/K'}}{2\Delta_r\sqrt{1+\sigma\frac{\Delta_{so}}{\Delta_r}}}\left(\|0,K'\sigma\rangle\} - \|K'\sigma,0\rangle\right)$
$\varepsilon_{4\sigma} = \varepsilon + t_{ee} + \Delta_r$	$\|s_{4\sigma}\rangle = \frac{1}{2}\sqrt{1+\sigma\frac{\Delta_{so}}{\Delta_r}}\left(\|K\sigma,0\rangle + \|0,K\sigma\rangle\right) + \frac{\Delta_{K/K'}}{2\Delta_r\sqrt{1+\sigma\frac{\Delta_{so}}{\Delta_r}}}\left(\|0,K'\sigma\rangle\} + \|K'\sigma,0\rangle\right)$

with $\Delta_r = \sqrt{\Delta_{so}^2 + \Delta_{K/K'}^2}$. Note that t_{ee} occurs in the expressions of the eigenenergies but not in the expressions of the eigenvectors because we have assumed that the left and rights dot have the same orbital energy ε.

B.2. Subspace of states occupied with two equal spins σ

One can treat separately the subspace of states occupied with two equal spins $\uparrow$ and the subspace of states occupied with two equal spins $\downarrow$. For a given spin direction $\sigma \in \{\uparrow,\downarrow\}$, the eigenenergies and corresponding eigenvectors of H_{ddot}^{eff} are:

eigenenergy	eigenvector
δ	$\left\|\tilde{S}_{1\sigma}\right\rangle = \frac{\Delta_{K/K'}}{2\tilde{\Delta}_r}\left(\|K'\sigma,K'\sigma\rangle - \|K\sigma,K\sigma\rangle\right) + \sigma\frac{\Delta_{so}}{\tilde{\Delta}_r}\|K'\sigma,K\sigma\rangle$
δ	$\left\|\tilde{S}_{2\sigma}\right\rangle = \frac{\Delta_{K/K'}}{2\tilde{\Delta}_r}\left(\|K'\sigma,K'\sigma\rangle - \|K\sigma,K\sigma\rangle\right) + \sigma\frac{\Delta_{so}}{\tilde{\Delta}_r}\|K\sigma,K'\sigma\rangle$
$\delta - 2\Delta_r$	$\left\|\tilde{S}_{3\sigma}\right\rangle = \frac{1}{2}\left(\sigma\frac{\Delta_{so}}{\Delta_r} - 1\right)\|K\sigma,K\sigma\rangle - \frac{1}{2}\left(1+\sigma\frac{\Delta_{so}}{\Delta_r}\right)\|K'\sigma,K'\sigma\rangle + \frac{\Delta_{K/K'}}{2\Delta_r}\left(\|K\sigma,K'\sigma\rangle + \|K'\sigma,K\sigma\rangle\right)$
$\delta + 2\Delta_r$	$\left\|\tilde{S}_{4\sigma}\right\rangle = \frac{1}{2}\left(1+\sigma\frac{\Delta_{so}}{\Delta_r}\right)\|K\sigma,K\sigma\rangle + \frac{1}{2}\left(1-\sigma\frac{\Delta_{so}}{\Delta_r}\right)\|K'\sigma,K'\sigma\rangle + \frac{\Delta_{K/K'}}{2\Delta_r}\left(\|K\sigma,K'\sigma\rangle + \|K'\sigma,K\sigma\rangle\right)$

with $\tilde{\Delta}_r = \sqrt{\Delta_{so}^2 + (\Delta_{K/K'}^2/2)}$. These states correspond to generalized triplet states with spin 1.

B.3. Subspace of states occupied with two opposite spins

We now discuss the subspace of states comprising the empty state $|0,0\rangle$ and states occupied with two opposite spins. It is practical to first define the eigenenergies and eigenvectors of H_{ddot}^{eff} for $t_{eh} = 0$:

