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On the Interaction of Classical Fields and Particles.

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Summary. — The interaction of a classical field with general tensorial character with a particle described by its trajectory is shown to lead in general to a Lorentz-Dirac-type equation, independently of the details of the interaction. The divergences of the theory are carefully analysed by a method we have proposed before, which is shown to be equivalent to other regularization schemes that have been used previously. We apply our results to several well-known interactions including linearized gravitation and symmetric electrodynamics with monopoles.

1. – Introduction.

We study here the interaction of a field with a general tensorial character with a particle described by its trajectory $Z(s)$. We consider a general class of interactions restricted only by assuming that the equation of motion for the particle should be linear in the field and the derivative of the field. If we further assume that the coupled equations of motion are obtained from a Lagrangian we prove that one is led to a Lorentz-Dirac-type equation for the trajectory, the different interactions have only the effect of changing the numerical coefficient in front of the Abraham force and the explicit value of the unobservable mass renormalization which we show is always a cut-off-dependent (divergent) quantity. The same conclusion is reached if the current of the field depends only on $\dot{Z}$. We perform our study using only the equations of motion and making a careful study of the divergences of the theory, which we show can all be absorbed by a mass renormalization. As is well known, the problem here is the divergence of the field on the world-line of the particles; we treat this in sect. **2** by a regularization technique we have proposed previously (1) which has a clear physical interpretation that we discuss in detail. We call this technique the double-zero–separation regularization. We show that this double-zero separation justifies and gives the physical meaning of the « analytic continuation » regularization proposed by BARUT and VILLARROEL (2-4), since both techniques are shown to be equivalent. In sect. **3** we compare the double-zero–separation method with the average procedure proposed by TEITELBOIM (5), which consists in defining the field on the world-line as the limit of the average of the field on a small sphere with centre on the particle in the rest system. One finds that the average method only modifies the numerical coefficient of a divergent term in the derivative of the field, and that this only modifies the unobservable mass renormalization.

In sect. **4** we introduce the Lagrangian formalism and show that the equation for the trajectory is always a Lorentz-Dirac–type equation. We also give explicit formulae for the numerical coefficient of the Abraham force and for the mass renormalization counterterm. We apply then our results to different cases studied previously in the literature and collect the results in table I. The cases of linearized gravitation and of symmetric electrodynamics with magnetic charges that require some minor modifications are treated in appendices B and C. We have treated only the case of massless fields, but the same techniques for handling the divergences can be used for massive fields, since the singularity of the retarded Green function on the light-cone is unchanged. The same comment applies to the electromagnetic interaction in the presence of a given gravitational field (4).

2. – The regularization method and its relation to the analytic-continuation technique.

The system of coupled equations we study here is (the dot stands for derivation with respect to the proper time s)

$$\Box\, \psi_{(\beta)}(x) = 4\pi\lambda \int \mathrm{d}s\, \delta^{(4)}\big(x - Z(s)\big)\, \Pi \dot{Z}_{(\beta)}(x) = j_{(\beta)}(x)\,, \tag{1}$$

$$m_B \ddot{Z}_\mu(s) = \lambda F_\mu\Big(\dot{Z}(s),\, \ddot{Z}(s),\, \psi_{(\beta)}\big(Z(s)\big),\, \partial_\nu \psi_{(\beta)}\big(Z(s)\big)\Big)\,, \tag{2}$$

(1) E. TIRAPEGUI: preprint K.U.L. (Leuven), to be published in *Amer. Journ. Phys.*
(2) A. BARUT: *Phys. Rev. D*, **10**, 3335 (1974).
(3) A. BARUT and D. VILLARROEL: *J. Phys. A*, **8**, 156 (1975).
(4) A. BARUT and D. VILLARROEL: *J. Phys. A*, **8**, 1537 (1975).
(5) C. TEITELBOIM: *Phys. Rev. D*, **4**, 345 (1971).

where (β) stands for a collection of indices and $\Pi\dot{Z}_{(\beta)}(s)$ for a product of $\dot{Z}_\alpha(s)$, a product of $g_{\mu\nu}$ (the metric tensor that we take as $g_{00}=+1$, $g_{ii}=-1$, $i=1,2,3$), a mixed product of $g_{\mu\nu}$ and $\dot{Z}_\alpha(s)$, or a sum of these three types, all with the correct tensorial character of the field $\psi_{(\beta)}(x)$. For classical electrodynamics $\psi_{(\beta)}(x)$ is the electromagnetic potential $A_\mu(x)$, $\Pi\dot{Z}_{(\beta)}(s)$ is $\dot{Z}_\mu(s)$, and $\lambda F_\mu = e\big(\partial_\mu A_\nu(Z(s)) - \partial_\nu A_\mu(Z(s))\big)\dot{Z}^\nu(s)$. For a tensor field $\psi_{(\beta)}$ is $U_{\mu\nu}(x)$ and $\Pi\dot{Z}_{(\beta)}(s)$ can be taken as $\dot{Z}_\mu(s)\dot{Z}_\nu(s)$, $g_{\mu\nu}$, or a sum of these two terms. From now on, we shall use the notation $r_{(\beta)}(s)=\Pi\dot{Z}_{(\beta)}(s)$. In order to obtain the equation of motion for the trajectory, we solve (1) for $\psi_{(\beta)}(x)$ in terms of $Z(s)$, using the retarded Green function $D_{\mathrm{R}}(x)$ and we replace the field in (2), the resulting differential equation for $Z(s)$ is the equation of motion of the particle. The difficulty with this program is, as is well known, that the field $\psi_{(\beta)}(x)$ and its derivative $\partial_\nu\psi_{(\beta)}(x)$ diverge on the trajectory, *i.e.* at the point $x=Z(s)$. This is precisely the problem we shall solve here for the case in which the force F_μ is linear in the field and its derivative, *i.e.* it is of the form

$$F_\mu = a_\mu{}^{\nu(\beta)}(\dot{Z}(s),\ddot{Z}(s))\partial_\nu\psi_{(\beta)}(Z(s)) + b_\mu{}^{(\beta)}(\dot{Z}(s),\ddot{Z}(s))\,\psi_{(\beta)}(Z(s)) + c_\mu(\dot{Z}(s),\ddot{Z}(s))\,. \tag{3}$$

The retarded Green function $D_{\mathrm{R}}(x)$ solution of $\Box D_{\mathrm{R}}(x)=\delta^{(4)}(x)$ is given by $(\theta(x^0)=0,\ x^0<0,\ \theta(x^0)=1,\ x^0>0)$

$$D_{\mathrm{R}}(x) = \frac{1}{2\pi}\theta(x)\delta(x^2) \tag{4}$$

and from (1) we obtain

$$\hat{\psi}_{(\beta)}(x) = \psi^{\mathrm{in}}_{(\beta)}(x) + \lambda\int \mathrm{d}^4x'\,D_{\mathrm{R}}(x-x')\,j_{(\beta)}(x')\,, \tag{5}$$

where $\psi^{\mathrm{in}}_{(\beta)}(x)$ is a free field ($\Box\,\psi^{(\mathrm{in})}(x)=0$) corresponding to the initial condition $\hat{\psi}_{(\beta)}(x^0\to-\infty,\boldsymbol{x})\to\psi^{(\mathrm{in})}_{(\beta)}(x)$. We write $\psi_{(\beta)}(x)$ for the retarded field, $\hat{\psi}_{(\beta)}(x)=\psi^{(\mathrm{in})}_{(\beta)}(x)+\psi_{(\beta)}(x)$, and from (4) and (5) we obtain

$$\psi_{(\beta)}(x) = 2\lambda\int \mathrm{d}s\,\theta(x^0-Z^0(s))\,\delta\big((x-Z(s))^2\big)\,r_{(\beta)}(s)\,. \tag{6}$$

If the field $\psi_{(\beta)}(x)$ is required to satisfy supplementary conditions, this of course induces corresponding constraints on $j_{(\beta)}(x)$ (in electrodynamics to the condition $\partial_\mu A^\mu(x)=0$ corresponds $\partial_\mu j^\mu(x)=0$) and consequently the retarded solution (5) will satisfy the subsidiary conditions when $\psi^{\mathrm{in}}_{(\beta)}(x)$ is chosen to satisfy them.

