AIP | Journal of Mathematical Physics

Approximate representations of SU(2) ordered exponentials in the adiabatic and stochastic limits

M. E. Brachet and H. M. Fried

Citation: J. Math. Phys. 28, 15 (1987); doi: 10.1063/1.527799
View online: http://dx.doi.org/10.1063/1.527799
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v28/i1
Published by the American Institute of Physics.

Approximate representations of SU(2) ordered exponentials in the adiabatic and stochastic limits

M. E. Brachet
Ecole Normale Supérieure, Paris, France

H. M. Fried[a)]
Physique Théorique, Université de Nice, 06034 Nice Cedex, France

(Received 24 June 1986, accepted for publication 10 September 1986)

Approximate representations for the SU(2) ordered exponential $U(t\,|E) = (\exp[i\int_0^t dt'\,\boldsymbol{\sigma}\cdot\mathbf{E}(t')])_+$, written as a functional of its input field $E(t)$, are derived in the adiabatic ($\rho \ll 1$) and stochastic ($\rho \gg 1$) limits, where $\rho \equiv |d\hat{E}/dt\,|/E$, $\hat{E} = \mathbf{E}/E$, $E = +(\mathbf{E}^2)^{1/2}$. An algorithm is set up for the adiabatic case, and fixed-point equations are obtained for situations of possible convergence. In the stochastic regime, "averaged" functions describing $U(t\,|E)$ are derived which reproduce its slowly varying dependence of large magnitude while missing, or approximating, rapid oscillations of small magnitude. Several functional integrals, analytic and machine are carried out over these approximate forms, and their results compared with the same functional integrals over the exact $U(t\,|E)$.

I. INTRODUCTION

Ordered exponentials are found in every branch of mathematical physics that deals with the causal time development of systems of more than one degree of freedom. Analytic treatments have typically been restricted to perturbative expansions, although computer calculations are now quite capable of dealing with any specific strong-coupling (SC) situation. However, when the variables in question are operators—numerical functions appearing in ordered exponentials and subsequently subjected to fluctuations as specified by an appropriate functional integral—the situation is much less clear. What would be most useful for such situations is a semianalytic approximation to the ordered exponential, which could then be inserted under the desired functional integral and its evaluation performed by some relevant approximation such as stationary phase. Functional integration aside, there are many instances when one would like to know the qualitative form of an ordered exponential as a functional of its input, without having to resort to a detailed numerical integration for each choice of input.

The purpose of this paper is to discuss and derive results, some of which have been previously quoted elsewhere,[1] for two classes of SC approximation to the ordered exponential solution of the differential equation

$$\frac{\partial U}{\partial t} = i\boldsymbol{\sigma}\cdot\mathbf{E}(t)U(t), \quad U(0) = 1, \tag{1.1}$$

where the σ_l denote 2×2 Pauli matrices, and the $E_l(t)$ are real, input functions. The unitary solution to (1.1) is

$$U(t) = \left(\exp\left[\int_0^t dt'\,\boldsymbol{\sigma}\cdot\mathbf{E}(t')\right]\right)_+ \equiv \sum_{n=0}^{\infty} \frac{i^n}{n!}\int_0^t dt_1\cdots\int_0^t dt_n(\boldsymbol{\sigma}\cdot\mathbf{E}(t_1)\cdots\boldsymbol{\sigma}\cdot\mathbf{E}(t_n))_+, \tag{1.2}$$

where the symbol $(\ )_+$ denotes an ordering of the t_i-dependent factors, with those containing later times standing to the left.

Perhaps the most interesting applications are associated with the generalization to SU(N), obtained by replacing the σ_l of (1.1) by the $N\times N$ Hermitian matrices λ_l which form the defining representation of SU(N). In principle, the analysis of this paper could be extended from SU(2) to SU(N); however, the specific details appear quite complicated, and have not yet been carried through. Some work on the SU(2) SC adiabatic limit has already appeared in rather special contexts,[2,4] which is here generalized in a nontrivial way; to the best of the author's knowledge, the material presented for the SC stochastic limit is new. Generalizations of the adiabatic limit to SU(N) are not difficult, and have been used in quite different contexts, for Navier–Stokes fluid flow,[5] $N = 3$, and in one approach to QCD,[6] for arbitrary N.

