One-dimensional Quantum Liquids

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Introduction

Objectives:

- Review the physics,
- To give an idea of the mathematics involved

(1+1)D liquids: systems with spectral gaps and **no local order parameter**.

However, I will start with a model which has a local order parameter in one of its phases, - the Quantum Ising model.

Because

- it encompasses most of the relevant physics and
- is one of the simpliest models of strong correlations.

Quantum Ising model

$$H = \sum_n \left[-J\sigma^z_n\sigma^z_{n+1} + h\sigma^x_n\right]$$

σ^z, σ^x are Pauli matrices. By Jordan-Wigner transformation (1928)

$$\sigma^x_n = F^+_n F_n - 1/2, \quad \sigma^z_n = (F^+_n + F_n)\exp\left[\mathrm{i}\pi\sum_{j<n} F^+_j F_j\right]$$

where $\{F_n, F^+_m\} = \delta_{nm}$ it is transformed into a model of free fermions:

$$H = \sum_p \epsilon(p) F^+_p F_p, \quad p = 2\pi N/L$$

where N - integer $J > h$, half-integer $J < h$.

$$\epsilon(p) = \sqrt{(J-h)^2 + 4Jh\sin^2(p/2)}$$

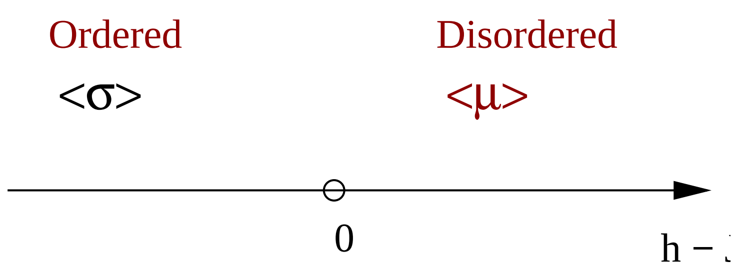
Duality: $h \to J, \sigma \to \mu$

Ising model can be recast in terms of dual variables

$$\mu^z_{n+1/2} = \prod_j^n \sigma^x_j, \quad \mu^x_{n+1/2} = \sigma^z_{n+1}\sigma^z_n$$

which are also Pauli matrices, but living on the dual lattice:

$$H = \sum_n \left[-h\mu^z_{n-1/2}\mu^z_{n+1/2} + J\mu^x_{n-1/2}\right]$$

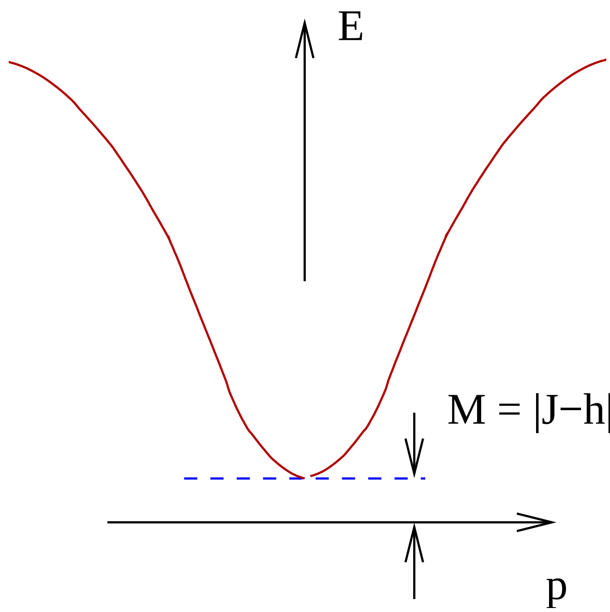


At $T = 0$ the Z_2 symmetry of the Ising model is spontaneously broken. At $h < J$ it is broken explicitly (local order parameter), at $h > J$ the order is hidden.