By a standard calculation (6) we obtain for $\partial_\nu \psi_{(\beta)}(x)$, $x \neq Z(s)$, the result

$$(7) \qquad \partial_\nu \psi_{(\beta)}(x) = 2\lambda \int \mathrm{d}s\, \theta(x^0 - Z^0(s))\, \delta\big((x - Z(s))^2\big) \frac{\mathrm{d}}{\mathrm{d}s} \frac{(x - Z(s))_\nu r_{(\beta)}(s)}{(x - Z(s))^\alpha \dot{Z}_\alpha(s)}\,.$$

We define now the linear operator I acting on the functions $f(x, Z(s))$ by

$$(8) \qquad If(x, Z(s)) = \int \mathrm{d}s\, \theta(x^0 - Z^0(s))\, \delta\big((x - Z(s))^2\big) f(x, Z(s))\,,$$

in term of which we can write

$$(9) \qquad \psi_{(\beta)}(x) = 2\lambda I f_{(\beta)}(x, Z(s))\,,$$

$$(10) \qquad \partial_\nu \psi_{(\beta)}(x) = 2\lambda I f_{\nu(\beta)}(x, Z(s))$$

with $f_j(x, Z(s))$, $j = (\beta)$, $\nu(\beta)$, defined as

$$(11) \qquad f_{(\beta)}(x, Z(s)) = r_{(\beta)}(s) = \Pi \dot{Z}_{(\beta)}(s)\,,$$

$$(12) \qquad f_{\nu(\beta)}(x, Z(s)) = \frac{\mathrm{d}}{\mathrm{d}s} \frac{(x - Z(s))_\nu r_{(\beta)}(s)}{(x - Z(s))^\alpha \dot{Z}_\alpha(s)}\,.$$

Our purpose is to evaluate (9) and (10) for $x \to Z(s_0)$ to replace them in (2). We remark now that $f_j(Z(s_0), Z(s))$ is a well-defined finite function for all s $\big(f_{(\beta)}(x, Z(s))$, because it does not depend on x and $f_{\nu(\beta)}(x, Z(s))$ because of its explicit expression one can calculate, see appendix$\big)$. Putting $s = s_0 + \tau$, and defining $\bar{f}_j(\tau)$ by

$$(13) \qquad f_j(Z(s_0), Z(s_0 + \tau)) = \bar{f}_j(\tau)\,,$$

one checks that $\bar{f}_j(\tau)$ and its derivatives are finite at $\tau = 0$. More explicitly one has (see appendix)

$$(14) \qquad \bar{f}_{(\beta)}(0) = r_{(\beta)}(s_0)\,, \qquad \left.\frac{\partial \bar{f}_{(\beta)}(\tau)}{\partial \tau}\right|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0)\,,$$

$$(15) \qquad \bar{f}_{\nu(\beta)}(0) = \frac{1}{2} \ddot{Z}_\nu(s_0) r_{(\beta)}(s_0) + \dot{Z}_\nu(s_0) \frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0)\,,$$

$$(16) \qquad \left.\frac{\partial \bar{f}_{\nu(\beta)}(\tau)}{\partial \tau}\right|_{\tau=0} = \frac{1}{3} \dddot{Z}_\nu(s_0) r_{(\beta)}(s_0) + \ddot{Z}_\nu(s_0) \frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0) + \\ + \dot{Z}_\nu(s_0) \frac{\mathrm{d}^2}{\mathrm{d}s_0^2} r_{(\beta)}(s_0) + \frac{1}{3} \ddot{Z}_\alpha(s_0) \ddot{Z}^\alpha(s_0) \dot{Z}_\nu(s_0) r_{(\beta)}(s_0)\,.$$

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(6) A. BARUT: *Electrodynamics and Classical Theory of Fields and Particles* (London, 1964).

The regularization method we have proposed (1) consists in the following. One has to compute $\psi_{(\beta)}(x)$ and $\partial_\nu \psi_{(\beta)}(x)$ for $x \to Z(s_0)$, and consequently (8) for $x \to Z(s_0)$ and $f = f_j$, but, as $f_j(x, Z(s))$ remains finite for $x \to Z(s_0)$, as we have just seen, we can put $x = Z(s_0)$ there thus writing $f_j(Z(s_0)), Z(s_0 + \tau)) = \bar{f}_j(\tau)$, the divergence coming from the other terms in the integral (8) which we have to regularize. We have then ($s = s_0 + \tau$) from (8) that

$$If_j(x, Z(s))\big|_{x \to Z(s_0)} = \int d\tau\, \theta\big(x^0 - Z^0(s_0 + \tau)\big)\, \delta\Big(\big(x - Z(s_0 + \tau)\big)^2\Big)\, \bar{f}_j(\tau)\big|_{x \to Z(s_0)}\,. \tag{17}$$

One has

$$X^0 - Z^0(s_0 + \tau)\big|_{x \to Z(s_0)} = -\tau \dot{Z}^0(s_0) + O(\tau^2)\,, \qquad \dot{Z}^0(s) = \frac{dt}{ds} = \frac{1}{\sqrt{1 - v^2}} > 0 \tag{18}$$

and

$$\big(X - Z(s)\big)^2\big|_{x \to Z(s_0)} = \big(Z(s_0) - Z(s_0 + \tau)\big)^2 = \tau^2 + o(\tau^3)\,. \tag{19}$$

We see then that in the integral (17) the integrand contributes only for small $\tau(\delta(\tau^2) = 0,\ \tau \neq 0)$, and consequently from (18) we can replace $\theta\big(x^0 - Z^0(s_0 + \tau)\big)$ by $\theta(-\tau)$. The origin of the divergence of (17) is now clear, since $\delta(\tau^2)$ is not defined because the argument of the δ-function has a double zero at $\tau = 0$. We regularize this δ-function replacing $\delta(\tau^2)$ by $\delta(\tau^2 - \eta^2)$ and defining (17) as the limit $\eta \to 0$. The physical interpretation of the regularization is clear when one remarks that the effect of putting $\delta(\tau^2 - \eta^2)$ is to separate the double

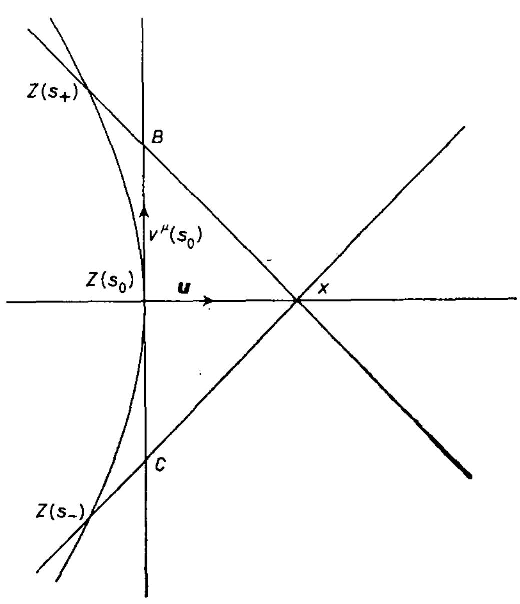

Fig. 1. – $v^\mu(s_0) = \dot{Z}^\mu(s_0) = (1, \mathbf{0})$, $x^\mu = Z^\mu(s_0) + \eta u^\mu$, $|\boldsymbol{x} - \boldsymbol{Z}(s_0)| = \eta$, $\tau_\pm = s_\pm - s_0 \sim |AB| = |AC| = \eta$.

zero in two simple zeros $\pm\eta$, but $(x - Z(s_0+\tau))^2 = 0$ has two simple zeros $\tau_\pm$, when $x \neq Z(s_0)$, and when $x \to Z(s_0)$ these two zeros τ_+ and τ_- coalesce to zero. Putting $x^\mu = Z^\mu(s_0) + \eta u^\mu$, $u^\mu \dot{Z}_\mu(s_0) = 0$, and interpreting $x \to Z(s_0)$ as $\eta \to 0$, that is approaching the world-line in the orthogonal plane to $\dot{Z}^\mu(s_0)$ (*i.e.* in some direction in the three-space of the rest system at proper time s_0, see fig. 1), one has that the two zeros of $(Z(s_0) + \eta u - Z(s_0+\tau))^2 = 0$ are $\tau_\pm = \pm\eta + O(\eta^2)$, thus justifying our regularization. We have then from (17) that

$$(20)\qquad If_j(X, Z(s))\big|_{X\to Z(s_0)} = \lim_{\eta\to 0}\int \mathrm{d}\tau\, \theta(-\tau)\, \delta(\tau^2-\eta^2)\, \bar{f}_j(\tau)$$

and, using $\theta(x) = \frac{1}{2}(1+\varepsilon(x))$, we obtain [1]

$$(21)\qquad If_j(X, Z(s))\big|_{X\to Z(s_0)} = \frac{1}{2\eta}\bar{f}_j(0) - \frac{1}{2}\left.\frac{\partial \bar{f}_j(\tau)}{\partial\tau}\right|_{\tau=0} + O(\eta)\,.$$

Replacing (21) in (9) and (10) and using expression (14), (15) and (16) for $\bar{f}_{(\beta)}$ and $\bar{f}_{\nu(\beta)}$, we obtain

$$(22)\qquad \psi_{(\beta)}(Z(s_0)) = \frac{\lambda}{\eta} r_{(\beta)}(s_0) - \lambda \frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0)\,,$$

$$(23)\qquad \partial_\nu \psi_{(\beta)}(Z(s_0)) = \frac{\lambda}{\eta}\left[\frac{1}{2}\ddot{Z}_\nu(s_0) r_{(\beta)}(s_0) + \dot{Z}_\nu(s_0)\frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0)\right] -$$
$$- \lambda\left[\frac{1}{3}\dddot{Z}_\nu(s_0) r_{(\beta)}(s_0) + \ddot{Z}_\nu(s_0)\frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0) + \dot{Z}_\nu(s_0)\frac{\mathrm{d}^2}{\mathrm{d}s_0^2} r_{(\beta)}(s_0) + \right.$$
$$\left. + \frac{1}{3}\ddot{Z}_\alpha(s_0)\ddot{Z}^\alpha(s_0)\dot{Z}_\nu(s_0) r_{(\beta)}(s_0)\right].$$

