THE UNITARY GAS:
SYMMETRY PROPERTIES AND APPLICATIONS

Yvan Castin, Félix Werner, Christophe Mora
LKB and LPA, Ecole normale supérieure (Paris, France)

Ludovic Pricoupko
LPTMC, Université Paris 6

GENERAL CONTEXT

The physical system:

- Fermionic atoms with two internal states $\uparrow, \downarrow$
- Short-range interactions between $\uparrow$ and $\downarrow$ controlled by a magnetic Feshbach resonance
- Arbitrary values for the numbers $N_\uparrow$, $N_\downarrow$
- Intense experimental studies (Thomas, Salomon, Jin, Ketterle, Grimm, Hulet, Zwierlein...), e.g. BEC-BCS crossover (Leggett, Nozières, Schmitt-Rink, Sa de Melo,...)

What is not discussed here:

- The actual many-body state of the system: superfluid or normal
- The particularly intriguing strongly polarized case $N_\uparrow \gg N_\downarrow$: Polaronic physics

OUTLINE OF THE TALK

- **What is the unitary gas ?**
- **Simple consequences of scaling invariance**
- **Dynamical consequences: $SO(2,1)$ hidden symmetry in a trap**
- **Separability in hyperspherical coordinates**
- **Does the unitary gas exist ?**

WHAT IS THE UNITARY GAS ?

DEFINITION OF THE UNITARY GAS

- Opposite spin two-body scattering amplitude

$$f_k = -\frac{1}{ik} \quad \forall k$$

- "Maximally" interacting: Unitarity of S matrix imposes $|f_k| \leq 1/k$.
- In real experiments with magnetic Feshbach resonance:

$$-\frac{1}{f_k} = \frac{1}{a} + ik - \frac{1}{2}k^2 r_e + O(k^4 b^3)$$

  unitary if "infinite" scattering length a and "zero" ranges:

$$k_{\rm typ}|a| > 100, k_{\rm typ}|r_e| \text{ and } k_{\rm typ} b < \frac{1}{100}$$

  imposing $|a| > 10$ microns for $r_e \sim b \sim$ a few nm.
- All these two-body conditions are only necessary.

THE ZERO-RANGE WIGNER-BETHE-PEIERLS MODEL

- Interactions are replaced by contact conditions.
- For $r_{ij} \to 0$ with fixed ij-centroid $\vec{C}_{ij} = (\vec{r}_i + \vec{r}_j)/2$ different from $\vec{r}_k, k \neq i, j$:

$$\psi(\vec{r}_1, \ldots, \vec{r}_N) = \left(\frac{1}{r_{ij}} - \frac{1}{a}\right) A_{ij}[\vec{C}_{ij}; (\vec{r}_k)_{k\neq i,j}] + O(r_{ij})$$

- Elsewhere, non interacting Schrödinger equation

$$E\psi(\vec{X}) = \left[-\frac{\hbar^2}{2m}\Delta_{\vec{X}} + \frac{1}{2}m\omega^2 X^2\right]\psi(\vec{X})$$

  with $\vec{X} = (\vec{r}_1, \ldots, \vec{r}_N)$.
- Odd exchange symmetry of ψ for same-spin fermion positions.
- Unitary gas exists iff Hamiltonian is self-adjoint.

EXERCISING WITH THE BETHE-PEIERLS MODEL

Scattering state of two particles:

$$\phi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f_k \frac{e^{ikr}}{r}$$

- For $r > 0$ this is an eigenstate of the non-interacting problem.
- Contact condition in $r = 0$:

$$\frac{f_k}{r} + (1 + ikf_k) + O(r) = \frac{A}{r} + O(r)$$

determines scattering amplitude f_k:

$$f_k = -\frac{1}{ik}$$

SIMPLE CONSEQUENCES OF SCALING INVARIANCE

SCALING INVARIANCE OF CONTACT CONDITIONS

$$\psi(\vec{X}) \underset{r_{ij}\to 0}{=} \frac{1}{r_{ij}} A_{ij}[\vec{C}_{ij}; (\vec{r}_k)_{k\neq i,j}] + O(r_{ij})$$