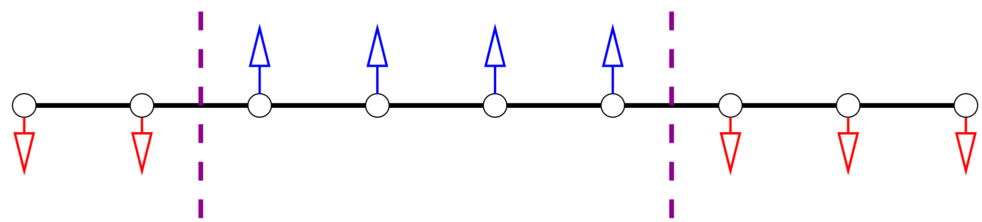
Continuum limit: $|J-h| << J, |p| << 1$. Then the excitation spectrum is relativistic:

$$\epsilon(p) = \sqrt{(J-h)^2 + 4Jh\sin^2(p/2)} \approx \sqrt{M^2 + v^2p^2}$$

where $M = (J-h), v = Jh$.



Excitations are solitons (domain walls) separating vacua with different sign of $\langle\sigma\rangle$ (or $\langle\mu\rangle$ if we are in the disordered phase!):



SIGN of M! - For the fermions the only difference is in boundary conditions, for the spins the difference is dramatic.

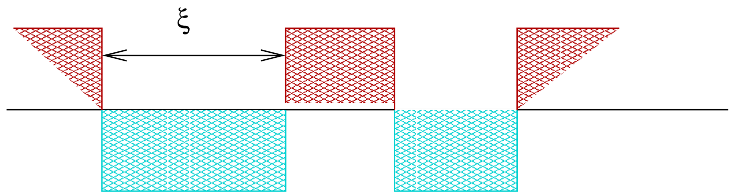
Qualitative picture of correlations at $T = 0$ ($h < J$):

$\langle\sigma\rangle \neq 0$, all correlation functions decay exponentially in Matsubara time and space:

$$\langle\langle\sigma(\tau, x)\sigma(0,0)\rangle\rangle \sim \exp\left(-2M\sqrt{\tau^2 + (x/v)^2}\right)$$

There is only one energy scale: spectral gap $M = |h - J|$.

At $T \neq 0$ the order is destroyed by solitons and antisolitons.



A new scale emerges: the average distance between domain walls

$$\xi \sim (TM)^{-1/2} \exp\left(\frac{M}{T}\right)$$

Problem: slow dynamics of σ at $T \neq 0$.

This problem is complicated only for σ^z which is nonlocal in fermions.

For $\sigma^x = F^+F - 1/2$ there is no problem: the dynamical correlation functions are ballistic! The heat transport in Ising model is ballistic.

Non-trivial correlation functions. The formfactor approach.

A convenient parametrization of the spectrum:

$$pv = M\sinh\theta, \quad \epsilon = M\cosh\theta$$

so that $\epsilon^2 - (pv)^2 = M^2$. Parameter θ is called rapidity.

Let us set $v = 1$. Ising model Hamiltonian:

$$H = \frac{L}{2\pi}\int \mathrm{d}\theta M\cosh\theta Z^+(\theta)Z(\theta)$$

where $Z(\theta) = [M\cosh\theta]^{1/2}F(p = M\sinh\theta)$.

$$\{Z^+(\theta), Z(\theta')\} = \delta(\theta - \theta')$$

Excited state of particles with rapidities $\theta_1, \theta_2, ...\theta_n$ is

$$|\theta_1, \theta_2, ...\theta_n\rangle$$

has energy $E = M\sum\cosh\theta_j$.

Example: T=0 correlation function of σ's

The matrix element of the σ^z operator

$$\langle 0|\sigma^z(0)|\theta_1,\theta_2,...\theta_n\rangle = CM^{1/8}\prod_{j>k}\tanh\left[(\theta_j-\theta_k)/2\right]$$

where $C \sim 1$ and $n = 2N$ for $h < J$ and $n = 2N + 1$ for $h > J$.

The Lehmann expansion for $\chi(\tau,x) = \langle\langle\sigma^z(\tau,x)\sigma^z(0,0)\rangle\rangle$:

$$\sum_n \frac{1}{N!}\int\prod_{j=1}^{N}\mathrm{d}\theta_j|\langle 0|\sigma(0,0)|\theta_1,...\theta_N\rangle|^2\times$$

$$\exp(-|\tau|M\sum\cosh\theta_j + \mathrm{i}xM\sum\sinh\theta_j)$$

Do Fourier transformation and continue $\mathrm{i}\omega_n = \omega + \mathrm{i}0$.