The SC situation may be defined by the requirement $\int_0^t dt'\,E(t') > 1$, $E = +\sqrt{\mathbf{E}^2}$, in contrast to the weak-coupling, or perturbative regime for which one assumes the converse, $\int_0^t dt'E(t') < 1$; in the latter case it is simple to derive a valid representation for $\ln U$ in terms of an expansion in multiple integrals over ascending powers of $\mathbf{E}(t')$. For the SC case, two distinct limiting regions can be defined, one for which $|d\hat{E}/dt\,|$ is "small" (the adiabatic, or quasistatic limit), and the opposite ("stochastic") situation for which it is "large." Clearly, if $\hat{E}(t) \equiv \mathbf{E}(t)/E(t)$ did not depend on time, and were fixed in one direction, a choice of coordinate axes could be made so that only one of the σ_l need appear, and the ordered exponential would become an ordinary exponential involving that σ_l. When $\hat{E}(t)$ varies with time, however, the problem becomes nontrivial, and naturally divides into these two quite different limits. By "large" or "small" one must mean the magnitude of $|d\hat{E}/dt\,|$ with respect to the only other relevant quantity of like dimension, $E(t)$; and hence if one defines $\rho(t) \equiv |d\hat{E}/dt\,|/E$, the SC adiabatic and stochastic limits are defined by $\rho \ll 1$ and $\rho \gg 1$,

[a)] Permanent Address: Physics Department, Brown University, Providence, Rhode Island 02912.

 0022-2488/87/010015-13$02.50

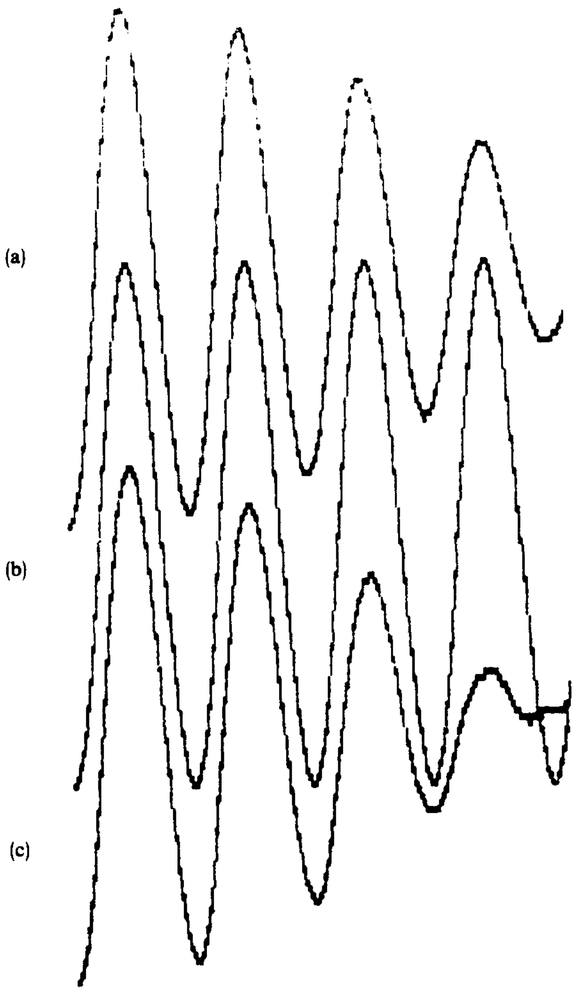

FIG. 1. Curves of (a) F_0, (b) $\cos Q_0$, and (c) $\cos Q_0 \cdot \cos q_1$; for the situation $\rho = 0.1$. (N.B. In all 18 figures, all curves plot the negative of every function indicated. Time increases from left to right.)

respectively. The word "stochastic" is appropriate because such behavior of ρ is expected in situations where a subsequent functional integration is performed with a "white-noise" Gaussian weighting; this will be fully discussed in Sec. IV.