- Domain of Hamiltonian is scaling invariant: If ψ obeys the contact conditions, so does ψ_λ with

$$\psi_\lambda(\vec{X}) \equiv \frac{1}{\lambda^{3N/2}}\psi(\vec{X}/\lambda)$$

- Consequences (also true for the ideal gas):

free space	box (periodic b.c.)	harm. trap
no bound state$^{(*)}$	$PV = 2E/3$ $^{(**)}$	virial $E = 2E_{\rm harm}$ $^{(***)}$

$^{(*)}$ If ψ of eigenenergy E, ψ_λ of eigenenergy E/λ^2. Square integrable eigenfunctions (after center of mass removal) correspond to point-like spectrum, for selfadjoint H. $^{(**)}$ $E(N, V\lambda^3, S) = E(N, V, S)/\lambda^2$, then take derivative in $\lambda = 1$. $^{(***)}$ For eigenstate ψ, mean energy of ψ_λ, $E_\lambda = \frac{\langle H_{\rm Laplacian}\rangle}{\lambda^2} + \langle H_{\rm harm}\rangle\lambda^2$, stationary in $\lambda = 1$.

DYNAMICAL CONSEQUENCES:

$\boldsymbol{SO(2,1)}$ HIDDEN SYMMETRY IN A TRAP

IN A TIME-DEPENDENT TRAP

- At $t = 0$: static trap $U(\mathrm{r}) = m\omega^2 r^2/2$, system in eigenstate $\psi_0(\vec{X})$ of energy E.
- For $t > 0$, arbitrary time dependence of trap spring constant, $\omega(t)$. Known solution for ideal gas:

$$\psi(\vec{X}, t) = \frac{e^{-i\theta(t)}}{\lambda^{3N/2}(t)} \exp\left[\frac{im\dot{\lambda}}{2\hbar\lambda} X^2\right] \psi_0(\vec{X}/\lambda(t))$$

with $\ddot{\lambda} = \omega^2\lambda^{-3} - \omega^2(t)\lambda$ and $\dot{\theta} = E\lambda^{-2}/\hbar$.

- This is a gauge plus scaling transform.
- The gauge transform also preserves contact conditions:

$$r_i^2 + r_j^2 = 2C_{ij}^2 + \frac{1}{2} r_{ij}^2$$

so solution also applies to unitary gas!

Y. Castin, Comptes Rendus Physique 5, 407 (2004).

IN THE MACROSCOPIC LIMIT

$$\psi(\vec{X},t) = \frac{e^{-i\theta(t)}}{\lambda^{3N/2}} \exp\left[\frac{im\dot{\lambda}}{2\hbar\lambda}X^2\right] \psi_0(\vec{X}/\lambda)$$

density $\rho(\vec{r},t) = \rho_0(\vec{r}/\lambda)/\lambda^3$	velocity field $\vec{v}(\vec{r},t) = \vec{r}\dot{\lambda}/\lambda$
local temp. $T(\vec{r},t) = T/\lambda^2$	pressure $P(\vec{r},t) = P_0(\vec{r}/\lambda)/\lambda^5$
local entropy per particle	$s(\vec{r},t) = s_0(\vec{r}/\lambda)$

This has to solve the hydrodynamic equations for a normal gas. Entropy production equation:

$$\rho k_B T(\partial_t s + \vec{v}\cdot\vec{\nabla}s) = \vec{\nabla}\cdot(\kappa\nabla T) + \zeta(\vec{\nabla}\cdot\vec{v})^2$$

$$+\frac{\eta}{2}\sum_{i,j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v}\right)^2$$

so the bulk viscosity is zero: $\zeta(\rho,T) = 0\ \forall T > T_c$. Reproduces the conformal invariance result of Son (2007).

