Lecture 1

2D quantum gases: the static case

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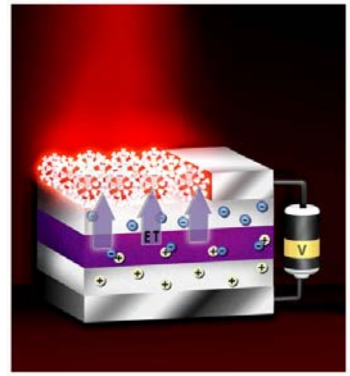
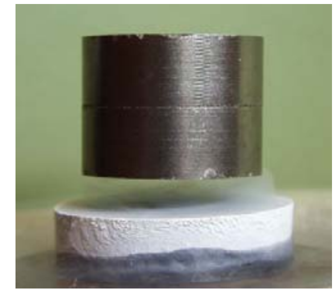
Review article:
I. Bloch, J. Dalibard, W. Zwerger,
Many-Body Physics with Ultracold Gases
arXiv:0704.3011

See also a very recent preprint:
L. Giorgetti, I. Carusotto, Y. Castin
arXiv:0705.1226

Low dimension quantum physics



Quantum wells and MOS structures

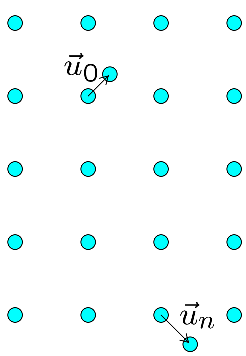


High T_c superconductivity

also Quantum Hall effect, films of superfluid helium, …

Physics in Flatland

Peierls 1935, Mermin – Wagner - Hohenberg 1966



No long range order
at finite temperature

in 2D: $\langle(\vec{u}_n - \vec{u}_0)^2\rangle \propto \log(n)$

whereas this tends to a
finite value in 3D

no real crystalline order
in 2D at finite T

The kind of order a physical system can possess
is profoundly affected by its dimensionality.

1.

The 2D Bose gas:

Ideal vs. interacting
Homogeneous vs. trapped

The ideal Bose gas

The uniform case at the thermodynamic limit

In 3D: when the phase space density reaches $n\lambda^3 = 2.612$, BEC occurs

In 2D: for any phase space density $n\lambda^2$, the occupation of the various states in non singular, and no BEC is expected.

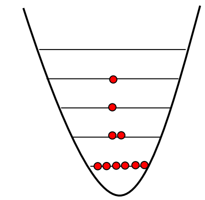
$$n\lambda^2 = -\ln\left(1 - e^{\mu / kT}\right)$$

μ varies from $-\infty$ to 0

Thermodynamic limit: $n = \frac{N}{L^2} = \text{Cst.} \qquad N, L \to \infty$

The ideal Bose in a harmonic potential

In 3D, BEC occurs when $N = 1.2 \left(\frac{k_B T}{\hbar\omega}\right)^3$



In 2D, BEC occurs when $N = 1.6 \left(\frac{k_B T}{\hbar\omega}\right)^2$

Bagnato – Kleppner (1991)

Does harmonic trapping make 2D and 3D equivalent?

What about interaction?

The density in the trap (semi-classical approach)

$$\rho(\vec{r}, \vec{p}) = \frac{1}{h^d} \frac{1}{\exp\left(\left(\frac{p^2}{2m} + V(r) - \mu\right)/kT\right) - 1}$$

$$n(\vec{r}) = \int \rho(\vec{r}, \vec{p})\, d^d p$$

3D:

$$n(\vec{r})\, \lambda^3 = g_{3/2}\left(e^{(\mu - V(r))/kT}\right)$$

At BEC:

$$n(0)\, \lambda^3 = 2.612$$

2D:

$$n(\vec{r})\, \lambda^2 = -\ln\left(1 - e^{(\mu - V(r))/kT}\right)$$

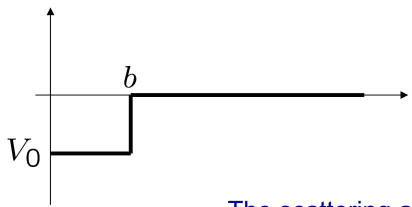
At BEC:

$$n(0)\, \lambda^2 = \infty$$

Note: $\int n(\vec{r})\, d^2 r = \frac{\pi^2}{6} \left(\frac{kT}{\hbar\omega}\right)^2$

Adding interactions…

Interactions in « true 2D »: the square well problem



$$\phi(\vec{r}) \sim e^{i\vec{k}\cdot\vec{r}} + \sqrt{\frac{i}{8\pi}}\, f(k)\, \frac{e^{ikr}}{\sqrt{kr}}$$

The scattering amplitude $f(k)$ is a dimension-less quantity that characterizes the scattering process

Scattering cross-section (length): $|f(k)|^2 / 4k$

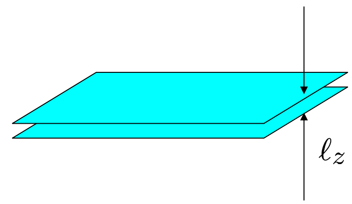
$$f(k) \sim \frac{4\pi}{-2\ln(ka_{2D}) + i\pi} \quad \text{when} \quad k \to 0$$

➡ no Born approximation for low energy scattering in 2D

How a 3D pseudopotential is transformed in 2D

Petrov-Holzmann-Shlyapnikov

a is the 3D scattering length associated to the 3D pseudopotential



$$f(k) \sim \frac{4\pi}{-2\ln(ka_2) + i\pi}$$

with $a_2 \simeq \ell_z \exp\left(-\sqrt{\frac{\pi}{2}}\frac{\ell_z}{a}\right)$

(real life: $\ell_z > R_e$)

a_2 is always positive: there is always a bound state in the 2D problem, irrespective of the sign of the 3D scattering length a

The relevant amplitude for experimentaly trapped 2D atomic gases

$$\frac{\hbar^2 k^2}{m} \sim \mu$$

$$\ell_z \gg a > 0$$

$$f(k) \simeq \sqrt{8\pi}\frac{a}{\ell_z} \sim 0.01 \text{ to } 0.15$$

How a 3D pseudopotential is transformed in 2D (II) ?