eigenenergy	eigenvector
0	$\|0,0\rangle$
δ	$\|s_1\rangle = \frac{\Delta_{K/K'}}{2\tilde{\Delta}_r}\left(\|\mathcal{C}_-(K\downarrow,K'\uparrow)\rangle - \|\mathcal{C}_-(K'\downarrow,K\uparrow)\rangle\right) + \frac{\Delta_{so}}{\tilde{\Delta}_r}\|\mathcal{C}_-(K'\downarrow,K'\uparrow)\rangle$
δ	$\|t_1\rangle = \frac{\Delta_{K/K'}}{2\tilde{\Delta}_r}\left(\|\mathcal{C}_+(K\downarrow,K'\uparrow)\rangle - \|\mathcal{C}_+(K'\downarrow,K\uparrow)\rangle\right) + \frac{\Delta_{so}}{\tilde{\Delta}_r}\|\mathcal{C}_+(K'\downarrow,K'\uparrow)\rangle$
δ	$\|s_2\rangle = \frac{\Delta_{K/K'}}{2\tilde{\Delta}_r}\left(\|\mathcal{C}_-(K\downarrow,K'\uparrow)\rangle - \|\mathcal{C}_-(K'\downarrow,K\uparrow)\rangle\right) + \frac{\Delta_{so}}{\tilde{\Delta}_r}\|\mathcal{C}_-(K\downarrow,K\uparrow)\rangle$
δ	$\|t_2\rangle = \frac{\Delta_{K/K'}}{2\tilde{\Delta}_r}\left(\|\mathcal{C}_+(K\downarrow,K'\uparrow)\rangle - \|\mathcal{C}_+(K'\downarrow,K\uparrow)\rangle\right) + \frac{\Delta_{so}}{\tilde{\Delta}_r}\|\mathcal{C}_+(K\downarrow,K\uparrow)\rangle$
$\delta - 2\Delta_r$	$2\|s_3\rangle = \left(\frac{\Delta_{so}}{\Delta_r} - 1\right)\|\mathcal{C}_-(K\uparrow,K'\downarrow)\rangle - \left(\frac{\Delta_{so}}{\Delta_r} + 1\right)\|\mathcal{C}_-(K'\uparrow,K\downarrow)\rangle + \frac{\Delta_{K/K'}}{\Delta_r}\left(\|\mathcal{C}_-(K\uparrow,K\downarrow)\rangle + \|\mathcal{C}_-(K'\uparrow,K'\downarrow)\rangle\right)$
$\delta - 2\Delta_r$	$2\|t_3\rangle = \left(\frac{\Delta_{so}}{\Delta_r} - 1\right)\|\mathcal{C}_+(K\uparrow,K'\downarrow)\rangle - \left(\frac{\Delta_{so}}{\Delta_r} + 1\right)\|\mathcal{C}_+(K'\uparrow,K\downarrow)\rangle + \frac{\Delta_{K/K'}}{\Delta_r}\left(\|\mathcal{C}_+(K\uparrow,K\downarrow)\rangle + \|\mathcal{C}_+(K'\uparrow,K'\downarrow)\rangle\right)$
$\delta + 2\Delta_r$	$2\|s_4\rangle = \left(\frac{\Delta_{so}}{\Delta_r} + 1\right)\|\mathcal{C}_-(K\uparrow,K'\downarrow)\rangle + \left(1 - \frac{\Delta_{so}}{\Delta_r}\right)\|\mathcal{C}_-(K'\uparrow,K\downarrow)\rangle + \frac{\Delta_{K/K'}}{\Delta_r}\left(\|\mathcal{C}_-(K\uparrow,K\downarrow)\rangle + \|\mathcal{C}_-(K'\uparrow,K'\downarrow)\rangle\right)$
$\delta + 2\Delta_r$	$2\|t_4\rangle = \left(\frac{\Delta_{so}}{\Delta_r} + 1\right)\|\mathcal{C}_+(K\uparrow,K'\downarrow)\rangle + \left(1 - \frac{\Delta_{so}}{\Delta_r}\right)\|\mathcal{C}_+(K'\uparrow,K\downarrow)\rangle + \frac{\Delta_{K/K'}}{\Delta_r}\left(\|\mathcal{C}_+(K\uparrow,K\downarrow)\rangle + \|\mathcal{C}_+(K'\uparrow,K'\downarrow)\rangle\right)$

where $|\mathcal{C}_\pm(\tau\sigma,\tau'\sigma')\rangle = (|\tau\sigma,\tau'\sigma'\rangle \pm |\tau'\sigma',\tau\sigma\rangle)/\sqrt{2}$. From the definition of $|\mathcal{C}_\pm(\tau\sigma,\tau'\sigma')\rangle$, one can view the states $|s_1\rangle$, $|s_2\rangle$, $|s_3\rangle$ and $|s_4\rangle$ as generalized singlet states whereas $|t_1\rangle$, $|t_2\rangle$, $|t_3\rangle$ and $|t_4\rangle$ can be viewed as generalized triplet

states with total spin 0. In the subspace $\{|0,0\rangle,|s_1\rangle,|t_1\rangle,|s_2\rangle,|t_2\rangle,|s_3\rangle,|t_3\rangle,|s_4\rangle,|t_4\rangle\}$, the hamiltonian (1) writes exactly:

$$\tilde{H}^{eff}_{ddot} = \delta(|s_1\rangle\langle s_1| + |s_2\rangle\langle s_2| + |t_1\rangle\langle t_1| + |t_2\rangle\langle t_2|) + (\delta - 2\Delta_r)(|t_3\rangle\langle t_3| + |s_3\rangle\langle s_3|) + (\delta + 2\Delta_r)(|t_4\rangle\langle t_4| + |s_4\rangle\langle s_4|) + \sqrt{2}t_{eh}(|s_4\rangle\langle 0,0| + |0,0\rangle\langle s_4| - |s_3\rangle\langle 0,0| - |0,0\rangle\langle s_3|)$$

Thus, the states $|s_1\rangle$, $|s_2\rangle$, $|t_1\rangle$,$|t_2\rangle$, $|t_3\rangle$ and $|t_4\rangle$ are eigenstates of H^{eff}_{ddot} while the states $|s_3\rangle$, and $|s_4\rangle$ and are hybridized with $|0,0\rangle$. This hybridization leads to three eigenstates $|V_1\rangle$, $|V_2\rangle$ and $|V_3\rangle$ with energy E_1, E_2 and E_3. Figure 1 shows the energy of the different doubly occupied eigenstates (with total spin 1 or 0, i.e. given by sections B.2 or B.3) as a function of δ. The triplet states have energies δ, $\delta - 2\Delta_r$ or $\delta + 2\Delta_r$ (blue lines), while the energies E_1, E_2 and E_3 (red, green and pink lines) show two anticrossings at $\delta = -2\Delta_r$ and $\delta = 2\Delta_r$.