The imaginary part of the retarded Green's function is

$$\Im m\chi(\omega > 0, q) =$$
$$\sum_N \frac{1}{N!}\int \prod_{j=1}^{N} \mathrm{d}\theta_j |\langle 0|\sigma(0,0)|\theta_1, ...\theta_N\rangle|^2 \times$$
$$\delta(\omega - M\sum \cosh\theta_j)\delta(q - M\sum \sinh\theta)$$

For a given frequency the sum contains only $\sim$ ω/M terms.

Let $h > J$ (disordered phase). The expansion starts with $N = 1$ term:

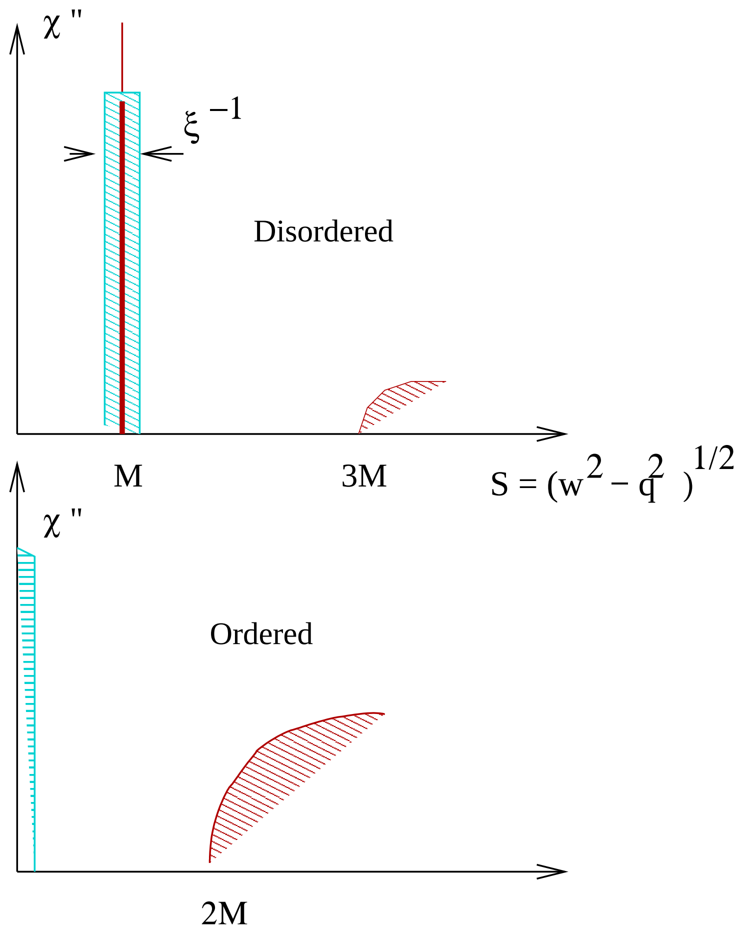
$$\Im m\chi(\omega < 3M, q) = CM^{-3/4}\frac{\delta(\omega - \sqrt{q^2 + M^2})}{\sqrt{q^2 + M^2}}$$

In the ordered state ($s^2 = \omega^2 - q^2$):

$$\Im m \chi(\omega < 4M, q) =$$

$$CM^{1/4} \int \mathrm{d}\theta \tanh^2\theta \frac{\delta(\omega - \sqrt{q^2 + 4M^2\cosh^2\theta})}{\sqrt{q^2 + 4M^2\cosh^2\theta}}$$

$$= CM^{1/4}\left(\frac{2M}{s}\right)^3 \sqrt{(s/2M)^2 - 1}$$



Matrix elements between excited states:

$$\langle u_1, ...u_m|\sigma^z(0)|v_1, ...v_n\rangle = CM^{1/4} \times \frac{\prod_{j>k} \tanh\left[(u_j - u_k)/2\right] \prod_{j>k} \tanh\left[(v_j - v_k)/2\right]}{\prod_{j,k} \tanh\left[(u_j - v_k)/2\right]}$$

Singularities (annihilation poles) appear when $u_j \to v_k$.