LADDER STRUCTURE OF THE SPECTRUM

- Infinitesimal change of ω for $0 < t < t_f$. For $t > t_f$:

$$\lambda(t) - 1 = \epsilon\, e^{-2i\omega t} + \epsilon^*\, e^{2i\omega t} + O(\epsilon^2)$$

so an udamped mode of frequency 2ω.

- Corresponding wavefunction change:

$$\psi(\vec{X}, t) = \Big[e^{-iEt/\hbar} - \epsilon e^{-i(E+2\hbar\omega)t/\hbar} L_+ \\ + \epsilon^* e^{-i(E-2\hbar\omega)t/\hbar} L_- \Big] \psi_0(\vec{X}) + O(\epsilon^2)$$

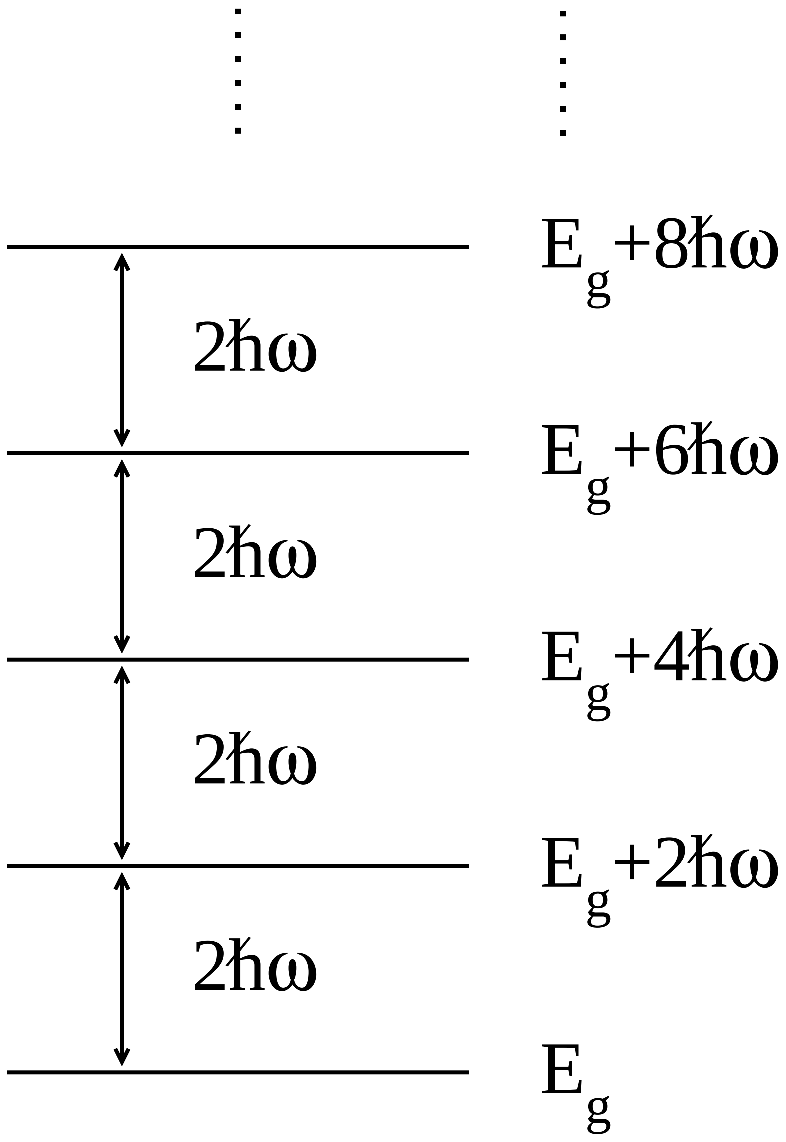
- Raising and lowering operators:

$$L_\pm = \pm i \left[\frac{3N}{2i} - i\vec{X} \cdot \partial_{\vec{X}} \right] + \frac{H}{\hbar\omega} - m\omega X^2/\hbar$$

(in red, generator of scaling transform)

- Spectrum=collection of semi-infinite ladders of step $2\hbar\omega$. $SO(2,1)$ hidden symmetry (Pitaevskii, Rosch, 1997).