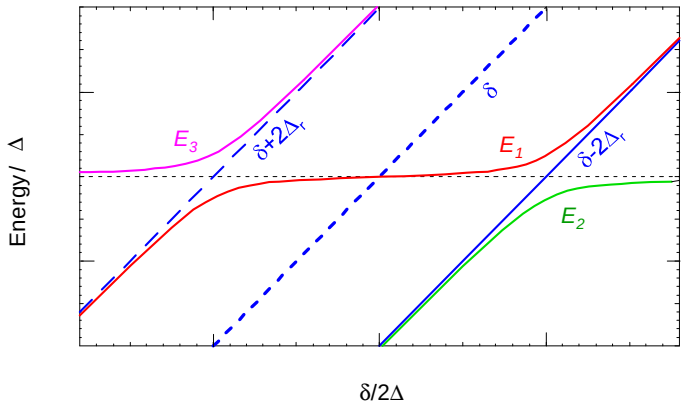

FIG. 1: Energy of the different doubly occupied eigenstates (with total spin 0 or 1) as a function of δ.

B.4. Expression of the CPS doubly occupied eigenstates for $\delta \sim 2\Delta_r$

In the main text, we work near the right anticrossing in Figure 1 ($\delta \sim 2\Delta_r$). For $\delta \sim 2\Delta_r$, a simplified expression of the relevant doubly occupied eigenstates of H^{eff}_{ddot} can be obtained by performing a diagonalization in the subpace $\{|0,0\rangle, |s_3\rangle,|t_3\rangle, \left|\tilde{S}_{3\uparrow}\right\rangle,\left|\tilde{S}_{3\downarrow}\right\rangle\}$. This leads to the CPS eigenstates $|V_1\rangle$, $|V_2\rangle$, $|T_0\rangle$, $|T_+\rangle$ and $|T_-\rangle$ discussed in the main text, which are defined by:

eigenenergy	eigenvector
$\delta - 2\Delta_r$	$\|T_0\rangle = \|t_3\rangle = \sum_\sigma \left\{\frac{1}{2}(\sigma\frac{\Delta_{so}}{\Delta_r} - 1)\|\mathcal{C}_+(K\sigma, K'\bar{\sigma})\rangle\right\} + \frac{\Delta_{K/K'}}{2\Delta_r}\sum_\tau \|\mathcal{C}_+(\tau\uparrow,\tau\downarrow)\rangle$
$\delta - 2\Delta_r$	$\|T_+\rangle = \frac{\|\tilde{S}_{3\uparrow}\rangle - \|\tilde{S}_{3\downarrow}\rangle}{\sqrt{2}} = \sum_\sigma \left\{\frac{1}{2}\left(\frac{\Delta_{so}}{\Delta_r} - \sigma\right)\frac{\|K\sigma,K\sigma\rangle - \|K'\bar{\sigma},K'\bar{\sigma}\rangle}{\sqrt{2}} + \sigma\frac{\Delta_{K/K'}}{2\Delta_r}\|\mathcal{C}_+(K\sigma,K'\sigma)\rangle\right\}$
$\delta - 2\Delta_r$	$\|T_-\rangle = \frac{\|\tilde{S}_{3\uparrow}\rangle + \|\tilde{S}_{3\downarrow}\rangle}{\sqrt{2}} = \sum_\sigma \left\{\frac{1}{2}\left(\frac{\Delta_{so}}{\Delta_r}\sigma - 1\right)\frac{\|K\sigma,K\sigma\rangle + \|K'\bar{\sigma},K'\bar{\sigma}\rangle}{\sqrt{2}} + \frac{\Delta_{K/K'}}{2\Delta_r}\|\mathcal{C}_+(K\sigma,K'\sigma)\rangle\right\}$
E_1	$\|V_1\rangle = \sqrt{1-v_1^2}\|0,0\rangle + v_1\|\mathcal{S}\rangle$
E_2	$\|V_2\rangle = \sqrt{1-v_2^2}\|0,0\rangle + v_2\|\mathcal{S}\rangle$

with

$$E_i = \frac{1}{2}\left(\delta - 2\Delta_r - (-1)^n\sqrt{8t_{eh}^2 + (\delta - 2\Delta_r)^2}\right)$$

$$v_n = \frac{2t_{eh}}{\sqrt{8t_{eh}^2 + (\delta - 2\Delta_r)(\delta - 2\Delta_r + (-1)^n\sqrt{8t_{eh}^2 + (\delta - 2\Delta_r)^2})}}$$

and

$$|\mathcal{S}\rangle = |s_3\rangle = \sum_\sigma \left\{ \frac{1}{2}\left(\frac{\Delta_{so}}{\Delta_r} - \sigma\right) |\mathcal{C}_-(K\sigma, K'\bar{\sigma})\rangle \right\} + \frac{\Delta_{K/K'}}{2\Delta_r} \sum_\tau |\mathcal{C}_-(\tau\uparrow, \tau\downarrow)\rangle$$

Note that $\sigma = \pm 1$ stands for $\sigma \in \{\uparrow, \downarrow\}$ in algebraic expressions. The state $|T_0\rangle$ corresponds to a generalized triplet state with zero spin, $|T_-\rangle$ and $|T_+\rangle$ correspond to a coherent mixture of two triplet states with finite spin, and the state $|\mathcal{S}\rangle$ corresponds to a generalized spin-singlet state. Note that without disorder ($\Delta_{K\leftrightarrow K'} = 0$), $|\mathcal{S}\rangle$ has components in $|\mathcal{C}_+(\tau\downarrow, \overline{\tau}\uparrow)\rangle$ only and $|T_-\rangle$ has components in $|\tau\sigma, \tau\sigma\rangle$ only. In the presence of disorder, $|\mathcal{S}\rangle$ also includes components in $\mathcal{C}_-(|\tau\uparrow, \tau\downarrow\rangle)$, and $|T_-\rangle$ has also components in $|K\sigma, K'\sigma\rangle$ and $|K'\sigma, K\sigma\rangle$. This enables a coupling between the states $\left|V_{1(2)}\right\rangle$ and $|T_-\rangle$ through H_{so}.