The 3D interaction energy:

$$E_{\text{int}} = \frac{N(N-1)}{2}\, g^{(3D)} \int |\psi(\vec{r})|^4\, d^3r \qquad g^{(3D)} = \frac{4\pi\hbar^2 a}{m}$$

The result $f(k) \simeq \sqrt{8\pi}\frac{a}{\ell_z}$ can be recovered simply by taking the 3D energy with trial wave functions $\psi(x,y,z) = \Phi(x,y)\, \frac{e^{-z^2/2\ell_z^2}}{(\pi\ell_z^2)^{1/4}}$

➡ $$E_{\text{int}} = \frac{N(N-1)}{2}\, g^{(2D)} \int |\Phi(\vec{r})|^4\, d^2r$$

with $$g^{(2D)} = \frac{\hbar^2}{m}\sqrt{8\pi}\frac{a}{\ell_z}$$

The role of (weak) interactions on BEC in a 3D trap

The simple shift due to mean field:

Repulsive interactions slightly decrease the central density, for given N and T

➡ For an ideal gas, the central density at the condensation point is

$$N = 1.2\left(\frac{k_B T}{\hbar\omega}\right)^3 \longleftrightarrow n(0)\lambda^3 = 2.6 \qquad \text{(semi-classical)}$$

➡ For a gas with repulsive interactions, one needs to put a bit more atoms to obtain the proper $n(0)$

Mean-field analysis: start from $n(r)\,\lambda^3 = g_{3/2}\left(e^{(\mu - V_{\text{trap}}(r))/kT}\right)$

$$V_{\text{trap}}(r) \longrightarrow V_{\text{eff}}(r) = V_{\text{trap}}(r) + 2g^{(3D)} n(r)$$

Note: In addition there is a shift of Tc due to complex non-mean field effects

The role of (weak) interactions on BEC in a 2D trap

The procedure similar to the 3D mean field analysis fails. Indeed,

$$N = 1.6\left(\frac{k_B T}{\hbar\omega}\right)^2 \quad \longleftrightarrow \quad n(0)\lambda^2 = \infty$$

where $n(r) = \int \rho(r,p)\frac{d^2p}{h^2}$ $\qquad \rho(r,p) = \left[e^{\beta(\frac{p^2}{2m}+\frac{m\omega^2 r^2}{2})} - 1\right]^{-1}$

One cannot make a perturbative treatment of interactions around this point!

Repulsive interactions tend to reduce the density, which cannot be infinite anymore at the center of the trap

Problem with 2D BEC in a harmonic trap (continued)

Treat the interactions at the mean field level: $V_{\text{eff}}(r) = \frac{m\omega^2 r^2}{2} + 2g n_{\text{mf}}(r)$

where the mean field density is obtained from the self-consistent equation

$$n_{\text{mf}}(r)\lambda^2 = -\ln\left(1 - e^{(\mu - V_{\text{eff}}(r))/kT}\right)$$

Two remarkable results

- One can accommodate an arbitrarily large atom number. Badhuri et al
- The effective frequency deduced from $V_{\text{eff}}(r) \simeq m\omega_{\text{eff}}^2 r^2/2$ tends to zero when $\mu \to 2g n_{\text{mf}}(0)$ Holzmann et al.

Similar to a 2D gas in a flat potential…

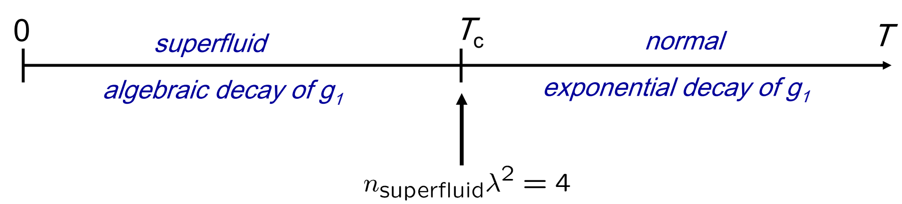
However there is more than just (no) BEC in a 2D interacting gas…

2.

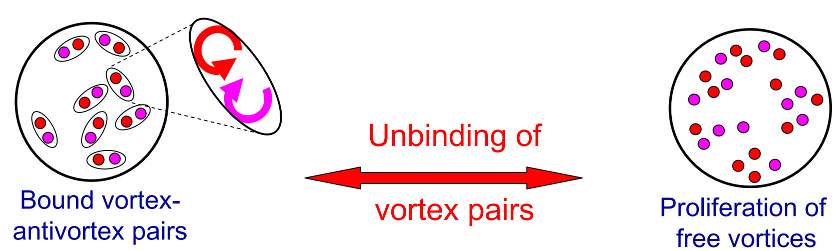
The superfluidity in a 2D Bose fluid and the BKT mechanism

Existence of a phase transition in a 2D Bose fluid

Berezinski and Kosterlitz –Thouless 1971-73
Nelson Kosterlitz 1977



$$g_1(|\vec{r} - \vec{r}\,'|) = \langle \psi^*(\vec{r})\psi(\vec{r}\,')\rangle$$

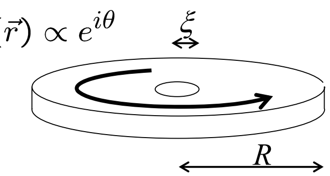


The KT mechanism for pedestrians

Probability for a vortex to appear as a thermal excitation?