LADDER STRUCTURE OF THE SPECTRUM (2)



USEFUL MAPPING AND SEPARABILITY

- Each energy ladder has a ground step of energy E_g, eigenfunction ψ_g.
- Integration of $L_- \psi_g = 0$ gives, with $\vec{X} = X\vec{n}$:

$$\psi_g(\vec{X}\,) = e^{-m\omega X^2/2\hbar} \times \left[X^{E_g/(\hbar\omega) - 3N/2} f(\vec{n}) \right]$$

- Limit $\omega \to 0$: mapping to zero energy free space solutions. N.B.: $E_g/(\hbar\omega)$ is a constant.
- Free space problem solved for $N = 3$ (Efimov, 1972)... so trapped case also solved (Werner, Castin, 2006).
- Also, this is separable in hyperspherical coordinates [Werner, Castin, PRA 74, 053604 (2006)].

SEPARABILITY IN HYPERSPHERICAL COORDINATES

SEPARABILITY IN INTERNAL COORDINATES

- Use Jacobi coordinates to separate center of mass $\vec{C}$
- Hyperspherical coordinates (arbitrary masses m_i):

$$(\vec{r}_1, \ldots, \vec{r}_N) \leftrightarrow (\vec{C}, R, \vec{\Omega}\,)$$

with $3N - 4$ hyperangles $\vec{\Omega}$ and the hyperradius

$$\bar{m} R^2 = \sum_{i=1}^{N} m_i (\vec{r}_i - \vec{C}\,)^2$$

where $\bar{m}$ is the mean mass.

- Hamiltonian is clearly separable:

$$H_{\text{internal}} = -\frac{\hbar^2}{2\bar{m}} \left[\partial_R^2 + \frac{3N-4}{R} \partial_R + \frac{1}{R^2} \Delta_{\vec{\Omega}} \right] + \frac{1}{2} \bar{m} \omega^2 R^2$$

Do the contact conditions preserve separability ?

- **For free space $E = 0$, yes, due to scaling invariance:**

$$\psi_{E=0} = R^{s-(3N-5)/2}\phi(\vec{\Omega})$$

$E = 0$ Schrödinger's equation implies

$$\Delta_{\vec{\Omega}}\phi(\vec{\Omega}) = -\left[s^2 - \left(\frac{3N-5}{2}\right)^2\right]\phi(\vec{\Omega})$$

with contact conditions. $s^2 \in$ discrete real set.

- **For arbitrary E, Ansatz with $E = 0$ hyperrangular part obeys contact conditions $[R^2 = R^2(r_{ij} = 0) + O(r_{ij}^2)]$:**

$$\psi = F(R)R^{-(3N-5)/2}\phi(\vec{\Omega})$$

- **Schrödinger's equation for a fictitious particle in 2D:**

$$EF(R) = -\frac{\hbar^2}{2\bar{m}}\Delta_R^{2D}F(R) + \left[\frac{\hbar^2 s^2}{2\bar{m}R^2} + \frac{1}{2}\bar{m}\omega^2 R^2\right]F(R)$$

SOLUTION OF HYPERRADIAL EQUATION $(N \geq 3)$

$$EF(R) = -\frac{\hbar^2}{2\bar{m}}\Delta_R^{2D}F(R) + \left[\frac{\hbar^2 s^2}{2\bar{m}R^2} + \frac{1}{2}\bar{m}\omega^2 R^2\right] F(R)$$

- Which boundary condition for $F(R)$ in $R = 0$? Wigner-Bethe-Peierls does not say.
- Key point: particular solutions $F(R) \sim R^{\pm s}$ for $R \to 0$.
- Case $s^2 > 0$: Defining $s > 0$, one discards as usual the divergent solution:

$$F(R) \underset{R\to 0}{\sim} R^s \longrightarrow E_q = E_{\mathrm{CoM}} + (s + 1 + 2q)\hbar\omega, \quad q \in \mathbb{N}$$