The equation of motion of the particle is obtained now replacing (22) and (23) in (2) with F_μ given by (3). One obtains

$$(24)\qquad m_B \ddot{Z}_\mu(s_0) = \lambda a_\mu{}^{\nu(\beta)}(\dot{Z}(s_0), \ddot{Z}(s_0))\left[\frac{\lambda}{\eta}\left(\frac{1}{2}\ddot{Z}_\nu(s_0) r_{(\beta)}(s_0) + \dot{Z}_\nu(s_0)\frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0)\right) - \right.$$
$$- \lambda\left(\frac{1}{3}\dddot{Z}_\nu(s_0) r_{(\beta)}(s_0) + \ddot{Z}_\nu(s_0)\frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0) + \dot{Z}_\nu(s_0)\frac{\mathrm{d}^2}{\mathrm{d}s_0^2} r_{(\beta)}(s_0) + \right.$$
$$\left.\left. + \frac{1}{3}\ddot{Z}_\alpha(s_0)\ddot{Z}^\alpha(s_0)\dot{Z}_\nu(s_0) r_{(\beta)}(s_0)\right)\right] +$$
$$+ \lambda b_\mu{}^{(\beta)}(\dot{Z}(s_0), \ddot{Z}(s_0))\left[\frac{\lambda}{\eta} r_{(\beta)}(s_0) - \lambda\frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0)\right] +$$
$$+ \lambda F_\mu\left(\dot{Z}(s_0), \ddot{Z}(s_0), \psi^{\mathrm{in}}_{(\beta)}(Z(s_0)), \partial_\nu\psi^{\mathrm{in}}_{(\beta)}(s_0)\right).$$

We summarize now the different steps leading to (24) and we introduce an abbreviated notation in order to compare with the analytic-continuation method of ref. ([2]). From (2) we obtain using (9) and (10) that $(x \to Z(s_0))$

$$(25)\qquad m_B \ddot{Z}_\mu(s_0) = 2\lambda^2 a_\mu{}^{\nu(\beta)}(\dot{Z}(s_0), \ddot{Z}(s_0)) If_{\nu(\beta)}(X, Z(s)) + \\ + 2\lambda^2 b_\mu{}^{(\beta)}(\dot{Z}(s_0), \ddot{Z}(s_0)) If_{(\beta)}(X, Z(s)) + \\ + \lambda F_\mu\big(\dot{Z}(s_0), \ddot{Z}(s_0), \psi^{\text{in}}_{(\beta)}(Z(s_0)), \partial_\nu \psi^{\text{in}}_{(\beta)}(Z(s_0))\big) = \\ = 2\lambda^2 \sum_{j=(\beta),\nu(\beta)} h_{\mu j}(\dot{Z}(s_0), \ddot{Z}(s_0)) If_j(X, Z(s)) + \lambda F^{\text{in}}_\mu ,$$

where $h_{\mu(\beta)} = b_\mu{}^{(\beta)}$, $h_{\mu\nu(\beta)} = a_\mu{}^{\nu(\beta)}$, and F^{in}_μ stands for (3) at the point s_0 with $\psi_{(\beta)}$ replaced by $\psi^{\text{in}}_{(\beta)}$. From (25) we obtain, using (21) (*i.e.* using the regularization method to compute $If_j(x, Z(s))$), that

$$(26)\qquad m_B \ddot{Z}_\mu(s_0) = 2\lambda^2 \sum_j h_{\mu j}(\dot{Z}, \ddot{Z}) \left(\frac{1}{2\eta} \bar{f}_j(0) - \frac{1}{2} \frac{\partial \bar{f}_j(\tau)}{\partial \tau}\bigg|_{\tau=0}\right) + \lambda F^{\text{in}}_\mu .$$

From (13) one has that

$$(27)\qquad \bar{f}_j(0) = f_j(Z(s_0), Z(s_0)), \qquad \frac{\partial \bar{f}_j(\tau)}{\partial \tau}\bigg|_{\tau=0} = \frac{\partial}{\partial s'} f_j(Z(s_0), Z(s'))\bigg|_{s'=s_0} ,$$

and consequently from (26) we obtain

$$(28)\qquad m_B \ddot{Z}_\mu(s_0) = \frac{\lambda^2}{\eta} \sum_j h_{\mu j}(\dot{Z}(s_0), \ddot{Z}(s_0)) f_j(Z(s_0), Z(s_0)) - \\ - \lambda^2 \sum_j h_{\mu j}(\dot{Z}(s_0), \ddot{Z}(s_0)) \frac{\partial}{\partial s'} f_j(Z(s_0), Z(s'))\bigg|_{s'=s_0} + \lambda F^{\text{in}}_\mu ,$$

which is then the equation of motion as given by the regularization method. Let us see now the analytic-continuation method of Barut and Villarroel. One can calculate directly $If_j(x, Z(s))$ from its definition (8) by a standard calculation ([6]) which gives

$$(29)\qquad If_j(X, Z(s)) = \frac{1}{2(x - Z(s_R))^\alpha \dot{Z}_\alpha(s_R)} f_j(x, Z(s_R)) ,$$

where s_R is the retarded proper time corresponding to x^μ (called s_- in fig. 1), *i.e.* s_R is determined by $(X - Z(s_R))^2 = 0$, $X^0 > Z^0(s_R)$. Replacing (29) in (25), we obtain $(x \to Z(s_0))$ that

$$(30)\qquad m_B \ddot{Z}_\mu(s_0) = 2\lambda^2 \sum_j h_{\mu j}(\dot{Z}(s_0), \ddot{Z}(s_0)) \frac{1}{2(x - Z(s_R))^\alpha \dot{Z}_\alpha(s_R)} f_j(X, Z(s_R)) + \lambda F^{\text{in}}_\mu .$$

One remarks that the retarded time corresponding to $x = Z(s_0)$ is $s_R = s_0$, so one puts $s_R = s_0$ in (30). The analytic-continuation method consists in considering eq. (30) in the point $x = Z(s_0 + u)$, thus replacing in (30) s_0 by $s_0 + u$ everywhere except in s_R which is put equal to s_0, and taking at the end the limit $u \to 0$. One has then

$$(31)\qquad m_B \ddot{Z}_\mu(s_0+u) = \lambda^2 \sum_j h_{\mu j}(\dot{Z}(s_0+u), \ddot{Z}(s_0+u)) \cdot \\ \cdot \frac{1}{(Z(s_0+u) - Z(s_0))^\alpha \dot{Z}_\alpha(s_0)} f_j(Z(s_0+u), Z(s_0)) + \lambda F^{\text{in}}_\mu .$$

A simple calculation shows that

$$(32)\qquad \frac{1}{(Z(s_0+u) - Z(s_0))^\alpha \dot{Z}_\alpha(s_0)} = \frac{1}{u + (u^3/6)\dddot{Z}(s_0)^\alpha \dot{Z}_\alpha(s_0) + O(u^4)} = \\ = \frac{1}{u}(1 + O(u^2)) .$$

Replacing (32) in (31) and doing there a transition of $-u$ (see ref. ([2]), footnote 8), one obtains

$$(33)\qquad m_B \ddot{Z}_\mu(s_0) = \lambda^2 \sum_j h_{\mu j}(\dot{Z}(s_0), \ddot{Z}(s_0)) \frac{1}{u}(1 + O(u^2)) f_j(Z(s_0), Z(s_0 - u)) + \lambda F^{\text{in}}_\mu .$$

Expanding $f_j(Z(s_0), Z(s_0 - u))$ around $u = 0$, and taking the limit $u \to 0$, one finally has

$$(34)\qquad m_B \ddot{Z}_\mu(s_0) = \frac{\lambda^2}{u} \sum_j h_{\mu j}(\dot{Z}(s_0), \ddot{Z}(s_0)) f_j(Z(s_0), Z(s_0)) - \\ - \lambda^2 \sum_j h_{\mu j}(\dot{Z}(s_0), \ddot{Z}(s_0)) \frac{\partial}{\partial s'} f_j(Z(s_0), Z(s'))\Big|_{s'=s_0} + \lambda F^{\text{in}}_\mu ,$$

which is just eq. (28), thus showing the equivalence of our double-zero-separation method with the analytic-continuation technique of Barut and Villorroel.

3. – Calculation of $\psi_{(\beta)}(Z(s_0))$ and $\partial_\nu \psi_{(\beta)}(Z(s_0))$ by the average method.

The average method proposed by TEITELBOIM ([5]) consists in computing the field on a sphere of radius η in the rest system of the particle at proper time s_0, then making the average over the surface of the sphere and finally taking the limit $\eta \to 0$ in order to define the field at the point $Z(s_0)$ of the trajectory. In a covariant language this involves computing $\psi_{(\beta)}(x)$ and $\partial_\nu \psi_{(\beta)}(x)$ at the point $x^\mu = Z^\mu(s_0) + \eta u^\mu$ with $u^2 = u_\mu u^\mu = -1$, $u^\mu \dot{Z}_\mu(s_0) = 0$ (*i.e.* in the hyperplane orthogonal to $v_\mu(s_0) = \dot{Z}_\mu(s_0)$ at s_0, see fig. 1), then doing the average over the directions u^μ and finally taking the limit $\eta \to 0$. The average of a function $f(u^\mu)$ is calculated by the formula [5]

$$\langle f(u^\mu)\rangle = \frac{1}{4\pi\eta^2}\int\limits_{\tilde{\Sigma}(\eta)} \mathrm{d}^2\sigma\, f(u^\mu)\,, \tag{35}$$

where $\tilde{\Sigma}(\eta)$ is the 2-dimensional surface that in the rest system at proper time s_0 corresponds to the surface of the sphere of radius η centred at the origin. The average (35) of the product of an odd number of u^μ vanishes (*i.e.* $\langle u^\mu\rangle = \langle u^\mu u^\nu u^\sigma\rangle = 0$) and for the product $u^\mu u^\alpha$ a simple calculation gives

$$\langle u_\mu u_\alpha\rangle = \tfrac{1}{3}\big(\dot{Z}_\mu(s_0)\dot{Z}_\alpha(s_0) - g_{\mu\alpha}\big)\,. \tag{36}$$

This procedure (formulae (35) and (36)) we call symmetrical integration.