These are signs of $T = 0$ 1^{st} order phase transition. Summing them up one obtains the blue peaks on Figs.

Part 2. Integrable models. General features.

Alexei Tsvelik, BNL, May 2007

Integrable models.

Suppose there is a Lorentz invariant integrable model with spectrum consisting of particles with masses M_j. Particles carry isotopic index a. The spectrum is

$$E = \sum_{j,a_j} M_j \cosh\theta_{a_j}, \quad P = \sum_{j,a_j} M_j \sinh\theta_{a_j}$$

Rapidities θ_{a_j} are conserved quantities. The eigenstate

$$|\theta_1, a_1; ...\theta_n, a_n\rangle = Z^+_{a_1}(\theta_1)...Z^+_{a_n}|0\rangle$$

In integrable systems all interactions are encoded in the commutation relations (Faddeev-Zamolodchikov algebra):

$$Z^a(\theta_1)Z^b(\theta_2) = S^{a,b}_{\bar{a},\bar{b}}(\theta_1 - \theta_2)Z^{\bar{b}}(\theta_2)Z^{\bar{a}}(\theta_1)$$

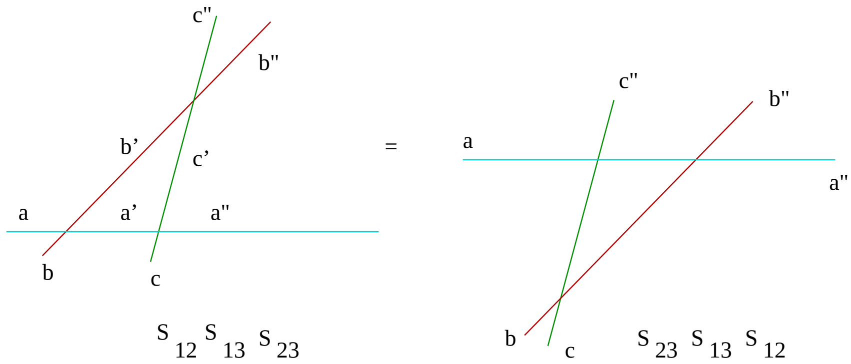
$$Z^+_a(\theta_1)Z^+_b(\theta_2) = S^{\bar{a},\bar{b}}_{a,b}(\theta_1 - \theta_2)Z^+_{\bar{b}}(\theta_2)Z^+_{\bar{a}}(\theta_1)$$

$$Z^a(\theta_1)Z^+_b(\theta_2) = S^{\bar{b},a}_{b,\bar{a}}(\theta_1 - \theta_2)Z^+_{\bar{b}}(\theta_2)Z^{\bar{a}}(\theta_1) +$$

$$+\delta^b_a\delta(\theta_{12})$$

where $S(\theta_{12})$ is the 2-particle scattering matrix.

The S-matrix satisfies the Yang-Baxter equations:



which are the associativity conditions for FZ algebra.

S-matrix also possesses

- Unitarity: its eigenvalues are phase factors: $\exp[\mathrm{i}\Phi(\theta_{12})]$, where Φ is real.
- Crossing symmetry (CPT invariance):

$$S_{a,b}^{\bar{a},\bar{b}}(\theta) = S_{\bar{b},a}^{b,\bar{a}}(\mathrm{i}\pi - \theta)$$

where red are indices obtained by charge conjugation.

SU(N) Chiral Thirring model

Hamiltonian density

$$\mathcal{H} = \mathrm{i}(-R_j^+\partial_x R_j + L_j^+\partial_x L_j) - g(R_j^+ L_j)(L_k^+ R_k)$$

where $j, k = 1, ...N$.