C. BIAS VOLTAGE WINDOW

The hamiltonian H_{ddot}^{eff} can be used provided there is no quasiparticle transport between the superconducting lead and the dots. This requires

$$-\Delta < -eV < \Delta \tag{2}$$

In the main text, we furthermore assume that electrons can go from the double dot to the normal metal leads but not the reverse. This is true provided the bias voltage V belongs to a certain range which we derive below.

(1) We recall that the double dot singly occupied states $|s_{i\sigma}\rangle$ have energies $\varepsilon_{i\sigma}$, with $i \in \{1, 2, 3, 4\}$, given in section B.1 of the supplemental material. We assume $\delta \sim 2\Delta_r$, so that the states $|V_1\rangle$ and $|V_2\rangle$ have significant components in $|0, 0\rangle$ and $|\mathcal{S}\rangle$ and their energies E_1 and E_2 are well approximated by the expressions given in section B.4. Since $|V_1\rangle$ and $|V_2\rangle$ have components in $|\mathcal{S}\rangle$, they can decay towards the singly occupied state $|s_{i\sigma}\rangle$ while an electron is transferred to the normal metal leads (and the reverse process is forbidden) if

$$-eV < E_{1(2)} - \varepsilon_{i\sigma} - \alpha k_B T \tag{3}$$

for $i \in \{1, 2, 3, 4\}$. Above, α is a dimensionless factor of order 1 which takes into account the temperature broadening of the levels.

(2) Since $|V_1\rangle$ and$|V_2\rangle$ have also components in $|0, 0\rangle$, a singly occupied state can decay towards $|V_1\rangle$ or $|V_2\rangle$ while an electron is transferred to the normal leads (and the reverse process is forbidden) if

$$-eV < \varepsilon_{i\sigma} - E_{1(2)} - \alpha k_B T \tag{4}$$

for $i \in \{1, 2, 3, 4\}$.

(3) Since we assume $\delta \sim 2\Delta_r$, the state $|V_3\rangle$ has a negligible component in $|0, 0\rangle$. Hence, it can be considered as a pure doubly occupied state, with energy $E_3 \simeq \delta + 2\Delta$. The other doubly occupied states have energies $\delta - 2\Delta_r$, $\delta + 2\Delta_r$, or δ (see section B.2 and B.3). The doubly occupied states (including $|V_3\rangle$) can relax to the singly occupied state $|s_{i\sigma}\rangle$ while an electron is transferred to the normal leads (and the reverse processes are forbidden) if

$$-eV < \delta \pm 2\Delta_r - \varepsilon_{i\sigma} - \alpha k_B T \tag{5}$$

and

$$-eV < \delta - \varepsilon_{i\sigma} - \alpha k_B T \tag{6}$$

Since $E_2 < \delta - 2\Delta_r < E_1, \delta < \delta + 2\Delta_r < E_3$ and $\varepsilon_{1\sigma} < \varepsilon_{2\sigma}, \varepsilon_{3\sigma} < \varepsilon_{4\sigma}$, the combination of Eqs. (2), (3), (4), (5) and (6) yields the constraint

$$-\Delta < -eV < \min(E_2 - \varepsilon_{4\sigma}, \varepsilon_{1\sigma} - E_1) - \alpha k_B T$$

We note $\delta = 2\varepsilon$. For $\delta \sim 2\Delta_r$, one has

$$E_2 - \varepsilon_{4\sigma} = -2\Delta_r - t_{ee} - \frac{1}{2}\sqrt{8t_{eh}^2 + (\delta - 2\Delta_r)^2} = \varepsilon_{1\sigma} - E_1 - 2\Delta_r$$

We conclude that we have to satisfy

$$-\Delta < -eV < -2\Delta_r - t_{ee} - \frac{1}{2}\sqrt{8t_{eh}^2 + (\delta - 2\Delta_r)^2} - \alpha k_B T$$

with Δ the BCS gap of the superconducting contact. Since the temperature T is much smaller than Δ_r in a typical experiment, we simplify this criterion as

$$-\Delta < -eV < -2\Delta_r - t_{ee} - \frac{1}{2}\sqrt{8t_{eh}^2 + (\delta - 2\Delta_r)^2}$$

With the parameters of Fig.2, $t_{ee} \ll \Delta_r$, $\delta \sim 2\Delta_r$ and Δ the BCS gap of NbN, this gives 1.8 meV $\lesssim eV \lesssim$ 3 meV.

To populate the state $|V_3\rangle$ and the triplet states other than $|T_-\rangle$, one needs to have a transition from a singly occupied state to one of these states. In the regime we consider above, this is not possible since this would require an electron to go from the normal metal leads to the double dot. Therefore, the state $|V_3\rangle$ and the triplet states other than $|T_-\rangle$ are not active. In contrast, the state $|T_-\rangle$ can be populated due to lasing transitions $|V_1\rangle \rightarrow |T_-\rangle$.