Let us compute now the field and its derivative, starting from formulae (8), (9) and (10). We put $x = Z(s_0) + \eta u$, $s = s_0 + \tau$, and define the function $\bar{f}_j(\eta, \tau)$ by

$$f_j(x, Z(s)) = f_j\big(Z(s_0) + \eta u, Z(s_0 + \tau)\big) = \bar{f}_j(\eta, \tau)\,. \tag{37}$$

We introduce now the function $\bar{f}'_j(\eta, \tau)$ by the decomposition

$$\bar{f}_j(\eta, \tau) = \bar{f}_j(0, \tau) + \bar{f}'_j(\eta, \tau)\,. \tag{38}$$

In sect. **2** we have called $\bar{f}_j(\tau)$ the function $\bar{f}_j(0, \tau)$, *i.e.* we have (see formula (13))

$$f_j\big(Z(s_0), Z(s_0 + \tau)\big) = \bar{f}_j(0, \tau) = \bar{f}_j(\tau)\,. \tag{39}$$

This function is finite and also its derivatives (see appendix). The values at $\tau = 0$ of the function and its first derivative, *i.e.* $\bar{f}_j(0, 0)$ and $\big(\partial\bar{f}_j(0, \tau)/\partial\tau\big)\big|_{\tau=0}$, are given explicitly in formulae (14), (15) and (16).

We consider now the function $\bar{f}'_j(\eta, \tau)$ defined by (38). From its definition we see that it vanishes at $\eta = 0$, *i.e.*

$$\bar{f}'_j(0, \tau) = 0\,. \tag{40}$$

For $j = (\beta)$ the function $\bar{f}'_{(\beta)}(\eta, \tau)$ vanishes identically, since $\bar{f}_{(\beta)}(\eta, \tau) = r_{(\beta)}(s_0 + \tau)$ is independent of η, and consequently from (38) it follows that $f'_{(\beta)}(\eta, \tau) \equiv 0$. For $\bar{f} = \nu(\beta)$ the computation starting from (12) gives the following form (see appendix):

$$\bar{f}'_{\nu(\beta)}(\eta,\tau) = \frac{1}{\tau^2} l_{\nu(\beta)}(\eta) + k_{\nu(\beta)}(\eta,\tau) \tag{41}$$

with

$$l_{\nu(\beta)}(\eta) = \eta u_\nu[1 + \eta u_\alpha \ddot{Z}^\alpha(s_0) + \eta^2(u_\alpha \ddot{Z}^\alpha(s_0))^2 + O(\eta^3)]\, r_{(\beta)}(s_0)\,, \tag{42}$$

and $k_{\nu(\beta)}(\eta,\tau)$ is regular in η and τ. One has from (40) and (42) that $k_{\nu(\beta)}(0,\tau)=0$ and more precisely this function is of the form (remember the development in η is in fact a development in ηu^μ see (37))

$$k_{\nu(\beta)}(\eta,\tau) = \eta u^\mu k^{(1)}_{\nu(\beta)\mu}(\tau) + \eta^2 u^\mu u^\varrho k^{(2)}_{\nu(\beta)\mu\varrho}(\tau) + O(\eta^3)\,, \tag{43}$$

where the $k^{(n)}_{\nu(\beta)\mu_1\ldots\mu_n}(\tau)$ are regular functions of τ, in particular $k^{(n)}_{\nu(\beta)\mu_1\ldots\mu_n}(0)$ are finite. We can now proceed to the calculation of $If(x, Z(s))$, where f will be f_j. From (8) and (37) we obtain, using $\theta(x) = \frac{1}{2}(1+\varepsilon(x))$, the expression

$$\begin{aligned} I\bar{f}(\eta,\tau) = {} & \tfrac{1}{2}\int d\tau\, \delta[(Z(s_0)+\eta u - Z(s_0+\tau))^2]\bar{f}(\eta,\tau) + \\ & + \tfrac{1}{2}\int d\tau\, \varepsilon(Z^0(s_0)+\eta u^0 - Z^0(s_0+\tau))\, \delta[(Z(s_0)+\eta u - Z(s_0+\tau))^2]\bar{f}(\eta,\tau)\,. \end{aligned} \tag{44}$$

Putting

$$G = (x - Z(s))^2 = (Z(s_0) + \eta u - Z(s_0+\tau))^2\,, \tag{45}$$

one has that the two roots τ_+ and τ_- of the equation $G=0$ are (from now on when we write $\dot{Z}, \ddot{Z}, \dddot{Z}$, one must understand the value of these functions at $s=s_0$)

$$\begin{aligned} \tau_\pm = \pm\eta\left(1 + \frac{1}{2}\eta u_\alpha \ddot{Z}^\alpha + \eta^2\left(\frac{3}{8}(u_\alpha\ddot{Z}^\alpha)^2 + \frac{1}{24}\ddot{Z}_\alpha\ddot{Z}^\alpha \pm \frac{1}{6}u_\alpha\dddot{Z}^\alpha\right)\right) + \\ + O(n^4) = \pm\eta + O(\eta^2)\,. \end{aligned} \tag{46}$$

One also has

$$X^0 - Z^0(s)\big|_{s=s_0+\tau_\pm} = Z^0(s_0) + \eta u^0 - Z^0(s_0+\tau)\big|_{\tau=\tau_\pm} = \mp\,\eta(\dot{Z}^0(s_0) \mp u^0) + O(\eta^2)\,. \tag{47}$$

But $(\dot{Z}^0(s_0) \mp u^0) > 0$ always, since the sign of the zero component of a light-cone vector is a Lorentz invariant and in the rest system $\dot{Z}^0(s_0) = 1$, $u^0 = 0$. This means that $\varepsilon(Z^0(s_0) + \eta u^0 - Z^0(s_0+\tau_\pm))$ in (44) gets effectively replaced by $\varepsilon(\mp\eta)$, since for η small the sign of (47) is that of $(\mp\eta)$. From (44) one obtains then

$$I\bar{f}(\eta,\tau) = \frac{1}{2}\left(\frac{\bar{f}(\eta,\tau_+)}{|G'|_+} + \frac{\bar{f}(\eta,\tau_-)}{|G'|_-}\right) + \frac{1}{2}\left(-\frac{\bar{f}(\eta,\tau_+)}{|G'|_+} + \frac{\bar{f}(\eta,\tau_-)}{|G'|_-}\right), \tag{48}$$

where

$$G'(\eta,\tau)=\frac{\mathrm{d}}{\mathrm{d}s}\,(x-Z(s))^2=-2\big(Z(s_0)+\eta u-Z(s_0+\tau)\big)_\alpha\dot Z^\alpha(s_0+\tau)\,, \tag{49}$$

and $|G'|_\pm=|G'(\eta,\tau_\pm)|$. Since I is a linear operator we have from (38) that

$$I\bar f_j(\eta,\tau)=I\bar f_j(0,\tau)+I\bar f'_j(\eta,\tau)\,, \tag{50}$$

and for $j=(\beta)$ one has $I\bar f_{(\beta)}(\eta,\tau)=I\bar f_{(\beta)}(0,\tau)$, because $\bar f'_{(\beta)}(\eta,\tau)\equiv 0$. Let us calculate now $I\bar f_j(0,\tau)$ explicitly starting from (48) where we replace $\bar f(\eta,\tau)$ by $\bar f_j(0,\tau)$. Using (see appendix)

$$\frac{1}{|G'|_\pm}=\frac{1}{2\eta}+\frac{1}{4}\,U_\alpha\ddot Z^\alpha+\eta\left[\frac{3}{16}\,(U_\alpha\ddot Z^\alpha)^2+\frac{1}{16}\,\ddot Z_\alpha\ddot Z^\alpha\pm\frac{1}{6}\,U_\alpha\dddot Z^\alpha\right]+O(\eta^2)\,, \tag{51}$$

one obtains from (48) that

$$\begin{aligned}2I\bar f_j(0,\tau)=&\left(\frac{1}{2\eta}+\frac{1}{4}\,U_\alpha\ddot Z^\alpha+O(\eta)\right)\left(\bar f_j(0,\eta)+\bar f_j(0,-\eta)+O(\eta^2)\right)+\\&+\left(\frac{1}{2\eta}+\frac{1}{4}\,U_\alpha\ddot Z^\alpha+O(\eta)\right)\left(-\bar f_j(0,\eta)+\bar f_j(0,-\eta)+O(\eta^2)\right).\end{aligned} \tag{52}$$