One has to calculate the vortex free energy *E-TS* and compare it with *kT*

$\psi(\vec{r}) \propto e^{i\theta}$



$$v = \frac{\hbar}{mr} \qquad n_s = \frac{N}{\pi R^2}$$

Energy: $$E = \int n_s \frac{mv^2}{2} 2\pi r \, dr \qquad \sim \frac{\pi \hbar^2}{m} n_s \log(R/\xi)$$

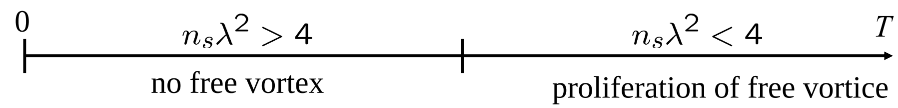
Entropy: $$S = k \log(W) \sim k \ln(R^2/\xi^2) = 2k \log(R/\xi)$$

Free energy of a vortex: $$\frac{E-TS}{kT} \sim \frac{1}{2}\left(n_s\lambda^2 - 4\right)\ln(R/\xi)$$

BKT transition for pedestrians (2)

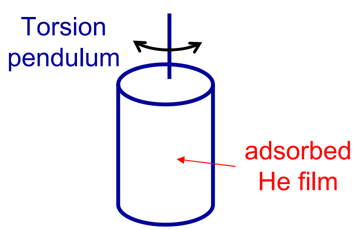
Free energy of a vortex: $$\frac{E-TS}{kT} \sim \frac{1}{2}\left(n_s\lambda^2 - 4\right)\ln(R/\xi)$$

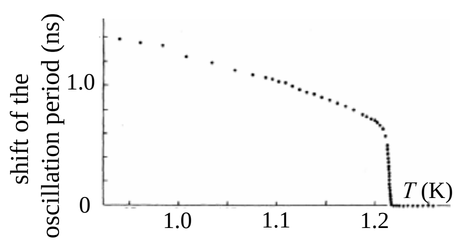
Thermodynamic limit in 2D: $R \to +\infty \qquad \rho^{(2D)} = \text{Ct.}$



Superfluidity in 2 dimensions

A 2D film of helium becomes superfluid at sufficiently low temperature (Bishop and Reppy, 1978)





"universal" jump to zero of superfluid density at $T = T_c$

$$\rho_s(T_c)\lambda^2 = 4 \longrightarrow \rho_s = 0$$

Also Safonov et al, 1998: variation of the recombination rate in a H film

When does the Kosterlitz-Thouless transition occurs?

The result $n_{\text{superfluid}}\lambda^2 = 4$ is only a partial answer, because $n_{\text{superfluid}}$ is a quantity that depends on temperature and it is usually unknown.

Analytical calculation by Fisher & Hohenberg, Monte-Carlo by Prokof'ev et al

$$n_{\text{total}}\lambda^2 = \ln\left(\frac{C}{\bar{g}}\right) \qquad C = 380 \pm 3 \qquad \bar{g} = \frac{m g^{(2D)}}{\hbar^2}$$

dimensionless interaction strength

For our setup with Rb cold atoms:

$$\bar{g} = 0.13 \quad \Rightarrow \quad n_{\text{total}}\lambda^2 \simeq 8.0$$

This is not "universal" by contrast to the 3D result $n_{\text{total}}\lambda^3 \simeq 2.6$

3.
The critical point in an atomic 2D Bose gas

Z. Hadzibabic, P. Krüger
B. Battelier, M. Cheneau, P. Rath, S. Stock

Phys. Rev. Lett. **95**, 190403 (2005)

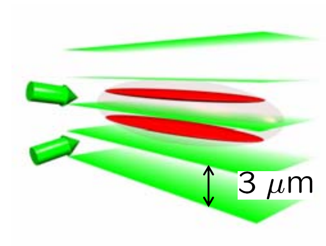
Nature **441**, 1118 (2006)

cond-mat/0703200

How to make a cold 2D gas?

Take a 3D condensate and slice it

superposition of a harmonic magnetic potential + periodic potential of a laser standing wave



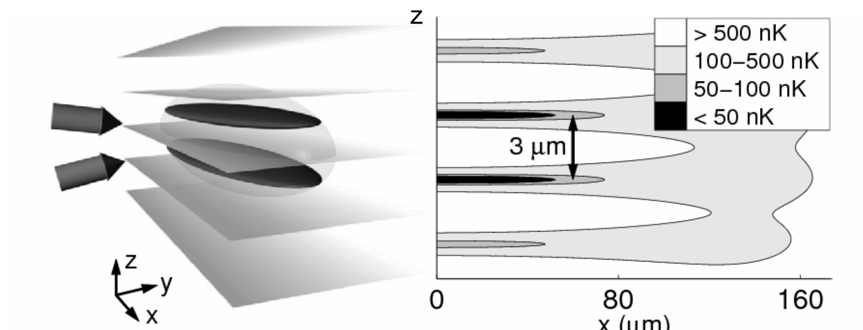
2 independent planes with 10^5 Rb atoms in each

other 2D experiments at MIT, Innsbruck, Oxford, Florence, Heidelberg, NIST, etc.

Why 2 planes? The crucial properties of a 2D Bose gas sit in its coherence function $g_1(x,y) = \langle \psi^*(x,y)\, \psi(0) \rangle$

→ accessible in an interference experiment

The trapping configuration: lattice + harmonic potential



Waist of the lasers along x: 120 μm

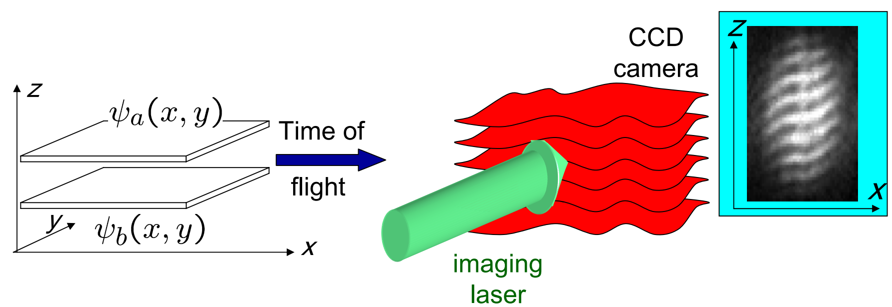
We vary the temperature between 50 and 100 nK (2 to « a few » planes)

Tunnelling between planes is negligible. However adjacent sites may exchange particles from the high energy tail of the distribution. This equalizes T and μ

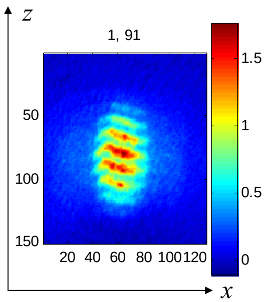
2D character of the gas: at center $\omega_z/(2\pi) = 3$ kHz