This is a model of Charge Density Wave. In 3D it would have a 2nd order phase transition into a state with

$$\Delta = \sum_j \langle R_j L_j^+ \rangle \neq 0$$

Bosonized version:

$$\mathcal{H} = \frac{1}{8\pi}\left[(4\pi\Pi_j)^2 + (\partial_x\Phi_j)^2\right] - \frac{g}{(4\pi a_0)^2}\sum_{j,k}\cos(\Phi_j - \Phi_k)$$

where $[\Phi_j, \Pi_k] = \mathrm{i}\delta_{jk}$.

$$(R, L)_j = \frac{\xi_j}{\sqrt{2\pi a_0}}\exp\left\{\frac{\mathrm{i}}{2}\left[\pm\Phi_j(x) + \frac{1}{4\pi}\int_{-\infty}^{x}\mathrm{d}y\Pi_j(y)\right]\right\}$$

where $\{\xi_j, \xi_k\} = 2\delta_{jk}$.

Field

$$\Phi = N^{-1/2}\sum_j \Phi_j$$

does not participate in the interaction and remains gapless.

The interaction scales to strong coupling if $g > 0$. At large N it is instructive to use $1/N$-approximation:

$$Z = \int D\Delta^* D\Delta D R^+ D R D L^+ D L \exp\{-\int \mathrm{d}\tau \mathrm{d}x \mathcal{L}\}$$

$$\mathcal{L} = \frac{|\Delta|^2}{2g} + (R_j^+, L_j^+) \left(\begin{array}{cc} \partial_\tau - \mathrm{i}\partial_x & \Delta \\ \Delta^* & \partial_\tau + \mathrm{i}\partial_x \end{array} \right) \left(\begin{array}{c} R_j \\ L_j \end{array} \right)$$

Integrating over fermions we get the action for Δ:

$$S = \int \mathrm{d}\tau \mathrm{d}x \frac{|\Delta|^2}{2g} - N \mathrm{Tr} \ln \left(\begin{array}{cc} \partial_\tau - \mathrm{i}\partial_x & \Delta \\ \Delta^* & \partial_\tau + \mathrm{i}\partial_x \end{array} \right)$$

The gradient expansion of this action produces the following Lagrangian density:

$$\mathcal{L} = \frac{N}{8\pi} \left[\frac{\partial_\mu \Delta^* \partial_\mu \Delta}{|\Delta|^2} + |\Delta|^2 \ln\left(\frac{|\Delta|^2}{M^2} \right) \right] + \ldots$$

The saddle point fixes

$$|\Delta| \approx M = (a_0)^{-1} g^{1/N} \exp(-2\pi/Ng)$$

$$\Delta = |\Delta| \mathrm{e}^{\mathrm{i}\Phi}$$

the phase Φ remains critical.

$$S = \frac{N}{8\pi} \int \mathrm{d}\tau \mathrm{d}x (\partial_\mu \Phi)^2$$

The total charge and current densities are

$$\rho = \sum_j (R_j^+ R_j + L_j^+ L_j) = \frac{N^{1/2}}{2\pi} \partial_x \Phi$$

$$j = \sum_j (R_j^+ R_j - L_j^+ L_j) = N^{1/2} \Pi$$

The transport is ballistic (sliding Charge Density Wave):

$$\sigma(\omega) = 2\pi N \delta(\omega)$$

The conductance of a finite wire is length independent:

$$G = \frac{e^2}{h} N$$

What can we say about other fields? Besides the gapless mode Φ the spectrum contains $N-1$ massive particles with masses

$$M_j = M\frac{\sin(\pi j/N)}{\sin(\pi/N)}, \quad j = 1, 2, ...N-1$$

They are bound states of the fundamental particle $j = 1$.

From the bosonization formula we deduce

$$R_j = \mathrm{e}^{\mathrm{i}\phi}\mathcal{R}_j, \quad L_j = \mathrm{e}^{-\mathrm{i}\bar{\phi}}\mathcal{L}_j$$

where

$$\langle \mathrm{e}^{\mathrm{i}\phi(\tau,x)}\mathrm{e}^{-\mathrm{i}\phi(0,0)}\rangle = (\tau - \mathrm{i}x)^{-1/N}$$

$$\langle \mathrm{e}^{\mathrm{i}\bar{\phi}(\tau,x)}\mathrm{e}^{-\mathrm{i}\bar{\phi}(0,0)}\rangle = (\tau + \mathrm{i}x)^{-1/N}$$

and $\mathcal{R}, \mathcal{L}$ annihilate particles from the massive sector. They have Lorentz spin

$$S = (1 - N^{-1})/2$$

Electron consists of charge and spin parts with different spectra: spin-charge separation.