When $\eta\to 0$ this gives

$$I\bar f_j(0,\tau)=\frac{1}{2\eta}\,\bar f_j(0,0)-\frac{1}{2}\left.\frac{\partial\bar f_j(0,\tau)}{\partial\tau}\right|_{\tau=0}+\frac{1}{4}\,U_\alpha\ddot Z^\alpha\bar f_j(0,0)\,. \tag{53}$$

The last term vanishes on symmetrical integration and one finally has

$$I\bar f_j(0,\tau)=\frac{1}{2\eta}\,\bar f_j(0,0)-\frac{1}{2}\left.\frac{\partial\bar f_j(0,\tau)}{\partial\tau}\right|_{\tau=0}, \tag{54}$$

which is exactly the result of the regularization method of sect. **1** (see (21)). We calculate now $I\bar f'_j(\eta,\tau)$ for $j=\nu(\beta)$. From (41) one has

$$I\bar f'_{\nu(\beta)}(\eta,\tau)=I\,\frac{1}{\tau^2}\,l_{\nu(\beta)}+Ik_{\nu(\beta)}(\eta,\tau)\,. \tag{55}$$

We show first that $Ik_{\nu(\beta)}(\eta,\tau)=0$. Using a short-hand notation we can write (43) as

$$k(\eta,\tau)=\eta\big(Uk^{(1)}(\tau)+\eta u^2k^{(2)}(\tau)+O(\eta^2)\big) \tag{56}$$

with $k^{(i)}(\tau)$ regular functions of τ. Using (51) and (56), one has that

$$(57)\qquad \frac{k(\eta,\tau_\pm)}{|G'|_\pm}=\left(\frac{1}{2\eta}+\frac{1}{4}U_\alpha\ddot{Z}^\alpha+O(\eta)\right)\left(\eta u k^{(1)}(\tau_\pm)+O(\eta^2)\right)=\frac{1}{2}uk^{(1)}(\tau_\pm)+O(\eta)\,.$$

But $k^{(1)}(\tau)$ contains no u^μ and consequently (57) vanishes by symmetrical integration. This shows then using (48) with $\bar{f}=k_{\nu(\beta)}(\eta,\tau)$ that $Ik_{\nu(\beta)}(\eta,\tau)=0$ and consequently

$$(58)\qquad I\bar{f}'_{\nu(\beta)}(\eta,\tau)=I\frac{1}{\tau^2}l_{\nu(\beta)}(\eta)\,.$$

We use the same notation as in (56) to write (42) in the form

$$(59)\qquad l(\eta)=\eta u l^{(1)}+\eta^2u^2l^{(2)}+\eta^3u^3l^{(3)}+O(\eta^4)\,.$$

On the other hand one has

$$(60)\qquad \frac{1}{\tau_\pm^2}=\frac{1}{\eta^2}\left(1-\eta u^\alpha\ddot{Z}_\alpha-\eta^2\left(\frac{1}{12}\ddot{Z}^\alpha\ddot{Z}_\alpha\pm\frac{1}{3}U^\alpha\dddot{Z}_\alpha\right)\right)+O(\eta)\,,$$

and consequently

$$(61)\qquad \frac{1}{\tau_\pm^2}l_{\nu(\beta)}(\eta)=\frac{1}{\eta}Ul^{(1)}+U^2l^{(2)}-(U^\alpha\ddot{Z}_\alpha)\,Ul^{(1)}+$$
$$+\eta\left(U^3l^{(3)}-(U^\alpha\ddot{Z}_\alpha)\,U^2l^{(2)}-\left(\frac{1}{12}\ddot{Z}^\alpha\ddot{Z}_\alpha\pm\frac{1}{3}U^\alpha\dddot{Z}_\alpha\right)Ul^{(1)}\right)+O(\eta^2)\,.$$

From (61) and (51) we obtain, after dropping terms containing an odd number of U^μ which will vanish by symmetrical integration, that

$$(62)\qquad \frac{(1/\tau_\pm^2)\,l_{\nu(\beta)}(\eta)}{|G'|_\pm}=\frac{1}{2\eta}\left(U^2l^{(2)}-\frac{1}{2}(U^\alpha\ddot{Z}_\alpha)\,Ul^{(1)}\right)\mp$$
$$\mp\frac{1}{6}(U^\alpha\dddot{Z}_\alpha)\,Ul^{(1)}\pm\frac{1}{6}(U^\alpha Z_\alpha)\,Ul^{(1)}+O(\eta)\,.$$

One can see from (62) that the finite terms cancel, and after replacing $l^{(1)}_{\nu(\beta)\mu}$ and $l^{(2)}_{\nu(\beta)\mu\alpha}$ by their values which can be read from (42) one obtains in the limit $\eta\to 0$ that

$$(63)\qquad \frac{(1/\tau_\pm^2)\,l_{\nu(\beta)}(\eta)}{|G'|_\pm}=\frac{1}{4\eta}U^\mu U^\alpha g_{\mu\nu}\ddot{Z}_\alpha r_{(\beta)}(s_0)+O(\eta)\,.$$

We can now use (48) to compute $I(1/\tau^2)\,l_{\nu(\beta)}(\eta)$ remarking that there the second term on the right-hand side of (48) vanishes, since (63) does not depend on $\tau_\pm$.

One has then ($\eta \to 0$)

$$I \frac{1}{\tau_\pm^2} l_{\nu(\beta)}(\eta) = \frac{1}{4\eta} U^\mu U^\alpha g_{\mu\nu} \ddot{Z}_\alpha(s_0) r_{(\beta)}(s_0) = -\frac{1}{12\eta} \ddot{Z}_\nu(s_0) r_{(\beta)}(s_0) , \tag{64}$$

where the last equality comes from symmetrical integration by means of (36). From (55) and (64) we obtain (recalling that $Ik_{\nu(\beta)}(\eta, \tau) = 0$) that

$$I\bar{f}'_{\nu(\beta)}(\eta, \tau) = -\frac{1}{12\eta} \ddot{Z}_\nu(s_0) r_{(\beta)}(s_0) . \tag{65}$$

We can now give the final result for $\psi_{(\beta)}(x)$ and $\partial_\nu \psi_{(\beta)}$, $x \to Z(s_0)$. From (9) and using (50), (54) and the fact that $\bar{f}'_{(\beta)}(\eta, \tau) \equiv 0$, we obtain

$$\psi_{(\beta)}(Z(s_0)) = 2\lambda I\bar{f}_{(\beta)}(0, \tau) = \lambda \left(\frac{1}{\eta} \bar{f}_{(\beta)}(0, 0) - \frac{\partial \bar{f}_{(\beta)}(0, \tau)}{\partial \tau} \bigg|_{\tau=0} \right), \tag{66}$$

which coincides with the result of the regularization method, since $\bar{f}_j(0, \tau) = \bar{f}_j(\tau)$ (see (21) and (22)). For $\partial_\nu \psi_{(\beta)}(Z(s_0))$ we obtain from (10), (50), (54) and (65) that

$$\partial_\nu \psi_{(\beta)}(Z(s_0)) = \lambda \left(\frac{1}{\eta} \bar{f}_{\nu(\beta)}(0, 0) - \frac{\partial \bar{f}_{\nu(\beta)}(0, \tau)}{\partial \tau} \bigg|_{\tau=0} \right) - \frac{\lambda}{6\eta} \ddot{Z}_\nu(s_0) r_{(\beta)}(s_0) . \tag{67}$$

The first term in (67) is the result of the regularization method (see (23)), while the second term coming from (65) is a new divergent contribution one obtains in the average method. This new term just changes the numerical coefficient of the first term of (23), and we shall see that this only affects the unobservable mass renormalization. We can conclude then that the average method is equivalent to our double-zero–separation procedure of sect. **2**. On the other hand, the average method can be shown to be equivalent to the usual technique using the energy-momentum tensor [7].

4. – Lagrangian formalism.

We shall derive here the equations of motion from the principle of stationary action. In order to obtain eq. (1) one takes the interaction Lagrangian $\lambda j_{(\beta)} \psi^{(\beta)}$ and consequently the action is

$$\mathscr{A} = -m_B \int ds + \frac{\varepsilon}{8\pi} \int d^4x \, \partial_\mu \psi_{(\beta)} \partial^\mu \psi^{(\beta)} + \frac{\varepsilon}{4\pi} \int d^4x \, j_{(\beta)}(x) \psi^{(\beta)}(x) , \tag{68}$$

[7] C. A. LOPEZ and C. TEITELBOIM: *Lett. Nuovo Cimento*, **2**, 225 (1971).

where ε can have the value $+1$ or -1, depending on the different cases we shall consider. The Euler-Lagrange equations for $\psi_{(\beta)}$ give (1) and to obtain the equation of motion for the trajectory one has to use the modified Euler-Lagrange equations (ref. [6], page 66)

$$\frac{\partial L}{\partial Z^\mu}-\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial \dot{Z}^\mu}-\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(L-\dot{Z}^\alpha\frac{\partial L}{\partial \dot{Z}^\alpha}\right)\dot{Z}_\mu\right]=0 \tag{69}$$

to take into account the supplementary condition $\dot{Z}^2=1$, with L given by

$$L=-m_B+\varepsilon\lambda\psi^{(\beta)}(Z(s))\,\Pi\dot{Z}_{(\beta)}(s)\,. \tag{70}$$