- Case $s^2 < 0$: To make the Hamiltonian self-adjoint, one is forced to introduce an extra parameter κ (inverse of a length, calculable via microscopic model). For $s = i|s|$:

$$F(R) \underset{R\to 0}{\sim} (\kappa R)^{s} - (\kappa R)^{-s}$$

- This breaks scaling invariance of the domain. In free space, a geometric spectrum of N-mers:

$$E_n \propto -\frac{\hbar^2\kappa^2}{\bar{m}} e^{-2\pi n/|s|}, \quad n \in \mathbb{Z}$$

For $N = 3$, this is the Efimov effect:

- Efimov (1971): Solution for three bosons $(1/a = 0)$. There exists a single purely imaginary $s_3 \simeq i \times 1.00624$.
- Efimov (1973): Solution for three arbitrary particles $(1/a = 0)$. Efimov trimers for two fermions (masse m, same spin state) and one impurity (masse m') iff (Petrov, 2003)

$$\alpha \equiv \frac{m}{m'} > \alpha_c(2;1) \simeq 13.6069$$

DOES THE UNITARY GAS EXIST ?

MINLOS'S THEOREM (1995)

Theorem: *In the $n+1$ fermionic problem, the Wigner-Bethe-Peierls Hamiltonian is self-adjoint and bounded from below iff*

$$(n-1)\frac{2\alpha(1+1/\alpha)^3}{\pi\sqrt{1+2\alpha}}\int_0^{\operatorname{asin}\frac{\alpha}{1+\alpha}} dt\, t \sin t < 1.$$

- α is mass ratio fermion/impurity
- Case $\alpha = 1$: No stable unitary gas for $n > 9$...
- Proof not included in Minlos' paper. Nobody (not even Minlos) was able to reproduce the "missing proof".
- Correggi, Dell'Antonio, Finco, Michelangeli, Teta (2012): Minlos'condition is sufficient for stability.
- Is is necessary ? A physical test: look for occurrence of $s^2 < 0$ for $n = 3$: four-body Efimov effect !?

ARE THERE EFIMOVIAN TETRAMERS ?

$$E_n^{(4)} \propto -\frac{\hbar^2 \kappa_4^2}{m} e^{-2\pi n/|s_4|} \;?$$

Negative results for bosons:

- **Amado, Greenwood (1973): "There is No Efimov effect for Four or More Particles". Explanation: Case of bosons, there exist trimers, tetramers decay.**
- **Hammer, Platter (2007), von Stecher, D'Incao, Greene (2009), Deltuva (2010): The four-boson problem (here $1/a = 0$) depends only on κ_3, no κ_4 to add.**
- **Key point: $N = 3$ Efimov effect breaks separability in hyperspherical coordinates for $N = 4$.**

Here, we are dealing with fermions.

OUR DEFINITION OF N-BODY EFIMOV EFFECT

- To find N-body Efimov effect, one simply needs to calculate the exponents s_N, that is to solve the Wigner-Bethe-Peierls model at zero energy:

$$\psi_{E=0}(\vec{r}_1, \ldots, \vec{r}_N) = R^{s_N-(3N-5)/2}\phi(\vec{\Omega})$$

- The N-body Efimov effect takes place iff one of the s_N^2 is < 0.
- This statement makes sense if $\Delta_{\vec{\Omega}}$ self-adjoint for the Wigner-Bethe-Peierls contact conditions: There should be no n-body Efimov effect $\forall n \leq N-1$.

