Properties of the BCS state

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We consider an ensemble of indistinguishable fermionic particles in a d-dimensional box of size L with periodic boundary conditions. Each particle has two possible orthogonal spin states: $|\uparrow\rangle$ and $|\downarrow\rangle$. Particles of opposite spin $\uparrow$ and $\downarrow$ interact through an attractive potential, while particles of same spin state do not interact.

Bardeen, Cooper and Schrieffer (BCS) were able to describe approximately the ground state of the gas using a variational approach. Here we will study some properties of this variational state.

1. Apply the following operator to the vacuum:
$$C^\dagger = \int d^d r_1 \int d^d r_2 \, \phi(r_1 - r_2) \, \hat{\psi}^\dagger_\uparrow(r_1) \, \hat{\psi}^\dagger_\downarrow(r_2), \tag{1}$$
where d is the space dimension. Show that $C^\dagger$ acting on the vacuum creates a pair of particles in a quantum state that we will write explicitly in first quantization.

2. It is useful to go to the k-representation. Show that one can write:
$$C^\dagger = \sum_k \phi_k \, a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow}, \tag{2}$$
and write explicitly the expression of the coefficients ϕ_k.

3. The first form of the BCS state is a coherent state of pairs
$$|\psi_{BCS}\rangle = \mathcal{N} e^{\gamma C^\dagger} |0\rangle, \tag{3}$$
where γ is in general a complex number but we will consider it to be real. Show that two other equivalent forms of $|\psi_{BCS}\rangle$ are given by
$$|\psi_{BCS}\rangle = \mathcal{N} \, \Pi_k e^{\Gamma_k a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow}} |0\rangle, \tag{4}$$
with $\Gamma_k = \gamma \phi_k$, and by
$$|\psi_{BCS}\rangle = \mathcal{N} \, \Pi_k (1 + \Gamma_k a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow}) |0\rangle \, . \tag{5}$$

4. Calculate the normalization factor $\mathcal{N}$ and find the following canonical form
$$|\psi_{BCS}\rangle = \Pi_k \, (u_k + v_k a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} |0\rangle, \tag{6}$$
with u_k and v_k that we will write explicitly.

5. To show the applicability of the Wick's theorem to calculate various observables in the BCS state we shall introduce yet another form of $|\psi_{BCS}\rangle$:

$$|\psi_{BCS}\rangle = \Pi_k \, e^{\theta_k (a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} - \text{ h.c.})}|0\rangle \equiv U|0\rangle, \tag{7}$$

with $u_k = \cos\theta_k$ and $v_k = \sin\theta_k$. Show that this form (7) is equivalent to the previous ones.

6. Let the operators b_k be any linear combination of a_α and $a^\dagger_\alpha$. Show that

$$\langle\psi_{BCS}|b_1 \dots b_k|\psi_{BCS}\rangle = \langle 0|\tilde{b}_1 \dots \tilde{b}_k|0\rangle, \tag{8}$$

with $\tilde{b_k} = U^\dagger b_k U$.

7. Show that

$$U^\dagger a_{k\uparrow} U = u_k a_{k\uparrow} + v_k a^\dagger_{-k\downarrow}, \tag{9}$$

$$U^\dagger a^\dagger_{-k\downarrow} U = -v_k a_{k\uparrow} + u_k a^\dagger_{-k\downarrow} \, . \tag{10}$$

In the light of this last result, explain why Wick's theorem is applicable in the BCS state.

8. Practice: Calculate the variance of the number of atoms in the BCS state and show that $\text{Var}(\hat{N}) \leq 2\langle\hat{N}\rangle$, which implies a sub-poissonian variance for the number of pairs.