One obtains then from (69), using $(\mathrm{d}/\mathrm{d}s)\Pi\dot{Z}_{(\beta)}=\ddot{Z}^\alpha(\partial/\partial\dot{Z}^\alpha)\Pi\dot{Z}_{(\beta)}$, since $\Pi\dot{Z}_{(\beta)}(s)$ depends on s only through the functions $\dot{Z}(s)$, that

$$\begin{aligned}(71)\qquad m_B\ddot{Z}_\mu=-\varepsilon\lambda\partial_\nu\psi^{(\beta)}\left[\Pi\dot{Z}_{(\beta)}g_\mu{}^\nu-\dot{Z}^\nu\frac{\partial\Pi\dot{Z}_{(\beta)}}{\partial\dot{Z}^\mu}-\dot{Z}^\nu\dot{Z}_\mu\left(\Pi\dot{Z}_{(\beta)}-\dot{Z}^\alpha\frac{\partial\Pi\dot{Z}_{(\beta)}}{\partial\dot{Z}^\alpha}\right)\right]+\\+\varepsilon\lambda\psi^{(\beta)}\left[\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial\Pi\dot{Z}_{(\beta)}}{\partial\dot{Z}^\mu}-\dot{Z}_\mu\dot{Z}^\alpha\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial\Pi\dot{Z}_{(\beta)}}{\partial\dot{Z}^\alpha}+\ddot{Z}_\mu\left(\Pi\dot{Z}_{(\beta)}-\dot{Z}^\alpha\frac{\partial\Pi\dot{Z}_{(\beta)}}{\partial\dot{Z}_\alpha}\right)\right].\end{aligned}$$

One easily checks that (71) is consistent with $\dot{Z}^\mu\ddot{Z}_\mu=0$, indeed the right-hand side vanishes identically when contracted with $\dot{Z}^\mu$. Introducing a variable T which has the value 0 for the double-zero–separation method of sect. **2** and the value 1 for the average method of sect. **3**, we obtain from (23) and (67) that

$$\begin{aligned}(72)\qquad \partial_\nu\psi^{(\beta)}(Z(s))=\frac{\lambda}{\eta}\left[\left(\frac{1}{2}-\frac{T}{6}\right)\ddot{Z}_\nu\Pi\dot{Z}^{(\beta)}+\dot{Z}_\nu\frac{\mathrm{d}}{\mathrm{d}s}\Pi\dot{Z}^{(\beta)}\right]-\\-\lambda\left[\frac{1}{3}\dddot{Z}_\nu\Pi\dot{Z}^{(\beta)}+\dot{Z}_\nu\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Pi\dot{Z}^{(\beta)}+\ddot{Z}_\nu\frac{\mathrm{d}}{\mathrm{d}s}\Pi\dot{Z}^{(\beta)}+\frac{1}{3}\ddot{Z}_\alpha\ddot{Z}^\alpha\dot{Z}_\nu\Pi\dot{Z}^{(\beta)}\right],\end{aligned}$$

while $\psi^{(\beta)}(Z(s))$ is given by (22). Replacing these expressions for $\psi^{(\beta)}$ and $\partial_\nu\psi^{(\beta)}$ in (71), we obtain the differential equation for the trajectory $Z(s)$. We shall now see that this equation is always the Lorentz-Dirac equation with a different numerical constant in front of the Abraham force $\Gamma_\mu=\dddot{Z}_\mu+a^2\dot{Z}_\mu$ ($a^2=\ddot{Z}_\alpha\ddot{Z}^\alpha$) which depends on the explicit expression of $\Pi\dot{Z}_{(\beta)}$. In order to prove this we remark that in (71) when $\partial_\nu\psi^{(\beta)}$ and $\psi^{(\beta)}$ are replaced the only terms with third-order derivatives (*i.e.* containing $\dddot{Z}$) will come from the first two terms in the second bracket of (72). These terms are multiplied by terms containing only first derivatives and as $\dddot{Z}^\alpha\dot{Z}_\alpha=-a^2$ the only third-order terms that remain will be those carrying the free index μ, *i.e.* containing $\dddot{Z}_\mu$, and these will be only on the contractions

$$-\varepsilon\lambda^2\left(-\frac{1}{3}\Pi\dot{Z}^{(\beta)}\Pi\dot{Z}_{(\beta)}\dddot{Z}_\mu+\frac{\partial\Pi\dot{Z}_{(\beta)}}{\partial\dot{Z}_\mu}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Pi\dot{Z}^{(\beta)}\right). \tag{73}$$

From (73) one can see that one has linearity in the third derivative, *i.e.* (73) is of the form $\lambda^2\gamma\dddot{Z}_\mu + f_\mu(\dot{Z}, \ddot{Z})$, where γ is just a numerical dimensionless constant and f some function containing only $\dot{Z}$ and $\ddot{Z}$. Moreover, for the purpose of calculating γ we can replace the second term in (73) by

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \Pi\dot{Z}_{(\beta)}}{\partial \ddot{Z}_\mu}\frac{\mathrm{d}}{\mathrm{d}s}\Pi\dot{Z}^{(\beta)}\right),$$

since the difference contains only $\dot{Z}$ and $\ddot{Z}$. One has then

$$\lambda^2\gamma\dddot{Z}_\mu = \varepsilon\lambda^2\left[\frac{1}{3}\Pi\dot{Z}^{(\beta)}\Pi\dot{Z}_{(\beta)}\dddot{Z}_\mu - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}}{\mathrm{d}s}\Pi\dot{Z}^{(\beta)}\frac{\partial \Pi\dot{Z}_{(\beta)}}{\partial \ddot{Z}^\mu}\right)\right], \tag{74}$$

and this equation will explicitly determine γ when the form of $\Pi\dot{Z}_{(\beta)}$ is known. According to the remarks we have done we see that (71) is of the form

$$m_B\ddot{Z}_\mu = \lambda^2\left(\gamma\dddot{Z}_\mu + X_\mu(\dot{Z}, \ddot{Z}) + \frac{1}{\eta}\,Y_\mu(\dot{Z}, \ddot{Z})\right). \tag{75}$$

Since $a^2 = \ddot{Z}_\alpha\ddot{Z}^\alpha$ is the only scalar one can construct with $\dot{Z}$ and $\ddot{Z}$ we have that X_μ and Y_μ are of the form

$$X_\mu(\dot{Z}, \ddot{Z}) = \dot{Z}_\mu X_1(a^2) + \ddot{Z}_\mu X_2(a^2), \tag{76}$$

$$Y_\mu(\dot{Z}, \ddot{Z}) = \dot{Z}_\mu Y_1(a^2) + \ddot{Z}_\mu Y_2(a^2), \tag{77}$$

where $X_i(a^2)$, $Y_i(a^2)$ are polynomial in a^2. If we contract (75) with $\dot{Z}^\mu$ we obtain zero and consequently

$$0 = \lambda^2\left(-\gamma a^2 + X_1(a^2) + \frac{1}{\eta}\,Y_1(a^2)\right). \tag{78}$$

We use now dimensional arguments ($c = 1$ here) and, recalling that γ is a dimensionless number, we see that the dimensions of the different quantities in (78) are $[\gamma a^2] = [X_1(a^2)] = L^{-2}$, $[\eta] = L$, $[Y_1(a^2)] = L^{-1}$. Since $X_1(a^2)$ and $Y_1(a^2)$ are polynomials in a^2 we conclude that $X_1(a^2) = \gamma a^2$ and $Y_1(a^2) = 0$. Replacing now (76) and (77) in (75) and using again a dimensional analysis, we conclude that $X_2(a^2) = 0$ and $Y_2(a^2) = \delta =$ dimensionless number. We have then

$$m_B\ddot{Z}_\mu = \lambda^2\gamma(\dddot{Z}_\mu + a^2\dot{Z}_\mu) + \lambda^2\frac{\delta}{\eta}\ddot{Z}_\mu. \tag{79}$$

Putting $\delta m = \lambda^2\delta/\eta$ and defining the renormalized mass $m = m_B - \delta m$, we can write (79) as

$$m\ddot{Z}_\mu = \lambda^2\gamma(\dddot{Z}_\mu + a^2\dot{Z}_\mu), \tag{80}$$

where γ is determined by (74). Equation (80) remains finite when $\eta \to 0$ and is just the Lorentz-Dirac equation as announced. We recall that (80) has to be supplemented with the term $\lambda F^{\text{in}}_\mu$ that corresponds to the boundary condition at $X^0 \to -\infty$, since we are using retarded Green's functions (see the lines following eq. (5) and (3), (24) and (34)). The number δ can be computed explicitly, for this it is enough to collect from (71) with $\partial_\nu \psi^{(\beta)}$ and $\psi^{(\beta)}$ replaced all the divergent terms (*i.e.* with the factor $1/\eta$) multiplying $\ddot{Z}_\mu$. We have ($\doteq$ means here equality of the factor multiplying $\ddot{Z}_\mu$)

$$\delta \ddot{Z}_\mu \doteq \varepsilon \left[\left(\frac{1}{2} + \frac{T}{6}\right)\ddot{Z}_\mu \Pi\dot{Z}_{(\beta)} \Pi\dot{Z}^{(\beta)} + \frac{\mathrm{d}}{\mathrm{d}s}\left(\Pi\dot{Z}_{(\beta)} \frac{\partial \Pi\dot{Z}^{(\beta)}}{\partial \dot{Z}^\mu}\right) - \right.$$
$$\left. - \ddot{Z}_\mu\left(\dot{Z}^\alpha \Pi\dot{Z}^{(\beta)} \frac{\partial \Pi\dot{Z}_{(\beta)}}{\partial \dot{Z}^\alpha}\right)\right]. \tag{81}$$

Note that $\Pi\dot{Z}_{(\beta)}(\partial \Pi\dot{Z}^{(\beta)}/\partial\dot{Z}^\mu)$ is necessarily of the form $\alpha\dot{Z}_\mu$, with α a number, and consequently the two last terms in (81) cancel and we have

$$\delta = \varepsilon\left(\frac{1}{2} + \frac{T}{6}\right)\Pi\dot{Z}^{(\beta)}\Pi\dot{Z}_{(\beta)}\,. \tag{82}$$

With our definition of δm a $\delta < 0$ means an interaction that increases the mass of the bare particle. The analysis we have done shows in fact that the mass renormalization δm is always divergent (it comes from the term in η^{-1}).