THE $3+1$ FERMIONIC PROBLEM
(Castin, Mora, Pricoupenko, 2010)

- Three fermions (mass m, same spin state) and one impurity (mass m')
- Our def. of 4-body Efimov effect requires a mass ratio

$$\alpha \equiv \frac{m}{m'} < \alpha_c(2;1) \simeq 13.6069$$

- Calculate $E=0$ solution in momentum space. An integral equation for Fourier transform of A_{ij}:

$$0 = \left[\frac{1+2\alpha}{(1+\alpha)^2}(k_1^2+k_2^2) + \frac{2\alpha}{(1+\alpha)^2}\vec{k}_1\cdot\vec{k}_2\right]^{1/2} D(\vec{k}_1,\vec{k}_2)$$

$$+ \int \frac{d^3k_3}{2\pi^2} \frac{D(\vec{k}_1,\vec{k}_3)+D(\vec{k}_3,\vec{k}_2)}{k_1^2+k_2^2+k_3^2+\frac{2\alpha}{1+\alpha}(\vec{k}_1\cdot\vec{k}_2+\vec{k}_1\cdot\vec{k}_3+\vec{k}_2\cdot\vec{k}_3)}$$

- D has to obey fermionic symmetry.

RESULTS

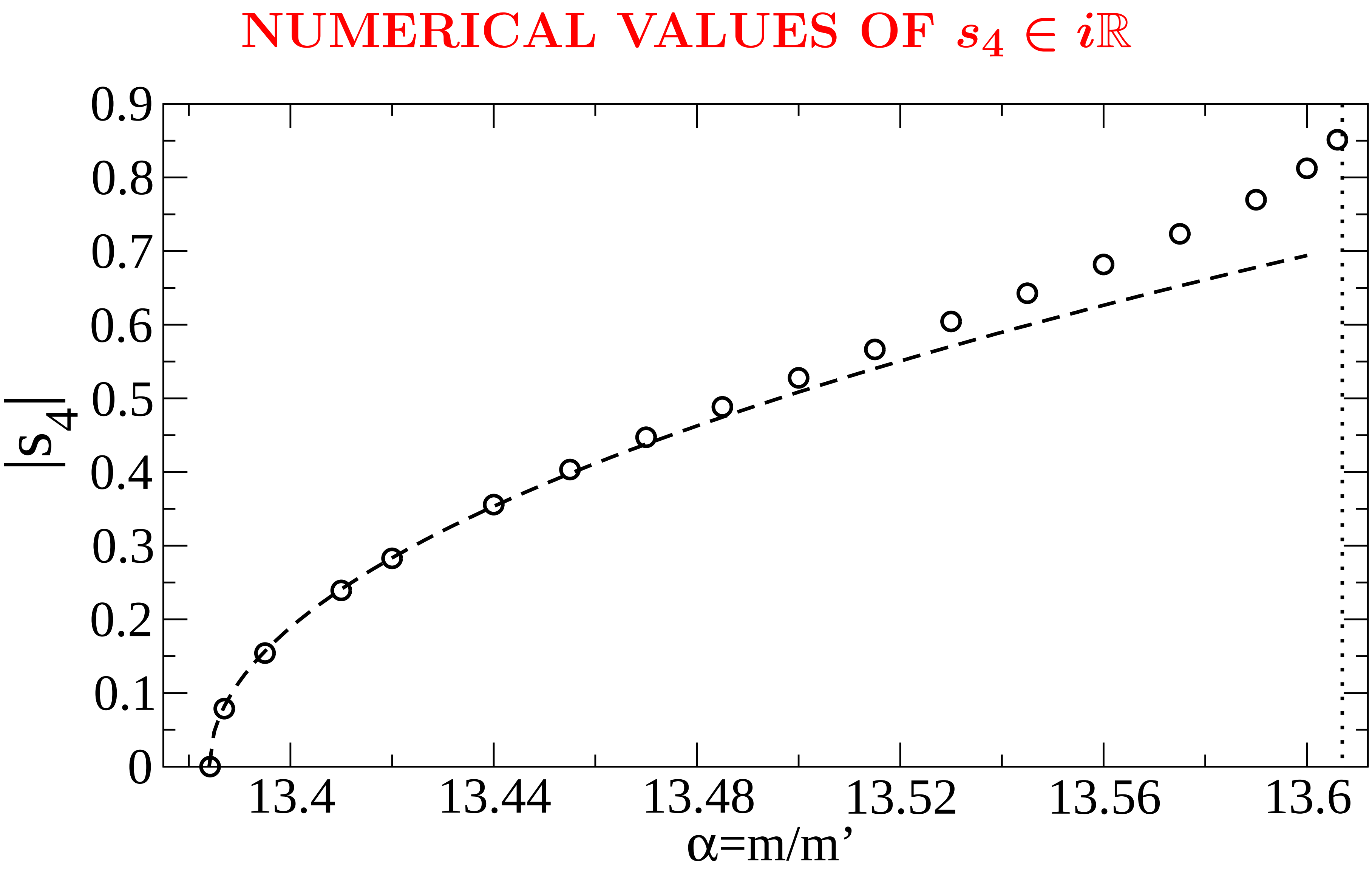
- Four-body Efimov effect obtained for a single s_4, in channel $l = 1$ with even parity. Corresponding ansatz:

$$D(\vec{k}_1, \vec{k}_2) = \vec{e}_z \cdot \frac{\vec{k}_1 \times \vec{k}_2}{||\vec{k}_1 \times \vec{k}_2||} (k_1^2 + k_2^2)^{-(s_4+7/2)/2} F(k_2/k_1, \theta)$$

in the interval of mass ratio

$$\alpha_c(3;1) \simeq 13.384 < \alpha < \alpha_c(2;1) \simeq 13.607$$

- Strong disagreement with Minlos' critical mass ratio for $n = 3$, $\alpha_c^{\mathrm{Minlos}} \simeq 5.29$
- In experiments: Use optical lattice to tune effective mass of ^{40}K and ^{3}He* away from $\alpha \simeq 13.25$


NUMERICAL VALUES OF $s_4 \in i\mathbb{R}$
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$|s_4|$
13.4
13.44
13.48
13.52
13.56
13.6
α=m/m’

CONCLUSION ON SYMMETRIES OF THE UNITARY GAS

- Unitary gas = gas of particles with interactions of infinite s-wave scattering length and negligible (true or effective) range
- Described by Wigner-Bether-Peierls zero-range model: Free Hamiltonian plus contact conditions
- Several physical properties result from scaling invariance of the model: E.g. undamped breathing mode of frequency 2ω in an isotropic harmonic trap $\longrightarrow$ vanishing of bulk viscosity.
- Existence of unitary gas (even for fermions) not evident; may be destroyed by generalized N-body Efimov effect.
- In the $n+1$ fermionic problem, sequence of critical mass ratios:

  $\alpha_c(2;1) = 13.6069\ldots \quad \alpha_c(3;1) = 13.384\ldots \quad \alpha_c(4;1) =?$

Our publications on the subject

- Y. Castin, "Exact scaling transform for a unitary quantum gas in a time dependent harmonic potential", Comptes Rendus Physique 5, 407 (2004).
- F. Werner, Y. Castin, "Unitary Quantum Three-Body Problem in a Harmonic Trap", Phys. Rev. Lett. 97, 150401 (2006).
- F.Werner, Y. Castin, "Unitary gas in an isotropic harmonic trap: Symmetry properties and applications", Phys. Rev. A 74, 053604 (2006).
- Y. Castin, C. Mora, L. Pricoupenko, "Four-Body Efimov Effect for Three Fermions and a Lighter Particle", Phys. Rev. Lett. 105, 223201 (2010).
- C. Mora, Y. Castin, L. Pricoupenko, "Integral equations for the four-body problem", Comptes Rendus Physique 12, 71 (2011).
- Y. Castin, F. Werner, "The Unitary Gas and its Symmetry Properties", contribution to the Springer Lecture Notes in Physics "BCS-BEC Crossover and the Unitary Fermi gas", ed. Wilhelm Zwerger (Springer, Berlin, 2011).