$$(\mathcal{R},\mathcal{L})^+_j|0\rangle = F^{\pm}_j(\theta_1,...\theta_n)Z^+_{a_1}(\theta_1)...Z^+_{a_n}(\theta_n)|0\rangle$$

where from the Lorentz invariance

$$F^{\pm}_j(\theta_1,...\theta_n) = \mathrm{e}^{\pm S(\theta_1+...\theta_n)/n}F_j(\{\theta_{pq}\})$$

The lowest state is

$$\langle\theta|(\mathcal{R},\mathcal{L})^+_j|0\rangle = A^{1/2}\mathrm{e}^{\pm S\theta} \quad (1)$$

where $A \sim 1$ is a numerical coefficient,
$S = (1 - 1/N)/2$.

The contribution to the single-electron Green's function :

$$\langle\langle R_j(\tau,x)R^+_j(0,0)\rangle\rangle = \frac{A}{(\tau - \mathrm{i}x)}(Mr)^{(1-1/N)}K_{(1-1/N)}(Mr)$$

where $r^2 = \tau^2 + x^2$.

Part 3. We continue about the Thirring model

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It is remarkable that the interactions generate a spectral gap, though, contrary to what happens in $D > 1$, no local order parameter is formed:

$$\langle RL^+ \rangle = 0$$

This agrees with Mermin-Wagner theorem: the U(1) symmetry in the charge sector is continuous and continuous symmetry cannot be broken in 1D.

Notice that $\langle RL^+ \rangle = 0$, but solely due to the charge sector, since

$$\langle \mathrm{e}^{\mathrm{i}\phi} \mathrm{e}^{-\mathrm{i}\bar{\phi}} \rangle = 0$$

From the Green's function one can extract

Tunneling Density of States

$$\rho(\omega) = A\int_0^{\cosh^{-1}(\omega/M)} \mathrm{d}x \frac{\cosh\left[(1-1/N)x\right]}{(\omega/M - \cosh x)^{(1-1/N)}}$$

$$\rho(\omega) = \Im m\left\{\int \mathrm{e}^{\mathrm{i}\Omega\tau} G(\tau, x=0)\mathrm{d}\tau\right\}_{\mathrm{i}\Omega\rightarrow\omega+\mathrm{i}0}$$

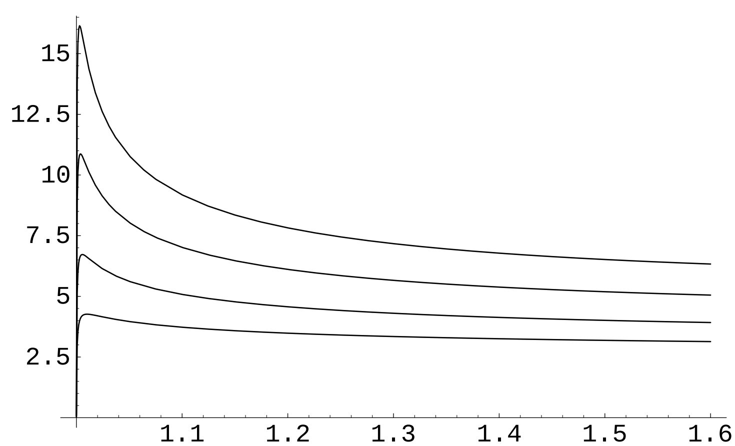

Figure 1: $\rho(\omega)$ for N=3,4,6,10.

From now on we'll concentrate on N=2 case.