TABLE I.

$\Pi\dot{Z}_{(\beta)}$	Field	ε	γ	δ
1	φ scalar	1	$\frac{1}{3}$	$+\frac{1}{2}$
				$+\frac{2}{3}$
$\dot{Z}_\alpha$	A_α electromagnetic	-1	$\frac{2}{3}$	$-\frac{1}{2}$
				$-\frac{2}{3}$
$\dot{Z}_\alpha\dot{Z}_\beta$	$U_{\alpha\beta}$ tensor	1	$-\frac{5}{3}$	$+\frac{1}{2}$
				$+\frac{2}{3}$
$\frac{1}{2}g_{\alpha\beta}$	$U_{\alpha\beta}$ tensor	-1	$-\frac{1}{3}$	$-\frac{1}{2}$
				$-\frac{2}{3}$

Let us apply now our results to some well-known interactions. The values of γ and δ are collected in table I which is to be compared with the corresponding table of [3]. In the last column of table I the first number corresponds to the value of δ in the double-zero-separation method ($T = 0$) and the second number to the value in the average method ($T = 1$).

We remark that the equations of the scalar field we treat here can be considered as a weak-field approximation to the covariant scalar theory of gravitation [8]. In that approximation the quantity $A(x)^3$ at page 222 of [8] can be replaced by one and the coupled equations for the field and the trajectory reduce to ours. The case of linearized gravitation is postponed to appendix B, since there the coupled equations for the field and the particle are not deduced from a variational principle. Finally we consider in appendix C the case of symmetric electrodynamics with magnetic monopoles, since there the occurrence of two trajectories requires some slight modifications.

5. – Conclusions.

We have shown in this paper that the regularization method we have proposed [1] (double-zero separation) with its intuitive interpretation is equivalent to the « analytic continuation » regularization [2-4] for which no justification was originally given.

A physically reasonable way to define the field on the world-line consists in doing an average over all possible spacelike directions [5], we have proved that the double-zero–separation regularization is also equivalent to this procedure. Finally, we have shown that the structure of the equation of motion for the trajectory is largely independent of the nature of the field and of the interaction, in fact one always obtains a Lorentz-Dirac–type equation and only the numerical coefficient of the Abraham force changes in the different cases. Moreover, we prove that only a mass renormalization is needed to absorb all the divergences.

* * *

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APPENDIX A

We show briefly here how one obtains the results concerning $\bar{f}_{\nu(\beta)}(\eta, \tau)$ used in the text. A dot means d/ds and we have by successive derivation of $\dot{Z}^2 = 1$ that $\dot{Z}_\mu \ddot{Z}^\mu = 0$ and $\ddot{Z}_\mu \ddot{Z}^\mu + \dot{Z}_\mu \dddot{Z}^\mu = 0$. We put $x^\mu = Z^\mu(s_0) + \eta U^\mu$, $U_\mu U^\mu = -1$, $U_\mu \dot{Z}^\mu(s_0) = 0$, $\tau = s - s_0$. We also write $a^2 = \ddot{Z}_\alpha \ddot{Z}^\alpha$ and remind

(8) J. L. PIETENPOL and D. SPEISER: *Ann. Soc. Sci. Brux.*, **86**, 220 (1972).

that all functions in the expansions are taken at $s = s_0$. One has

$$\bar{f}_{\nu(\beta)}(\eta, \tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\bar{F}_{\nu(\beta)}(\eta, \tau) = \frac{\mathrm{d}}{\mathrm{d}s}\frac{(x - Z(s))_\nu r_{(\beta)}(s)}{(x - Z(s))^\alpha \dot{Z}_\alpha(s)} . \tag{A.1}$$

The expansions of the numerator and denominator of (A.1) are

$$(x - Z(s))_\nu r_{(\beta)}(s) = \left(\eta U_\nu - \tau \dot{Z}_\nu - \frac{\tau^2}{2}\ddot{Z}_\nu - \frac{\tau^3}{6}\dddot{Z}_\nu + O(\tau^4)\right)\cdot \left(r_{(\beta)} + \tau \frac{\mathrm{d}}{\mathrm{d}s_0} r_{(\beta)}(s_0) + \frac{\tau^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}s_0^2} r_{(\beta)}(s_0) + O(\tau^3)\right), \tag{A.2}$$

$$\frac{1}{(x - Z(s))^\alpha \dot{Z}_\alpha(s)} = -\frac{1}{\tau}\left[1 + \eta U_\alpha \ddot{Z}^\alpha + \frac{\eta\tau}{2} U_\alpha \dddot{Z}^\alpha + \frac{\tau^2}{6} a^2 + \eta^2 (U_\alpha \ddot{Z}^\alpha)^2 + O(\eta\tau^2) + O(\eta^3) + O(\tau^3)\right]. \tag{A.3}$$

From (A.1), (A.2) and (A.3) one checks immediately that $\bar{f}_{\nu(\beta)}(0, \tau)$ is finite and that $\bar{f}'_{\nu(\beta)}(\eta, \tau)$ is not of the form (41) of sect. **3**. The computation of the term in τ^{-2} gives (42) and to calculate $\bar{f}_{\nu(\beta)}(0, \tau) = \bar{f}_{\nu(\beta)}(\tau)$ and its derivative one just puts $\eta = 0$ in (A.1) and makes a Taylor expansion in τ, the results are (15) and (16) of sect. **2**.

Appendix B

The coupled equations of motion of linearized gravitation come from an approximation scheme of Einstein field equations developed in ([9]). One has (ref. ([9]), formula (47))

$$m\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(\dot{Z}_\mu - \left(\frac{1}{2}\psi_{\alpha\beta}\dot{Z}^\alpha\dot{Z}^\beta - C\right)\dot{Z}_\mu\right) + \psi_{\mu\beta}\dot{Z}^\beta\right] = \frac{1}{2} m\,\partial_\mu \psi_{\alpha\beta}\dot{Z}^\alpha\dot{Z}^\beta , \tag{B.1}$$

while in the equation for the field the quantity $r_{(\beta)} = \Pi\dot{Z}_{(\beta)}$ is given by (ref. ([9]), formula 62))

$$r_{\alpha\beta} = 2(g_{\alpha\beta} - \dot{Z}_\alpha\dot{Z}_\beta) . \tag{B.2}$$

Equation (B.1) for the trajectory does not come from our Lagrangian of sect. **4**, and consequently we cannot apply the results of that section. But (B.1) is linear in $\psi_{\alpha\beta}$ and $\partial_\mu\psi_{\alpha\beta}$ and we can then use the analysis of sect. **2** and **3**. In fact, when one replaces $\psi_{\alpha\beta}$ and $\partial_\mu\psi_{\alpha\beta}$ in (B.1), one is in the same situation

([9]) P. Havas: *Phys. Rev.*, **128**, 1104 (1966).

as in sect. **4**, since (B.1) vanishes identically when contracted with $\dot{Z}^\mu$, and we know repeating the same argument that this leads to a Lorentz-Dirac–type equation (there is no δm in this case, since the constant C can be adjusted to cancel the divergent term). In order to calculate the number γ one just has to look for the terms containing $\dddot{Z}$. In $\partial_\nu \psi_{\alpha\beta}$ these terms are

$$(\mathrm{B.3}) \qquad (\partial_\nu \psi_{\alpha\beta})^{(3)} = -2mG[\tfrac{1}{3}\dddot{Z}_\nu(g_{\alpha\beta} - 2\dot{Z}_\alpha \dot{Z}_\beta) - 2\dot{Z}_\nu(\dddot{Z}_\alpha \dot{Z}_\beta + \dot{Z}_\alpha \dddot{Z}_\beta)] .$$

When (B.3) is replaced in (B.1) the terms with $\dddot{Z}$ will be the last term on the left-hand side of (B.1) and the term on the right-hand side, and one obtains for γ the equation

$$(\mathrm{B.4}) \qquad Gm^2\gamma + 4Gm^2 = -Gm^2(\tfrac{1}{3}(1-2)) ,$$

which gives $\gamma = -\frac{11}{3}$. From this example it is clear that one will obtain a Lorentz-Dirac–type equation for more general situations than the one considered in sect. **4**, namely one needs that $\dot{Z}^\mu F_\mu \equiv 0$ (see eq. (2)), which is a consistency condition, and that eq. (75) hold with γ a dimensionless number. But (75) will always be true if for instance $a_\mu{}^{\nu(\beta)}(\dot{Z}, \ddot{Z})$ is a function only of $\dot{Z}$ as is the case here in (B.1). We remark here that the physical interpretation of the Lorentz-Dirac equation for gravitation puts problems. Because of the quadrupolar nature of the gravitational waves (due to the conservation of energy-momentum) one can expect that the radiation reaction must be different from the electromagnetic dipolar one (for a detailed discussion see [10], page 993).