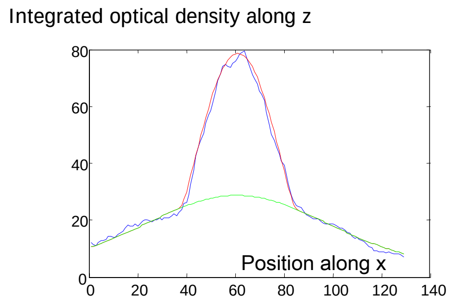
$$\hbar\omega_z \;>\; kT\,,\; \mu$$

Getting an interferogram for 2D gases



Overlap between expanding planar gases





Interference fringes

Central bright component
Halo in the periphery
} Characteristic of each gas individually

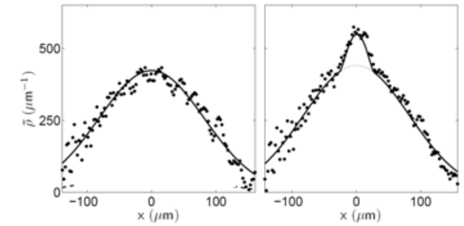
Bimodal fit:
Gauss + Thomas-Fermi give info on numbers, sizes, temperature …

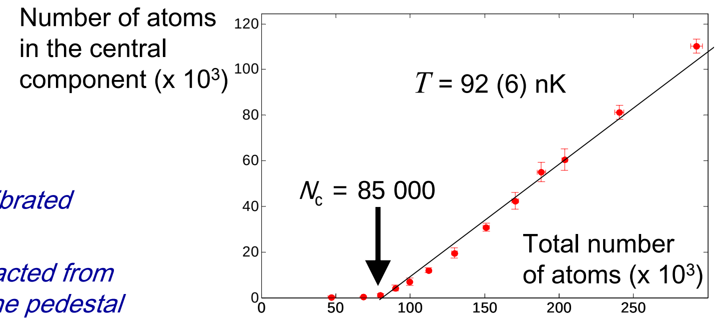
TF profile $\nRightarrow$ true BEC

Transition in a planar atomic gas

We maintain a constant temperature and vary the atom number

Apparition of a bi-modal distribution above a critical atom number

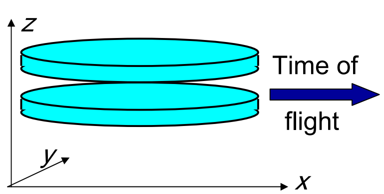


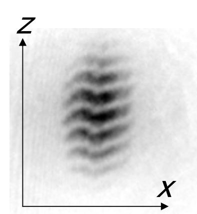


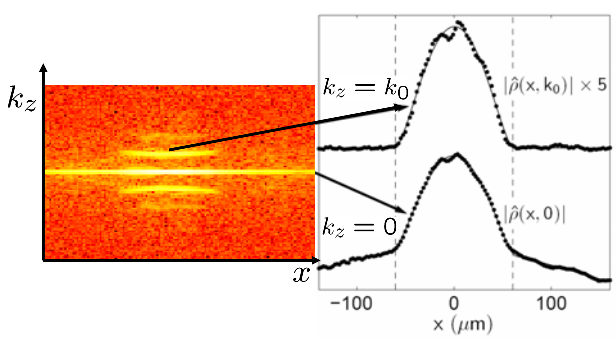
Atom number calibrated using 3D BEC

Temperature extracted from a gaussian fit of the pedestal

Interference between two planar gases

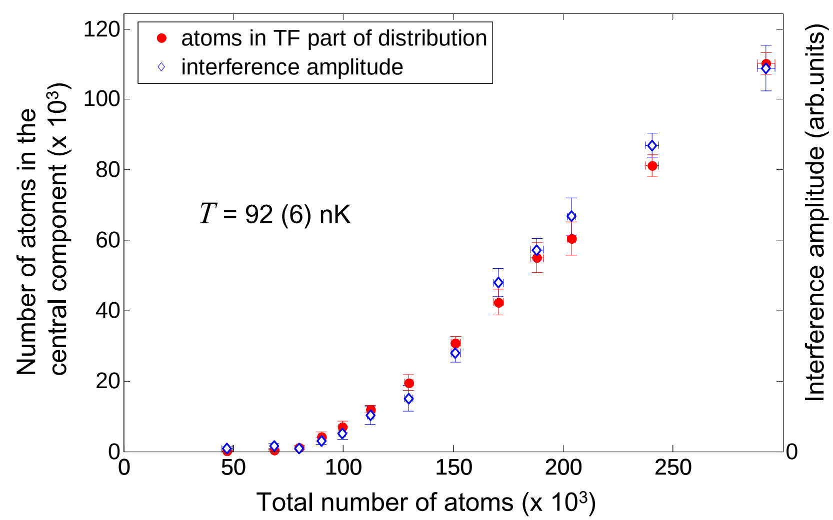






The “interfering” part coincides with the central part of the bimodal distribution

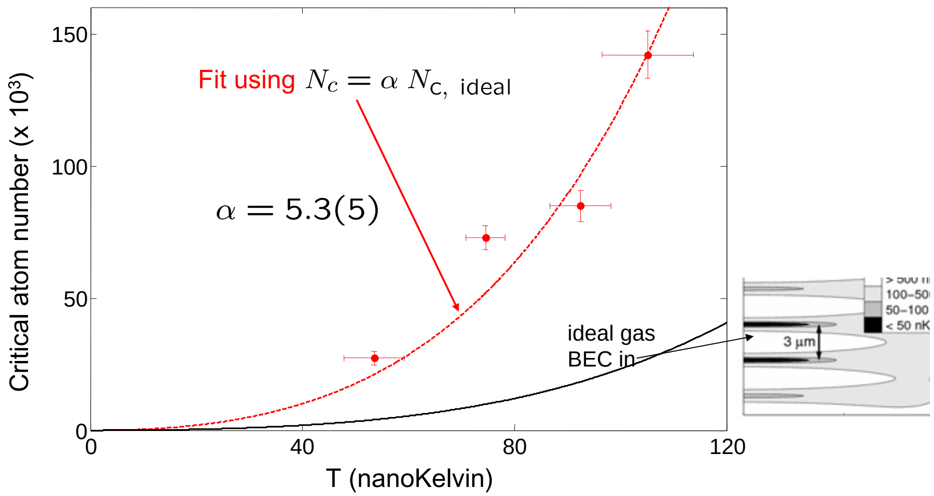
Bimodality and interferences



Within our accuracy, onset of bimodality and interference agree

Variation of the critical atom number with T

$$\hbar\omega_z = 150 \text{ nK}$$



5.3 larger than the ideal gas prediction

Can it be the Kosterlitz-Thouless transition point?