D. EFFECT OF THE SPIN-ORBIT COUPLING

In order to discuss the effect of the term h_{so} appearing in Eq. (2) of the main text, it is practical to redefine the eigenvectors of H_{ddot}^{eff} in the subspace of states occupied with two equal spins as

eigenenergy	eigenvectors
δ	$\|T_a\rangle = \frac{\alpha_+}{\sqrt{\alpha_-^2+\alpha_+^2}} \frac{\|\tilde{S}_{1\uparrow}\rangle - \|\tilde{S}_{2\downarrow}\rangle}{\sqrt{2}} - \frac{\alpha_-}{\sqrt{\alpha_-^2+\alpha_+^2}} \frac{\|\tilde{S}_{2\uparrow}\rangle - \|\tilde{S}_{1\downarrow}\rangle}{\sqrt{2}}$
δ	$\|T_b\rangle = \frac{\alpha_-}{\sqrt{\alpha_-^2+\alpha_+^2}} \frac{\|\tilde{S}_{1\uparrow}\rangle - \|\tilde{S}_{2\downarrow}\rangle}{\sqrt{2}} - \frac{\alpha_+}{\sqrt{\alpha_-^2+\alpha_+^2}} \frac{\|\tilde{S}_{2\uparrow}\rangle - \|\tilde{S}_{1\downarrow}\rangle}{\sqrt{2}}$
δ	$\|T_{1-}\rangle = \frac{\|\tilde{S}_{1\uparrow}\rangle + \|\tilde{S}_{2\downarrow}\rangle}{\sqrt{2}}$
δ	$\|T_{2-}\rangle = \frac{\|\tilde{S}_{2\uparrow}\rangle + \|\tilde{S}_{1\downarrow}\rangle}{\sqrt{2}}$
$\delta - 2\Delta_r$	$\|T_+\rangle$ already defined in section B.4
$\delta - 2\Delta_r$	$\|T_-\rangle$ already defined in section B.4
$\delta + 2\Delta_r$	$\|\tilde{S}_{4\uparrow}\rangle$
$\delta + 2\Delta_r$	$\|\tilde{S}_{4\downarrow}\rangle$

with

$$\alpha_\mp = \lambda_L - \lambda_R \mp \frac{\Delta_{so}}{\Delta_r}(\lambda_L + \lambda_R)$$

For $\delta \sim 2\Delta_r$ and the bias voltage conditions considered in section C, the states $|V_1\rangle$ and $|V_2\rangle$ can be populated but not $|V_3\rangle$. Only two of the triplet states, namely $|T_-\rangle$ and $|T_b\rangle$, are coupled to $|V_1\rangle$ and $|V_2\rangle$ by h_{so}. One has

$$\langle T_-| h_{so} |V_{1(2)}\rangle = \mathbf{i} v_{1(2)} \frac{\Delta_{K\leftrightarrow K'}}{\Delta_r}(\lambda_L - \lambda_R) \tag{7}$$

which corresponds to Eq. (3) of the main text, and

$$\langle T_b| h_{so} |V_{1(2)}\rangle = \frac{\mathbf{i} v_{1(2)} \Delta_{so}^2 (\lambda_R^2 - \lambda_L^2)}{\sqrt{\tilde{\Delta}_r \left(\Delta_{so}^2(\lambda_L^2 + \lambda_R^2) + \frac{\Delta_{K/K'}^2}{2}(\lambda_L - \lambda_R)^2 \right)}} \tag{8}$$

The couplings to $|T_-\rangle$ and $|T_b\rangle$ are both subradiant since they vanish for $\lambda_L = \lambda_R$. For $\Delta_{K\leftrightarrow K'} = 0$, the coupling between $|T_-\rangle$ and $|V_{1(2)}\rangle$ vanishes, whereas the coupling between $|T_b\rangle$ and $|V_{1(2)}\rangle$ persists:

$$\lim_{\Delta_{K\leftrightarrow K'} \rightarrow 0} \langle T_b| h_{so} |V_{1(2)}\rangle = \frac{\mathbf{i} v_{1(2)} (\lambda_R^2 - \lambda_L^2)}{\sqrt{\lambda_L^2 + \lambda_R^2}} \tag{9}$$

We conclude that it is not necessary to use a finite $\Delta_{K\leftrightarrow K'}$ to obtain a coupling between $\left|V_{1(2)}\right\rangle$ and a triplet state. It is nevertheless impossible to use the transitions $\left|V_{1(2)}\right\rangle \leftrightarrow |T_b\rangle$ for lasing since $|T_b\rangle$ is higher in energy than $|V_1\rangle$ and $|V_2\rangle$.

For $\delta \sim -2\Delta_r$, the state $|V_3\rangle$ can be populated. When $\Delta_{K\leftrightarrow K'} = 0$, we find that only one of the triplet states, namely $|T_a\rangle$, is coupled to $|V_3\rangle$, with a coupling element

$$\lim_{\Delta_{K\leftrightarrow K'}\to 0} \langle T_a| h_{so} |V_3\rangle = \frac{\mathbf{i}v_3(\lambda_R^2 - \lambda_L^2)}{\sqrt{\lambda_L^2 + \lambda_R^2}} \tag{10}$$

When $\Delta_{K\leftrightarrow K'} \neq 0$, one has:

$$\langle T_a| h_{so} |V_3\rangle = \frac{\mathbf{i}v_3\Delta_{so}^2(\lambda_R^2 - \lambda_L^2)}{\sqrt{\tilde{\Delta}_r\left(\Delta_{so}^2(\lambda_L^2 + \lambda_R^2) + \frac{\Delta_{K/K'}^2}{2}(\lambda_L - \lambda_R)^2\right)}} \tag{11}$$