Appendix C

We study here the equations of symmetric electrodynamics, *i.e.* of the interaction of an electric charge and a monopole (magnetic charge) with the electromagnetic field. Let $\varepsilon^{\mu\nu\varrho\sigma}$ be the completely antisymmetric tensor with $\varepsilon^{0123} = 1$, then we define the dual $\bar{B}_{\mu\nu}$ of an antisymmetric tensor $B_{\mu\nu}$ by $\bar{B}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\varrho\sigma} B^{\varrho\sigma}$. The coupled equations of motion for the system are [11]

$$(\mathrm{C.1}) \qquad \begin{cases} \partial_\mu F^{\mu\nu} = 4\pi j^\nu = 4\pi e \displaystyle\int \mathrm{d}s\, \delta^{(4)}(x - Z(s))\, \dot{Z}_\nu , \\ \partial_\mu \bar{F}^{\mu\nu} = 4\pi \bar{j}^\nu = 4\pi \bar{e} \displaystyle\int \mathrm{d}s\, \delta^{(4)}(x - Y(s))\, \dot{Y}^\nu , \end{cases}$$

$$(\mathrm{C.2}) \qquad \begin{cases} m_B \ddot{Z}^\mu = e F_{\mu\nu}(Z(s))\, \dot{Z}^\nu , \\ \bar{m}_B \ddot{Y}^\mu = \bar{e} \bar{F}_{\mu\nu}(Y(s))\, \dot{Y}^\nu . \end{cases}$$

[10] Ch. W. Misner, K. S. Thorne and J. A. Wheeler: *Gravitation* (San Francisco, Cal., 1973).

[11] R. A. Brandt and J. R. Primack: *Phys. Rev. D*, **15**, 1798 (1977).

These equations describe the interaction of the electric charge e with trajectory $Z(s)$, the magnetic charge $\bar{e}$ with trajectory $Y(s)$ and the electromagnetic field. We introduce now the Cabibbo and Ferrari potentials ([12]) A_μ and $\bar{A}_\mu$ by

$$\text{(C.3)} \qquad \begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \varepsilon_{\mu\nu\varrho\sigma}\partial^\varrho \bar{A}^\sigma \,, \\ \bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu + \varepsilon_{\mu\nu\varrho\sigma}\partial^\varrho A^\sigma \,, \end{cases}$$

and impose the Lorentz condition $\partial_\mu A^\mu = \partial_\mu \bar{A}^\mu = 0$ so that eqs. (C.1) are equivalent to

$$\text{(C.4)} \qquad \begin{cases} \Box A_\mu = 4\pi e \int \mathrm{d}s\, \delta^{(4)}(x - Z(s))\, \dot{Z}_\mu \,, \\ \Box \bar{A}_\mu = 4\pi \bar{e} \int \mathrm{d}s\, \delta^{(4)}(x - Y(s))\, \dot{Y}_\mu \,. \end{cases}$$

We note that the system of coupled equations satisfies duality invariance: $F_{\mu\nu} \to \bar{F}_{\mu\nu}$, $\bar{F}_{\mu\nu} \to -F_{\mu\nu}$, $e \to \bar{e}$, $\bar{e} \to -e$, $m_B \to \bar{m}_B$, $\bar{m}_B \to m_B$. We split now the field into two parts as one always does when dealing with more than one particle. In order to do this we can write (C.3) in the form

$$\text{(C.5)} \qquad F_{\mu\nu} = F^{(e)}_{\mu\nu} + F^{(\bar{e})}_{\mu\nu}$$

with $F^{(e)}_{\mu\nu}$ and $F^{(\bar{e})}_{\mu\nu}$ defined by

$$\text{(C.6)} \qquad F^{(e)}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \qquad \bar{F}^{(\bar{e})}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu \,.$$

We proceed now as usual computing the retarded solutions A_μ and $\bar{A}_\mu$ from (C.4) and replacing in (C.6) and then in (C.2) (of course the boundary condition $F^{\text{in}}_{\mu\nu}$ has to be added, but we omit it). One obtains then

$$\text{(C.7)} \qquad m_B \ddot{Z}_\mu = eF^{(e)}_{\mu\nu}(Z(s))\,\dot{Z}^\nu + eF^{(\bar{e})}_{\mu\nu}(Z(s))\,\dot{Z}^\nu \,,$$

$$\text{(C.7}') \qquad \bar{m}_B \ddot{Y}_\mu = \bar{e}\bar{F}^{(\bar{e})}_{\mu\nu}(Y(s))\,\dot{Y}^\nu + \bar{e}\bar{F}^{(e)}_{\mu\nu}(Y(s))\,\dot{Y}^\nu \,.$$

We remark now that the second term on the right-hand side of (C.7) and (C.7′) is regular and defined on the trajectory (it represents the field created by the other particle), while the first term is the usual one with the divergences that leads to the Abraham force. This is obvious for (C.7) and for (C.7′) it follows from duality invariance. These terms can be handled as we have done in the text and they give a Lorentz-Dirac–type expression with $\gamma = \frac{2}{3}$ and $\delta = -(\frac{1}{2} + T/6)$, so that, putting

$$m = m_B + \left(\frac{1}{2} + \frac{T}{6}\right)\frac{e^2}{\eta}\,, \qquad \bar{m} = \bar{m}_B + \left(\frac{1}{2} + \frac{T}{6}\right)\frac{\bar{e}^2}{\eta}\,,$$

([12]) N. Cabibbo and E. Ferrari: *Nuovo Cimento*, **23**, 1147 (1962).

one obtains from (C.7) and (C.7′) (incorporating the boundary condition $F^{\text{in}}_{\mu\nu}$)

$$\text{(C.8)}\qquad m\ddot{Z}_\mu = \tfrac{2}{3}e^2(\dddot{Z}_\mu + a^2\dot{Z}_\mu) + eF^{(\bar{e})}_{\mu\nu}(Z(s))\dot{Z}^\nu + eF^{\text{in}}_{\mu\nu}(Z(s))\dot{Z}^\nu\,,$$

$$\text{(C.8}'\text{)}\qquad \bar{m}\ddot{Y}_\mu = \tfrac{2}{3}\bar{e}^2(\dddot{Y}_\mu + a^2\dot{Y}_\mu) + \bar{e}\bar{F}^{(e)}_{\mu\nu}(Y(s))\dot{Y}^\nu + \bar{e}\bar{F}^{\text{in}}_{\mu\nu}(Y(s))\dot{Y}^\nu\,,$$

where $F^{(\bar{e})}_{\mu\nu}$ is the retarded field produced by the monopole and $F^{(e)}_{\mu\nu}$ the retarded field produced by the electric charge. These equations were previously obtained in ([13]). It has been pointed out that when they are considered for a heavy fixed monopole in the nonrelativistic limit (and neglecting the Abraham force in (C.8)) then angular-momentum conservation is violated for some set of measure of initial conditions. As pointed out by TROOST ([14]), the reason is that the equation of motion is not defined on the corresponding trajectory and he solved the situation giving the charge e a supplementary rotation degree of freedom, but this means that the charge has some structure and is not pointlike anymore. One can also argue that the conflicting trajectories are to be defined as limit of trajectories corresponding to initial conditions infinitesimally near to the singular ones, this also eliminates the paradox.

([13]) F. ROHRLICH: *Phys. Rev.*, **150**, 1104 (1966).

([14]) W. TROOST: *Solution of a paradox concerning the motion of an electric and a magnetic point charge*, preprint K.U.L.-TF-78/006 (Leuven).

● RIASSUNTO (*)

Si mostra che l'interazione di un campo classico di carattere generale tensoriale con una particella descritta dalla sua traiettoria porta in generale ad un'equazione del tipo di Lorentz-Dirac, indipendentemente dai dettagli dell'interazione. Si analizzano attentamente le divergenze della teoria per mezzo di un metodo che è stato proposto precedentemente da noi, e che si mostra essere equivalente ad altri schemi di regolarizzazione che sono stati usati precedentemente. Si applicano i risultati a molte ben note interazioni, comprendendo la gravitazione linearizzata e l'elettrodinamica simmetrica con monopoli.

(*) *Traduzione a cura della Redazione.*

О взаимодействии классических полей и частиц.

Резюме (*). — Показывается, что взаимодействие классического поля, имеющего тензорный характер, с частицей, описываемой траекторией, приводит в общем виде к уравнению типа Лоренца-Дирака вне зависимости от характера взаимодействия. Анализируются расходимости этой теории, используя метод, предложенный нами ранее, который является эквивалентным другим схемам регуляризации. Мы применяем наши результаты к некоторым хорошо известным взаимодействиям, включая линеаризованную гравитацию и симметричную электродинамику с монополями.

(*) *Переведено редакцией.*