To simplify the initial analyses as much as possible, and because it is always possible to reduce the problem to an equation of form (1.1) with a two-component $\mathbf{E}(t)$ vector, we first discuss both adiabatic and stochastic limits treating $\mathbf{E}$ as a vector in the (x, y) plane. The results of these investigations may be briefly summarized as follows, and will be described using the form of U which is convenient for numerical integration, $U(t) = F_0(t) + i\boldsymbol{\sigma}\cdot\mathbf{F}(t)$.

A. The adiabatic limit

There exists here a sequence of corrections which can be written for (F_0, F_i), and which should approach the exact (that is, numerically integrated) solutions rapidly, if the first two approximations are at all representative. One can write an algorithm that can be used to generate successive approximations; and if (which we do not prove) convergence exists, then the solutions are given in terms of four simultaneous fixed-point equations. For brevity, we here

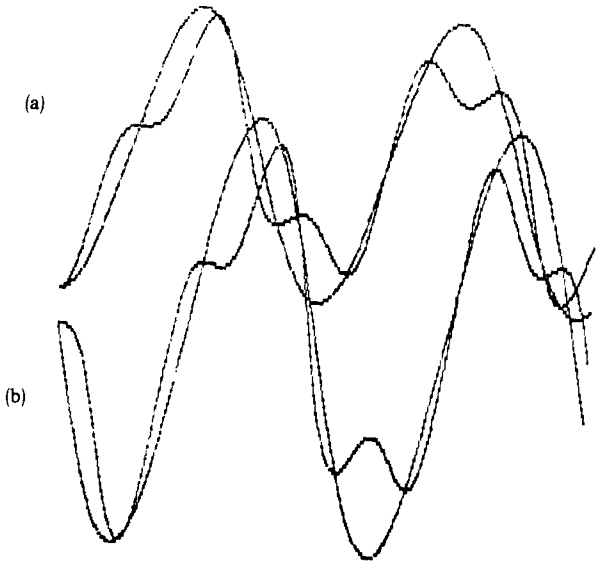

FIG. 2. Superpositions of (a) F_0 and $\overline{F}_0$, and of (b) F_3 and $\overline{F}_3$; for $\rho = 1$.

only mention the form of solutions to $F_0 = \frac{1}{2}\mathrm{Tr}[U]$, with representations for $\mathbf{F}$ and all details reserved for the subsequent text.

In the adiabatic limit, $\rho = 0$, one finds

$$F_0(t) = \cos Q_0(t), \quad Q_0(t) = \int_0^t dt'\, E(t'). \tag{1.3}$$

For a constant magnitude E, F_0 varies as a simple cosine, with frequency $E/2\pi$. As ρ is increased from zero, but $\rho \ll 1$, this essential form remains but is modulated by a smaller competing frequency; e.g., if we suppose that $\omega = |d\hat{E}/dt|$ is also independent of time, and is chosen such that $\rho = \omega/E \simeq 0.1$, then the modulation will cause F_0 to shrink to zero after five cycles or so, then increase again, and repeat the same pattern. Variations of $E(t)$ and/or $\omega(t)$ will change the details but not the overall behavior, as long as $\rho \ll 1$.

Use of the algorithm discussed in the text leads to the first correction to (1.3) given by

$$F_0(t) = [\cos Q_0(t)]\left[\cos\left(\omega\int_0^t dt'|\sin Q_0(t')|\right)\right], \tag{1.4}$$

again assuming, for simplicity, that ω and E are constant. As pictured in Fig. 1, the numerically calculated F_0 is compared with the approximations of (1.3) and (1.4); and one can, in fact, see that (1.4) provides a bit too much modulation. If the procedure converges, the next approximation should modify that discrepancy, etc. We have not attempted further numerical work, and do not yet know whether the fixed-point equations written in Sec. II have solutions for certain $E(t)$.