For a uniform system with our interaction strength, the KT transition is expected to occur for

$$n_{\text{total}}\lambda^2 \simeq 8.0$$

Local density approximation + gaussian density profile (as seen experimentally)

$$n_{\text{total}}(0)\lambda^2 = 8.0 \qquad n(\vec{r}) = n(0)\, e^{-V(\vec{r})/kT}$$

$$N_{\text{c, KT}} = 4.9\; N_{\text{c, ideal}}$$

This factor 4.9 has to be compared with the experimental factor 5.3: not bad…

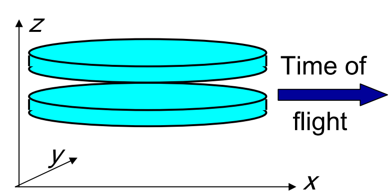
4.
Quasi-long range order in a superfluid 2D Bose gas

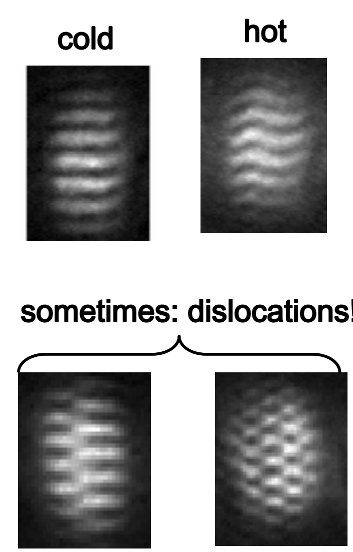
Z. Hadzibabic, P. Krüger
B. Battelier, M. Cheneau, S. Stock

Phys. Rev. Lett. **95**, 190403 (2005)
Nature **441**, 1118 (2006)

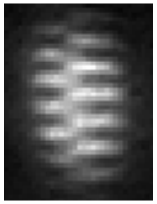
Theory: Shlyapnikov-Gangardt-Petrov, Holtzman *et al.,* Kagan *et al.*, Stoof *et al.*, Mullin *et al.*, Simula-Blackie, Hutchinson *et al.* Polkovnikov-Altman-Demler

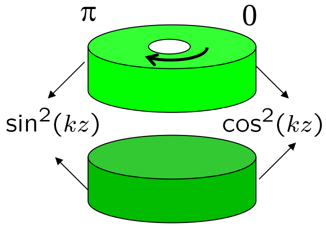
The superfluid phase investigated using matter-wave interferometry





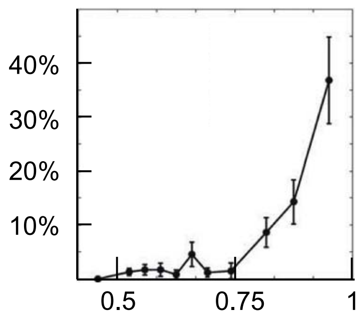
Dislocation = evidence for free vortices





Similar results at NIST

fraction of images showing a dislocation



temperature control (arb.units)

0 = full contrast at center

1 = no contrast at center

Investigation of the long range order: $g_1(\vec{r}) = \langle \psi^*(\vec{r})\, \psi(0) \rangle$

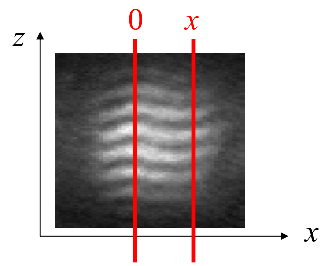
The interference signal between $\psi_a(x,y)$ and $\psi_b(x,y)$ gives

$$|\psi_a|^2 + |\psi_b|^2 + \psi_a^*\psi_b e^{ikz} + \psi_a\psi_b^*\, e^{-ikz}$$

$\psi_a^*\psi_b \rightarrow \kappa(x,y)$

$$\langle \kappa(x,y)\kappa^*(0) \rangle = \langle \psi_a^*(x,y)\psi_b(x,y)\psi_a(0)\psi_b^*(0) \rangle$$

$$= \langle \psi_a^*(x,y)\psi_a(0) \rangle \, \langle \psi_b(x,y)\psi_b^*(0) \rangle = |g_1(x,y)|^2$$

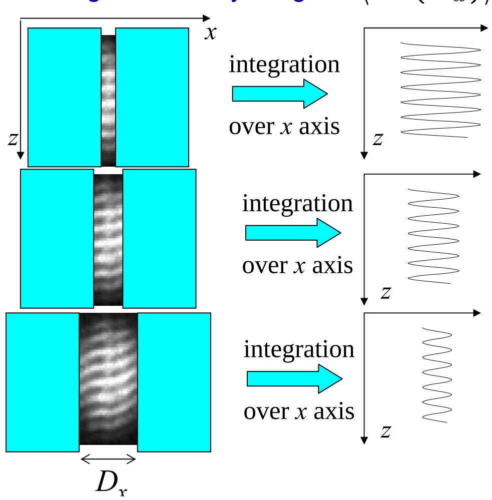


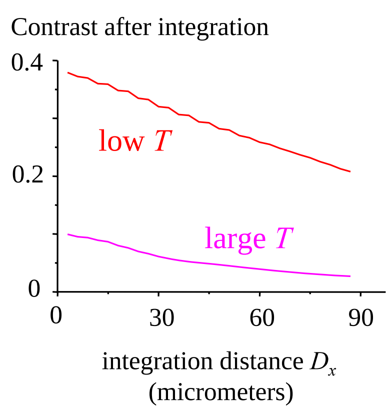
Investigation of the long range order (2)

Integrated contrast: $C(D_x) = \dfrac{1}{D_x}\displaystyle\int_0^{D_x} \kappa(x)\, dx$

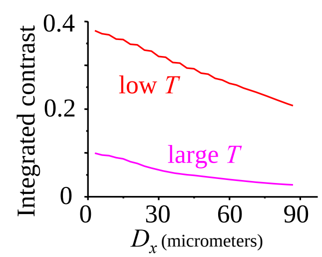
Polkovnikov, Altman, Demler

average over many images: $\langle C^2(D_x) \rangle \sim \dfrac{1}{D_x}\displaystyle\int_0^{D_x} (g_1(x,0))^2\, dx$