These equations are analogue to Eqs. (9) and (8). In principle, if the cavity frequency is matching with $E_3 - \delta \sim 2\Delta_r + \sqrt{2}t_{eh}$, there can be lasing transitions from $|V_3\rangle$ to $|T_a\rangle$ since $|V_3\rangle$ is higher in energy than $|T_a\rangle$. The $|V_3\rangle \leftrightarrow |T_a\rangle$ transitions are subradiant since $\langle T_a| h_{so} |V_3\rangle$ cancels for $\lambda_L = \lambda_R$. Importantly, $\langle T_a| h_{so} |V_3\rangle$ remains finite for $\Delta_{K\leftrightarrow K'} = 0$. Therefore, $\Delta_{K\leftrightarrow K'} \neq 0$ is not a fundamental constraint to obtain a subradiant lasing transition in our system. Nevertheless, in practice, the frequency of the $|V_3\rangle \leftrightarrow |T_a\rangle$ transition is not likely to match with the cavity frequency because $\Delta_{K/K'}, \Delta_{so} \gg 2\pi\nu_0$ is expected [2, 4]. We have chosen to discuss the lasing transition $|V_1\rangle \leftrightarrow |T_-\rangle$ at $\delta \sim 2\Delta_r$ because it corresponds to a frequency of the order of $\sqrt{2}t_{eh}$, which is expected to be much smaller [15].

E. OTHER POSSIBLE LASING TRANSITIONS

- In principle, there can be radiative transitions from $|T_-\rangle$ to $|V_2\rangle$, which are taken into account by Eq.(7) of the main text. However, the lasing threshold corresponding to this transition is not reached in the regime of parameters we consider, because the population of state $|T_-\rangle$ is negligible for $\delta > 2\Delta_r$.
- Since the parameter δ depends on the dots' gate voltages, cavity photons can also couple to the CPS through the operator $\hat{\delta}_{diff} = \sum_{\tau,\sigma,\tau',\sigma'} |\tau\sigma,\tau'\sigma'\rangle\langle\tau\sigma,\tau'\sigma'|$. One finds $\langle V_1| \hat{\delta}_{diff} |V_2\rangle = \sqrt{2}t_{eh}/\sqrt{8t_{eh}^2 + (\delta - 2\Delta_r)^2}$. Hence, in principle, there can be lasing between the states $|V_1\rangle$ and $|V_2\rangle$. Nevertheless, this can be avoided by using $E_0 = 2\pi\hbar\nu_0 < 2\sqrt{2}t_{eh}$ or by tuning properly δ, so that $E_0 \neq E_1 - E_2$.
- In principle, spin-orbit interaction can also induce lasing transitions inside the CPS singly occupied charge sector, corresponding to energy differences $2\Delta_r$, $2\Delta_r + 2t_{ee}$, $2\Delta_r - 2t_{ee}$ and $2t_{ee}$. Since the scale Δ_r is expected to be much larger than E_0, only the transitions with frequency $t_{ee}/\pi\hbar$ can possibly match with the cavity frequency ν_0. However, it is rather unlikely to have such a matching in practice. In the main text we assume $\nu_0 \neq t_{ee}/\pi\hbar$. This criterion can be checked experimentally by extracting t_{ee} from the data.

F. EVALUATION OF THE SPIN/PHOTON COUPLING IN A CARBON-NANOTUBE BASED QUANTUM DOT

In this section, we estimate the spin/photon coupling $\lambda_{L(R)}$ which can be obtained in the single-wall carbon-nanotube based quantum dot $L(R)$ thanks to spin-orbit coupling.

F.1. Electronic wavefunction in the absence of inter-subband coupling elements

The position $\overrightarrow{u}$ of an electron on the nanotube is marked with a longitudinal coordinate ξ and an azimuthal angle φ, i.e. $\overrightarrow{u} = \xi\overrightarrow{z} + R\cos[\varphi]\overrightarrow{x} + R\sin[\varphi]\overrightarrow{y}$ with R the nanotube radius. We write electronic wavevectors under the form

$$|\Psi\rangle = e^{i\kappa\varphi} |\psi\rangle \otimes |\sigma\rangle \tag{12}$$

where κ is the electronic circumferential wavevector, $|\sigma\rangle$ denotes the spin part of the wavefunction and $|\psi\rangle$ is the ξ-dependent orbital part, which has a structure in sublattice space. We use the spin index $\sigma \in \{\uparrow, \downarrow\}$ and the sublattice index $\tau \in \{K, K'\}$ or equivalently $\sigma \in \{+, -\}$ and $\tau \in \{+, -\}$ in algebraic expressions. For a zigzag nanotube, $\langle \xi \, | \varphi \rangle$ is an eigenvector of

$$H_{SWNT} = \hbar v(\tau\kappa s_1 - i s_2 \frac{\partial}{\partial \xi}) + \sigma\tau\Delta_{so}^0 s_0 + \sigma\tau\Delta_{so}^1 s_1 - \Delta_g s_1 + V(\xi) s_0$$

with $V(\xi)$ a longitudinal confinement potential and $\{s_0, s_1, s_2, s_3)$ the identity and Pauli operators in sublattice space. We have used the same conventions as in Refs. [2, 11] to write H_{SWNT}. The motion of electrons along the nanotube circumference is quantized, i.e.

$$\kappa = (N + \frac{\tau\eta}{3})\frac{1}{R}$$

with N the subband index and $\eta \in \{-1, 0, 1\}$ a parameter which depends on the nanotube chiral vector. We have introduced in the above hamiltonian intra-subband spin-orbit coupling terms in Δ_{so}^1, Δ_{so}^0 and a spin-independent term in Δ_g, which are derived e.g. in Refs. [5, 6]. The constants Δ_{so}^1 and Δ_{so}^0 are first order in the atomic spin-orbit interaction V_{so} and the nanotube curvature R^{-1}, whereas Δ_g is proportional to R^{-2}.