B. The stochastic limit

As ρ increases, the forms of the exact solutions change dramatically. For $\rho \sim 1$, with constant ω, E, the exact F_0 is displayed in Fig. 2, and bears no resemblance to its form in the adiabatic limit. As ρ is increased further, for $\rho \gg 1$ there is great simplification with F_0 taking the form of small, rapid ω-oscillations superimposed upon a cosine of larger magnitude and much slower frequency $\sim E^2/2\pi\omega$. When E and ω are themselves time-independent, the slowly varying behavior of F_0 can become considerably more complicated than a simple cosine.

Whatever physical properties are being described by these equations, it is surely the larger, slower oscillations—the "average" functional dependence—which should contain physically significant information, and not the smaller, faster oscillations riding on the "averaged" behavior. It is then a matter of some interest to be able to extract the "averaged" behavior of F_0 in any stochastic situation where $\rho \gg 1$; and such an "averaged" $F_0(t)$ can be crudely represented by $\bar{F}_0(t) = \cos(\int_0^t dt' E/\rho)$, a curve that misses all the rapid fluctuations of frequency $\omega/2\pi$ and order of magnitude $1/\rho$ in F_0, but reproduces its slower-frequency and large-magnitude behavior. (The phrase "of order" used in this paper means "of order relative to the slowly varying, averaged forms of $\bar{F}_{0,3}$," which are assumed to be correct when specifying the size of the small, rapidly fluctuating corrections. This is an operational definition of accuracy, not an absolute one.) Various forms of this "averaging," for the F_i as well as F_0, are illustrated in the associated figures. In fact, one can construct a simple argument, using unitarity, to include the rapid fluctuations correct to order $1/\rho$; but the main thrust of our discussion in Sec. III is to derive simple forms for $\bar{F}_0$, $\bar{F}_i$ in the stochastic limit. Slightly more complicated forms are derived in the text for use with smaller values of $\rho > 1$, and they even bear a certain resemblance to the exact forms for $\rho \lesssim 1$.

In Sec. IV we turn to the application of these "averaged" approximations in the stochastic limit generated by white-noise Gaussian (WNG) functional integration, by first comparing the known result of exact WNG integration over U with the result of WNG integration over $\bar{F}_0$, which can also be done exactly (and has an amusing form reminiscent of a Heisenberg, nearest-neighbor, spin–spin interaction). To within a spurious phase factor, which can be easily understood and "renormalized" away, both expressions agree. Other more general examples of functional integration over the adiabatic and stochastic approximations are also considered, and are compared to numerical results performed on the Saclay CRAY.

A generalization of the stochastic-limit approximations to a three-dimensional input $\mathbf{E}(t)$ is written in Sec. V, and is presented there along with the relevant, associated figures. Finally, in the next section, a "fine tuning" of the first stochastic averaged functions is performed, resulting in curves that reproduce the exact numerical integrations in an uncanny way, including the small rapid oscillations correct to order $1/\rho$. The last section is devoted to a very brief summary, and the posing of some relevant questions for future study.

II. AN ALGORITHM FOR THE ADIABATIC LIMIT

In the extreme adiabatic limit $\rho = 0$, corresponding to $d\hat{E}/dt = 0$, all the complexity of the problem disappears. For, as noted above one can choose an arbitrary spatial axis to lie along the direction of $\hat{E}$, and the ordered exponential becomes an ordinary exponential, so that $U(t) \Rightarrow \cos G + i\boldsymbol{\sigma}\cdot\hat{\mathbf{E}} \sin G$, with $G(t) = \int_0^t dt'\, E(t')$. The adiabatic algorithm which we now construct should involve forms close to this limiting case.