Investigation of the long range order (3)



Polkovnikov, Altman, Demler

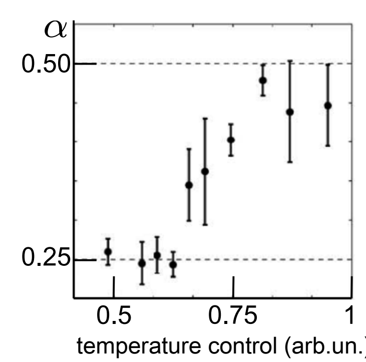
fit $\langle C^2(D_x) \rangle$ by $\dfrac{1}{(D_x)^{2\alpha}}$

- Just below the transition, $g_1(r)$ decays algebraically like $r^{-1/4}$

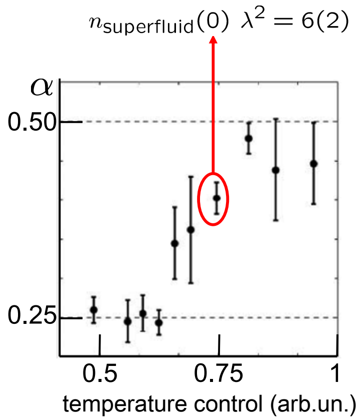
$\alpha = 1/4$

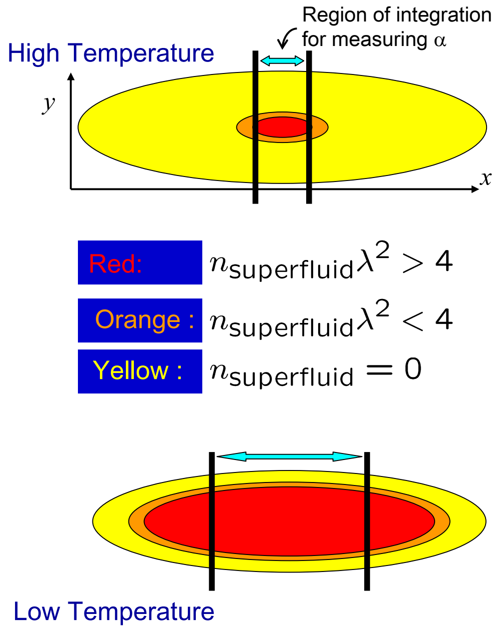
- Just above the transition, $g_1(r)$ decays exponentially

$\alpha = 1/2$



The inhomogeneous density profile in the trap

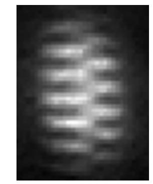




The 2D atomic Bose gas at present…

Several signatures of a Kosterlitz-Thouless cross-over have been identified

- Apparition of a bi-modal structure (superfluid core?)
- proliferation of vortices
- loss of long range order



The observed phenomena do not match the ideal Bose gas condensation

Good agreement with quantitative predictions of the KT mechanisms

One important remaining question:
direct test the superfluidity of the core

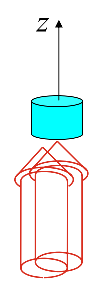
Lecture 2

2D Bose gases: the rotating case

Jean Dalibard,
Laboratoire Kastler Brossel*,
ENS Paris

* Research unit of CNRS, ENS, and UPMC

Physics in a rotating frame



Cylindrically symmetric trap potential in the xy plane:

$$V(\vec{r}) = \frac{1}{2} m\omega^2 r^2 \qquad r^2 = x^2 + y^2$$

Stir at frequency Ω (with a rotating laser beam for example)

$$\delta V(\vec{r}) = \frac{\epsilon}{2} m\omega^2 \left(X^2 - Y^2\right) \qquad \epsilon \sim \text{a few } \%$$

Hamiltonian in the rotating frame:

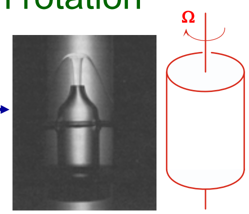
$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2 - \Omega L_z$$

$$= \frac{\left(\vec{p} - \vec{A}\right)^2}{2m} + \frac{1}{2} m \left(\omega^2 - \Omega^2\right) r^2 \qquad \vec{A} = m\vec{\Omega} \times \vec{r}$$

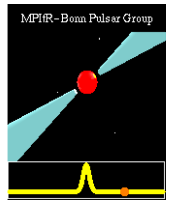
Same physics as charged particles in a magnetic field + harmonic confinement

Macroscopic quantum objects in rotation

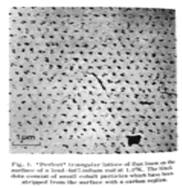
Rotating bucket with superfluid liquid helium



Neutron stars and pulsars



Superconductor in a magnetic field



Rotating nuclei

1.

Bose gases in rotation

and vortex nucleation

Classical vs. quantum rotation

Rotating classical gas

velocity field of a rigid body $\vec{v} = \vec{\Omega} \times \vec{r}$ → $\vec{\nabla} \times \vec{v} = 2\vec{\Omega}$

Rotating a quantum macroscopic object

macroscopic wave function: $\psi(\vec{r}) = \sqrt{\rho(\vec{r})}\, e^{i\phi(\vec{r})}$

In a place where $\rho(\vec{r}) \neq 0$, irrotational velocity field: $\vec{v} = \frac{\hbar}{m}\vec{\nabla}\phi$