For a problem uniform in the ξ direction ($V(\xi) = 0$), the eigenstates $|\psi\rangle = \left|\psi_{\tau,N,k,\sigma,b}^0\right\rangle$ of the above hamiltonian satisfy $H_{SWNT}\left|\psi_{\tau,N,k,\sigma}^0\right\rangle = E_{\tau,N,k,\sigma}\left|\psi_{\tau,N,k,\sigma}^0\right\rangle$, with k the electrons longitudinal wavevector. One can check:

$$\langle \xi \, | \psi_{\tau,N,k,\sigma}^0 \rangle = \begin{pmatrix} u_{\tau,N,k,\sigma} \\ 1 \end{pmatrix} e^{ik\xi} = \begin{pmatrix} b\frac{\hbar v_F \tau\kappa + \sigma\tau\Delta_{so}^1 - \Delta_g - i\hbar v_F k}{\sqrt{(\hbar v_F \tau\kappa + \sigma\tau\Delta_{so}^1 - \Delta_g)^2 + (\hbar v_F k)^2}} \\ 1 \end{pmatrix} e^{ik\xi}$$

and

$$E_{\tau,N,k,\sigma} = \sigma\tau\Delta_{so}^0 + b\sqrt{\left(\hbar v_F \tau\kappa + \sigma\tau\Delta_{so}^1 - \Delta_g\right)^2 + (\hbar v_F k)^2} \tag{13}$$

with $b = \pm 1$ for the conduction/valence band. In the following, we use $b = 1$. In order to define a quantum dot, we take into account a rectangular confinement potential

$$V(\xi) = \begin{cases} V_{conf} \text{ for } \xi < 0 \\ 0 \text{ for } 0 < \xi < L \\ V_{conf} \text{ for } \xi > L \end{cases}$$

We obtain confined electronic states $|\psi\rangle = |\Psi_{\tau,N,n,\sigma}\rangle$, with n the index corresponding to a longitudinal confinement of electrons. More precisely, one has

$$\langle \xi \, | \psi_{\tau,N,n,\sigma} \rangle = \begin{cases} \frac{A_{\tau,N,n,\sigma}}{\sqrt{2\pi}} e^{\tilde{k}_1^n \xi} \begin{pmatrix} u_{\tau,N,-i\tilde{k}_1^n,\sigma} \\ 1 \end{pmatrix} \text{ for } \xi < 0 \\ \frac{C_{\tau,N,n,\sigma}}{\sqrt{2\pi}} e^{ik_1^n \xi} \begin{pmatrix} u_{\tau,N,k_1^n,\sigma} \\ 1 \end{pmatrix} + \frac{D_{\tau,N,n,\sigma}}{\sqrt{2\pi}} e^{-ik_1^n \xi} \begin{pmatrix} u_{\tau,N,-k_1^n,\sigma} \\ 1 \end{pmatrix} \text{ for } 0 < \xi < L \\ \frac{B_{\tau,N,n,\sigma}}{\sqrt{2\pi}} e^{\tilde{k}_1^n (L-\xi)} \begin{pmatrix} u_{\tau,N,i\tilde{k}_1^n,\sigma} \\ 1 \end{pmatrix} \text{ for } \xi > L \end{cases}$$

with $k_1^n > 0$ and $\mathrm{Re}[\tilde{k}_1^n] > 0$. The wavevectors k_1^n and $\tilde{k}_1^n$ can be obtained from the energy-conservation condition

$$E_{\tau,N,k_1^n,\sigma} = E_{\tau,N,-i\tilde{k}_1^n,\sigma} + V_{conf} \tag{14}$$

the constraint

$$n\frac{\pi}{L} < k_1^n < (n+1)\frac{\pi}{L}$$

and the secular condition

$$\exp\left(i2k_1^n L\right) = \frac{\left(u_{\tau,N,i\tilde{k}_1^n,\sigma} - u_{\tau,N,-k_1^n,\sigma}\right)\left(u_{\tau,N,-i\tilde{k}_1^n,\sigma} - u_{\tau,N,k_1^n,\sigma}\right)}{\left(u_{\tau,N,-i\tilde{k}_1^n,\sigma} - u_{\tau,N,-k_1^n,\sigma}\right)\left(u_{\tau,N,i\tilde{k}_1^n,\sigma} - u_{\tau,N,k_1^n,\sigma}\right)}$$

which results from the continuity of $|\psi_{\tau,N,n,\sigma}\rangle$ at $\xi = 0$ and $\xi = L$. The constants $A_{\tau,N,n,\sigma}$, $B_{\tau,N,n,\sigma}$, $C_{\tau,N,n,\sigma}$, and $D_{\tau,N,n,\sigma}$ can be obtained from the continuity of $|\psi_{\tau,N,n,\sigma}\rangle$ and its normalization condition, i.e. $\int_{-\infty}^{\infty} d\xi \left|\langle\psi_{\tau,N,n,\sigma}\,|\psi_{\tau,N,n,\sigma}\rangle\right|^2 = 1$.