In the bosonized version the Lagrangian density is

$$\mathcal{L} = \frac{1}{8\pi}\left[v_c^{-1}(\dot{\Phi})^2 + v_c(\partial_x\Phi)^2\right] +$$
$$\frac{(1+g)}{8\pi}\left[v^{-1}(\dot{\Phi}_s)^2 + v(\partial_x\Phi_s)^2\right] - \frac{g}{(2\pi a_0)^2}\cos\Phi_s$$

where

$$\Phi = \frac{\Phi_1+\Phi_2}{\sqrt{2}}, \Phi_s = \frac{\Phi_1-\Phi_2}{\sqrt{2}}$$

and I deliberately made the charge and spin velocities different. This is achieved by adding

$$g_f\left[(R_j^+R_j)^2 + (L_j^+L_j)^2\right]$$

to the original Hamiltonian.

The vacuum $\Phi_s = 0 \bmod 2\pi$. At $T=0$ the system is in one of these vacua, the discrete symmetry

$$\Phi_s(x) \to \Phi_s(x) + 2\pi n$$

is broken.

The average

$$\langle \cos(\Phi_s/2) \rangle \sim \pm M$$

depends on the choice vacuum, but the operator is nonlocal with respect to the fermions! Therefore it signifies a hidden order. The local order parameter

$$\Delta \equiv \sum_j R_j^+ L_j = \frac{1}{(2\pi a_0)} \mathrm{e}^{\mathrm{i}\Phi/2} \cos(\Phi_s/2)$$

has only quasi long range order:

$$\langle \Delta(\tau, x)\Delta^+(0,0) \rangle \approx \frac{a_0}{\sqrt{(v_c\tau)^2 + x^2}} \Delta_0^2$$

$$\Delta_0 = \langle \cos(\Phi_s/2) \rangle \sim M \sim a_0^{-1} g^{1/2} \mathrm{e}^{-\pi/g}$$

At $T \neq 0$ the the dynamical correlation function at $|x| >> M^{-1}$ is

$$\frac{\pi T}{\sqrt{\sinh[\pi T(x/v_c - t)]\sinh[\pi T(x/v_c + t)]}} \times$$

$$M^2 \exp\left[-\int \frac{\mathrm{d}p}{\pi}\mathrm{e}^{-\epsilon(p)/T}|x - t\frac{\partial\epsilon(p)}{\partial p}|\right]$$

where $\epsilon = \sqrt{M^2 + (pv)^2}$.

At $t = 0$ at large x

$$\sim \mathrm{e}^{-|x|/\xi}, \quad \xi^{-1} = \pi T/v_c + \int \frac{\mathrm{d}p}{\pi}\mathrm{e}^{-\epsilon(p)/T}$$

The spectral function at T=0 is

$$G_{RR}(\omega, q) \sim -\frac{\omega + v_c q}{\sqrt{M^2 + (v_c q)^2 - \omega^2}} \times$$

$$\left[\left(M + \sqrt{M^2 + (v_c q)^2 - \omega^2}\right)^2 - \frac{v - v_c}{v + v_c}(\omega + v_c q)^2\right]^{-1}$$