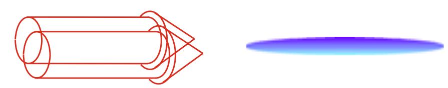
The only possibility to generate a non-trivial rotating motion is to nucleate quantized vortices (points in 2D or lines in 3D)

$$\oint \vec{v} \cdot d\vec{r} = \frac{nh}{m}$$

Feynman, Onsager, Pitaevskii

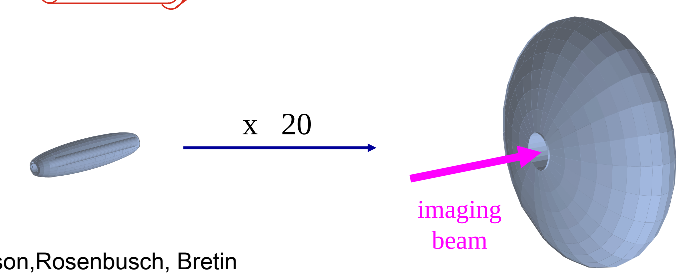
Vortices in a stirred condensate

Cylindrical trap
+
stirring

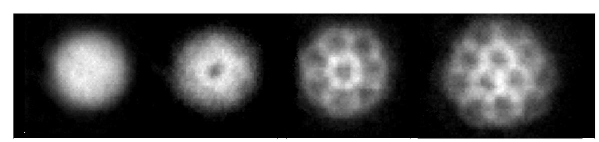


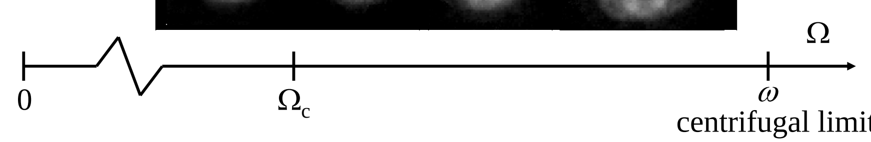
ENS, Boulder, MIT, Oxford

Time of flight analysis (25 ms)

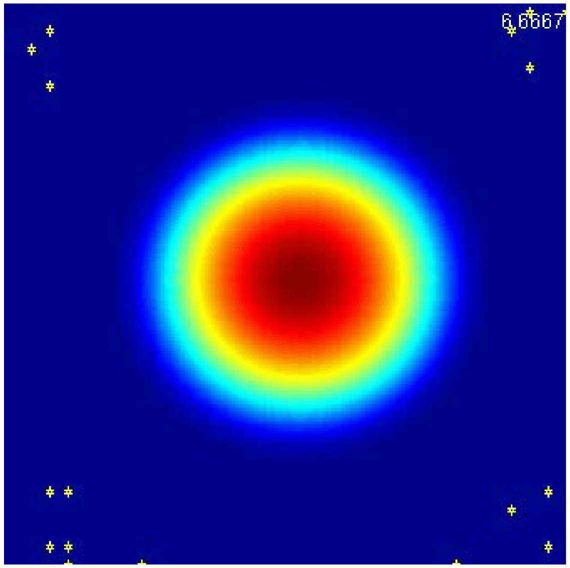


ENS: Chevy,Madison,Rosenbusch, Bretin





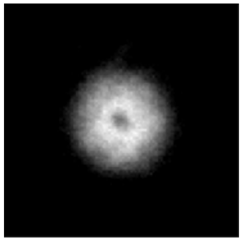
Nucleation of vortices: a simulation using Gross-Pitaevskii equation

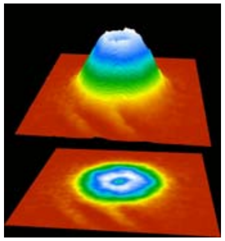


C. Lobo,
A. Sinatra,
Y. Castin,
Phys. Rev. Lett. **92**, 020403 (2004)

The single vortex case

After time-of-flight expansion:





Questions which have been answered:

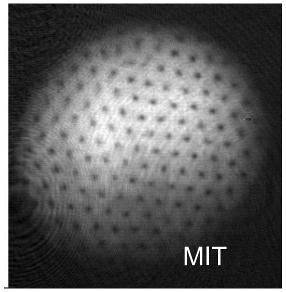
- Total angular momentum $N\hbar$ (i.e. $\hbar$ per particle) ?
- Is the phase pattern varying as $e^{i\theta}$?
- What is the shape of the vortex line?
- Can this line be excited (as a guitar string)? Kelvin mode

The intermediate rotation regime

The number of vortices is notably larger than 1.

However one keeps the rotation frequency Ω notably below ω

core size ξ << vortex spacing



Uniform surface density of vortices n_v with

$$\Omega = \frac{\pi\hbar}{m} n_v$$

Coarse-grain average for the velocity field

$$\vec{v} = \vec{\Omega} \times \vec{r}$$

The fast rotation regime $\Omega \sim \omega$

The centrifugal force nearly balances the trapping force.
The radius of the gas increases and the density ρ drops.

⇨ The vortex core $\xi = \frac{1}{\sqrt{8\pi\rho a}}$ increases to infinity ???

⇨ The vortex spacing $n_v^{-1/2} \propto \Omega^{-1/2}$ decreases

Are there still vortices? → *cf.* H_{c2} field for a superconductor?

Do they still form a regular lattice?

2.

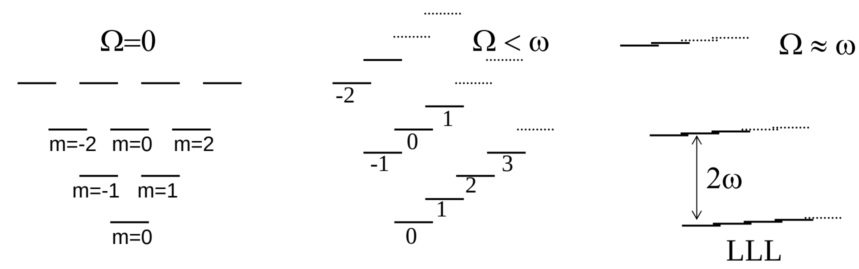
The fast rotation regime:

Physics in the Lowest Landau Level

Landau levels for a rotating gas

Isotropic harmonic trapping in the xy plane with frequency ω

Hamiltonian in the rotating frame: $H - \Omega L_z$

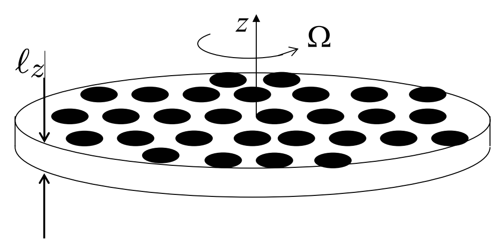


When $\Omega = \omega$, macroscopically degenerate ground state for the one-body hamiltonian (Rohshar, Ho)

$$H = \frac{\left(\vec{p} - \vec{A}\right)^2}{2m} \qquad \vec{A} = \vec{\Omega} \times \vec{r}$$