F.2. Effect of the inter-subband coupling elements

In this section, we discuss the coupling between electronic spins and cavity photons, mediated by the electromagnetic field of the cavity.

- We assume that the nanotube is parallel to the cavity central conductor. We take into account the interaction of electrons with the vector potential $\overrightarrow{A}$ of the cavity treated in the Coulomb gauge [11, 12]. We quantize the field $\overrightarrow{A}$ in terms of the photonic operators[13, 14]. This gives a coupling operator

$$H_{inter}^{A} = -\frac{e\hbar V_{rms}}{8\pi m_{eff} R\nu_0 d}(a + a^{\dagger})[(\mu_+ - \mu_-)\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\varphi}(\mu_+ - \mu_-)]$$

  We have used above $\sin(\varphi) = (\mu_+ - \mu_-)/2i$, where the operator $\mu_\pm = e^{\pm i\varphi}$ increases/decreases the index N.

- The subband and spin subspaces are coupled by a term [5, 6]

$$H_{inter}^{so} = -\Delta_{so}^{1} s_2 i(\sigma_-\mu_+ - \sigma_+\mu_-)$$

  which is first order in the atomic spin-orbit interaction V_{so} and the nanotube curvature R^{-1} (spin-orbit interaction in carbon nanotubes was also discussed in Refs. 7–11). Here, $\sigma_\pm$ is the operator increasing/decreasing the spin index in $|\sigma\rangle$.

- Note that the terms H_{inter}^{A} and H_{inter}^{so} apply to the full wavefunctions [see Eq. (12)]

$$|\Psi_{\tau,N,n,\sigma}\rangle = e^{i(N+\frac{\tau\eta}{3})\varphi}\,|\psi_{\tau,N,n,\sigma}\rangle \otimes |\sigma\rangle$$

  One can perform a first order perturbation of these wavefunctions by H_{inter}^{so}:

$$\left|\tilde{\Psi}_{\tau,N,n,\sigma}\right\rangle = |\Psi_{\tau,N,n,\sigma}\rangle + \sum_{n'} \lambda_{\tau,N,n,n'}^{-\sigma}\,|\Psi_{\tau,N+\sigma,n',-\sigma}\rangle$$

  with

$$\lambda_{\eta,\tau,N,n,n'}^{\sigma} = \frac{\langle\Psi_{\tau,N-\sigma,n',\sigma}|\,H_{inter}^{so}\,|\Psi_{\tau,N,n,-\sigma}\rangle}{E_{\tau,N,n,-\sigma} - E_{\tau,N-\sigma,n',\sigma}}$$

- The term H_{inter}^{A} couples the perturbed wavefunctions $\left|\tilde{\Psi}_{\tau,N,n,\sigma}\right\rangle$ for $\sigma =\uparrow$ and $\sigma =\downarrow$. This yields intra-subband spin/photon coupling elements, which write for the lower subband $N = 0$ which we consider:

$$\begin{aligned}\lambda_\tau &= \left\langle\tilde{\Psi}_{\tau,N=0,n,-1}(\overrightarrow{r})\right|\,H_{inter}^{A}\,\left|\tilde{\Psi}_{\tau,N=0,n,+1}(\overrightarrow{r})\right\rangle \\ &= P\left(\frac{\tau\eta}{3}\left(\lambda_{\tau,N=0,n,n}^{-} + {\lambda_{\tau,N=0,n,n}^{+}}^{*}\right) + \frac{1}{2}\left(\lambda_{\tau,N=0,n,n}^{-} - {\lambda_{\tau,N=0,n,n}^{+}}^{*}\right)\right)\end{aligned}$$

  with $P = ie\hbar V_{rms}/8\pi m_{eff} R\nu_0 d$.

- One can check that $\lambda_K = \lambda_{K'} = -i\lambda$ is purely imaginary due to the assumptions used above [zigzag nanotube and $V(\xi - \frac{L}{2}) = V(\frac{L}{2} - \xi)$]. One thus obtains the intra-subband spin/photon coupling term of Eq.(3) of the main text. For $L = 100$ nm, $R = 1$ nm, $\eta = 1$, $d = 5$ μm, $V_{rms} = 4$ μV, $\nu_0 = 3.64$ GHz, $E_{conf} = \hbar v_F/3R$, $m_{eff} = \hbar|\kappa|/v_F = 4.4\ 10^{-32}$ kg, and using the parameters $\Delta_{so}^{1} = -0.08$ meV, $\Delta_{so}^{0} = -0.32$ meV, and $\Delta_g = 5.7$ meV taken from Ref. 6, we obtain $\lambda \simeq 0.4$ MHz.

- In order to obtain a tunable spin/photon coupling, one can insert in one of the dots gate voltage supply a tunable capacitance made out of a single electron transistor (SET). The electric field seen by the nanotube can be modulated electrostatically by placing the SET in the blockaded or transporting regimes. This allows one to vary the couplings $\lambda_{L(R)}$ for dot $L(R)$.

- In the case $\lambda_{iK} \neq \lambda_{iK'}$, with i the dot index added in the main text, the results of the main text can be generalized straightforwardly by using

$$\left\langle T_- \right| h_{so} \left| V_n \right\rangle = v_n \frac{\Delta_{K\leftrightarrow K'}}{2\Delta_r} (\lambda_{LK} + \lambda_{LK'} - \lambda_{RK} - \lambda_{RK'}) \tag{15}$$

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