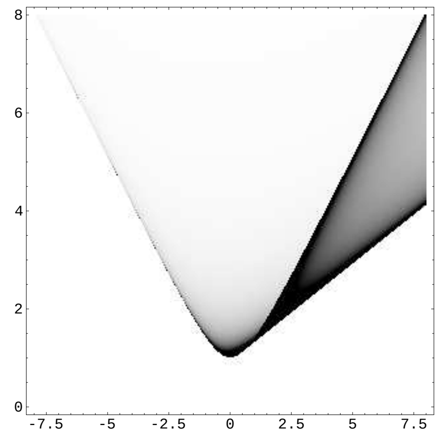

Figure 2: The spectral intensity for $v_c/v_s = 0.4$

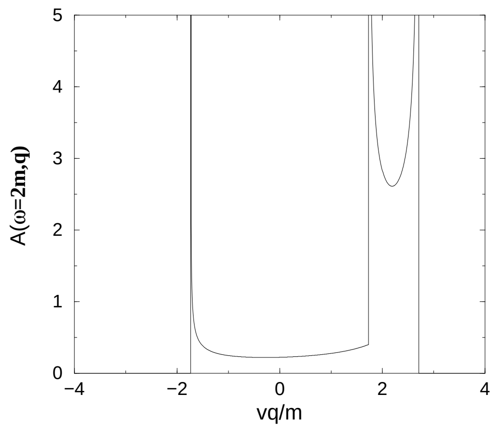

Figure 3: The spectral function as a function of momentum at $v_c/v_s = 0.4$

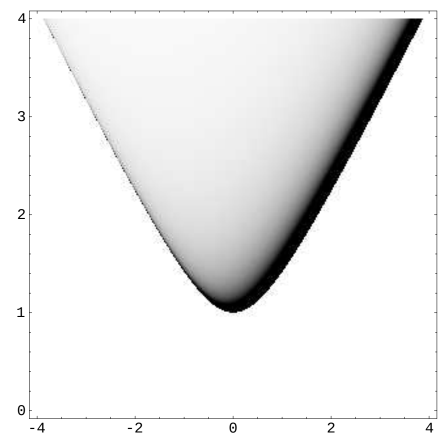

Figure 4: The spectral intensity for $v_c = v_s$.

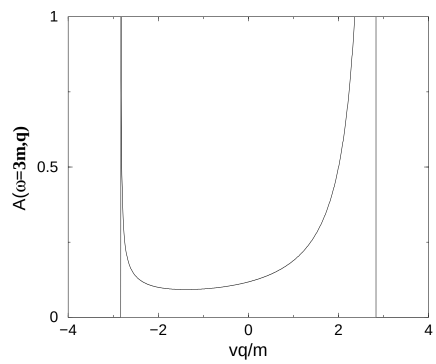

Figure 5: The spectral function as a function of momentum at $v_c = v_s$.

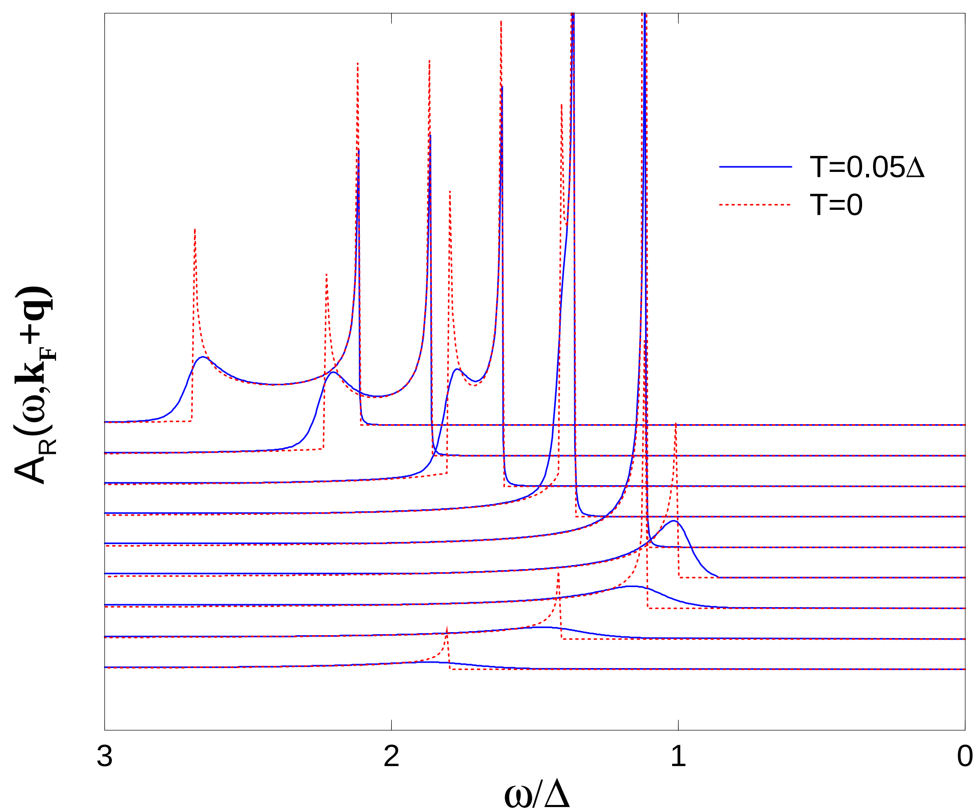

Figure 6: Spectral function for T=0 and T = 0.05Δ [Essler, Tsvelik (2002)]