Reaching the lowest Landau level

Assume that the z direction is frozen, with a extension ℓ_z

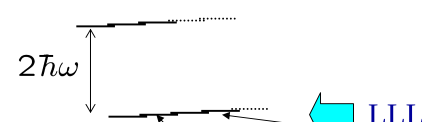


If the chemical potential and the temperature are much smaller than $2\hbar\omega$, the physics is restricted to the lowest Landau level

This LLL regime corresponds to $1 - \frac{\ell_z}{Na} < \frac{\Omega}{\omega}$ $\qquad$ a: scattering length

Typically: N=10 000 atoms, ℓ_z=0.5 μm, a =5 nm $\Rightarrow$ $\Omega > 0.99\ \omega$

The lowest Landau level



$e^{-r^2/2a_0^2}$ $\qquad$ $(x+iy)e^{-r^2/2a_0^2}$ $\qquad$ $(x+iy)^n e^{-r^2/2a_0^2}$

Characteristic length scale

$$a_0 = \sqrt{\hbar/m\omega}$$

General one-particle state in the LLL:

$$e^{-r^2/2a_0^2}\, P(x+iy) \longrightarrow e^{-r^2/2a_0^2} \prod_{j=1}^{n} (u - u_j)$$

Polynomial or analytic function

$u = x + iy$ $\qquad$ u_j : vortices

➡ The size of the vortices is comparable to their spacing

➡ The atom distribution is entirely determined from the vortex position

Ground state in the mean-field LLL approximation

Minimization of $E = E_{\text{kin}} + E_{\text{pot}} + E_{\text{int}} + E_{\text{rot}}$

$$E_{\text{int}} = \frac{N(N-1)g}{2} \int |\psi|^4 \qquad g = \frac{\hbar^2}{m}\sqrt{8\pi}\frac{a}{\ell_z}$$

In the LLL:

$$\begin{cases} E_{\text{rot}} = -N\Omega \int \psi^* [L_z \psi] \\ E_{\text{kin}} = E_{\text{pot}} = \dfrac{N\hbar\omega}{2} + \dfrac{N\omega}{2}\int \psi^*[L_z\psi] \end{cases}$$

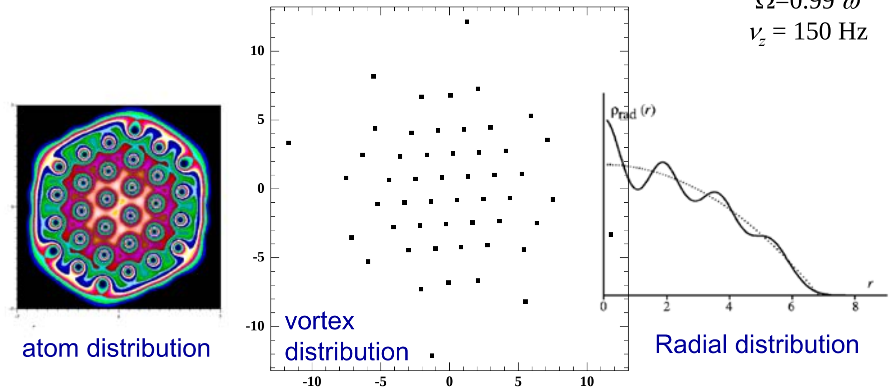
→ Minimization of $\int r^2|\psi|^2 + \Lambda|\psi|^4$

only 1 parameter: $\Lambda = \dfrac{Ng}{2(1-\Omega/\omega)}$

Aftalion, Blanc, Dalibard,
Phys. Rev. A **71**, 023611 (2005)

A typical LLL solution

1000 Rb atoms
Ω=0.99 *ω*
ν_z = 150 Hz



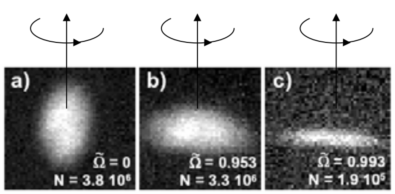
The distortion of the vortex distribution on the edges is essential for producing a quasi-parabolic atomic density profile

A. Aftalion, X. Blanc, and J. Dalibard,
Phys. Rev. A **71**, 023611 (2005)

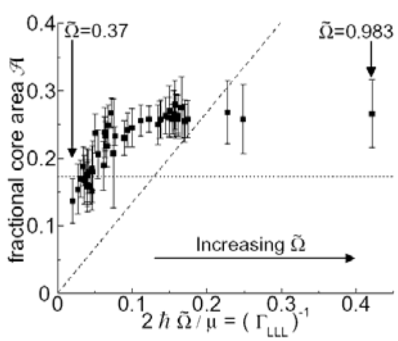
also Watanabe, Baym, Pethick
Komineas, Read, Cooper
Sheehy, Radzihovsky

How to reach the LLL experimentally

Evaporative spin-up method (Boulder): first rotate the gas at a moderate frequency and then evaporate of atoms with small angular momentum.



side view of the rotating BEC



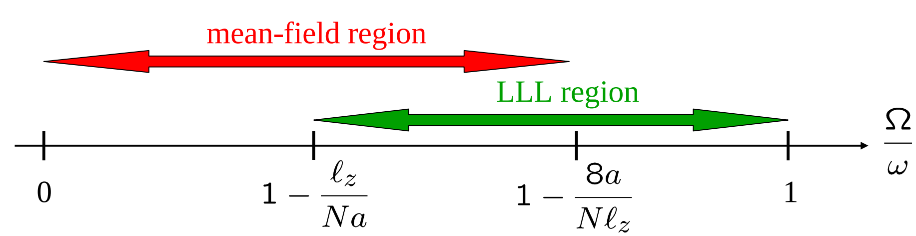
V. Schweikhard *et al.*, PRL **92**, 040404 (2004)

Limits of the mean field approach

When Ω increases even more than the LLL threshold, the number of vortices N_{v} increases and becomes similar to the atom number N

It occurs for $\dfrac{\Omega}{\omega} \geq 1 - \dfrac{8a}{N\ell_z}$

Total angular momentum: $L_z \sim N^2\hbar$ ($N\hbar$ per particle)



Note that $a < \ell_z$ for a 3D description of the binary collisions