Suppose that $\hat{E}(t)$ is a slowly varying unit vector, in the sense of very small ρ; then it is reasonable to choose as a first guess for $U(t)$ the same limiting form

$$U_0(t) = \exp[i\boldsymbol{\sigma}\cdot\mathbf{Q}_0(t)], \tag{2.1}$$

where $\hat{Q}_0(t) = \hat{E}(t)$ and

$$Q_0(t) = |\mathbf{Q}_0(t)| = \int_0^t dt'\, E(t'), \quad E = +\sqrt{\mathbf{E}^2}.$$

This is not correct, but it is unitary, and its deviation from the exact U can be expressed by a unitary $V(t)$: if $U(t) = U_0(t)\cdot V(t)$ with U_0 given by (2.1), then V must satisfy the exact differential equation

$$\frac{\partial V}{\partial t} = i\boldsymbol{\sigma}\cdot\left(\mathbf{E} - \hat{Q}_0 \frac{dQ_0}{dt}\right)V - iQ_0 \int_0^1 d\mu\, e^{-i\mu\boldsymbol{\sigma}\cdot\mathbf{Q}_0}\boldsymbol{\sigma}\cdot \frac{d\hat{Q}_0}{dt}\, e^{+i\mu\boldsymbol{\sigma}\cdot\mathbf{Q}_0}\cdot V, \tag{2.2}$$

or

$$\frac{\partial V}{\partial t} = i\boldsymbol{\sigma}\cdot\mathbf{E}_1 V,$$

with

$$\mathbf{E}_1(t) = \mathbf{E} - \hat{Q}_0\frac{dQ_0}{dt} + \frac{1}{2}\left\{\sin(2Q_0)\,\frac{d\hat{Q}_0}{dt} - [1-\cos(2Q_0)]\hat{Q}_0\times\frac{d\hat{Q}_0}{dt}\right\}. \tag{2.3}$$

We write (2.3) in the form $\mathbf{E}_1 = \mathscr{E}(Q_0,\hat{Q}_0;\mathbf{E})$, and note that while the first two rhs terms of (2.3) will cancel for the specific choice of Q_0 and $\hat{Q}_0$, the functional form of (2.3) will be useful later on. Under the initial condition $V(0) = 1$, the exact solution to (2.2) is

$$V(t) = \left(\exp\left[i\int_0^t dt'\boldsymbol{\sigma}\cdot\mathbf{E}_1(t')\right]\right)_+. \tag{2.4}$$

But if, in the $\rho \ll 1$ regime, the U_0 of (2.1) is a reasonable first approximation to U, then a reasonable approximation to (2.4) should be given by

$$V_1(t) = e^{i\boldsymbol{\sigma}\cdot\mathbf{q}_1(t)}, \tag{2.5}$$

where

$$\hat{q}_1(t) = \hat{E}_1(t), \quad q_1(t) = |\mathbf{q}_1(t)| = \int_0^t dt'|\mathbf{E}_1(t')|. \tag{2.6}$$

With this approximation, we have an "improved" estimate of $U(t)$,

$$U_1(t) \equiv U_0 V_1 = e^{i\boldsymbol{\sigma}\cdot\mathbf{Q}_0(t)}\cdot e^{i\boldsymbol{\sigma}\cdot\mathbf{q}_1(t)}. \tag{2.7}$$

But the combination of (2.7) is unitary, and can be rewritten in a manifestly unitary form as

$$U_1(t) = e^{i\boldsymbol{\sigma}\cdot\mathbf{Q}_1(t)}, \tag{2.8}$$

with

$$Q_1(t) \equiv |\mathbf{Q}_1(t)| = \arccos[\cos Q_0\cdot\cos q_1 - (\hat{Q}_0\cdot\hat{q}_1)\sin Q_0\cdot\sin q_1],$$

or

$$Q_1(t)=Q(Q_0,q_1;\hat{Q}_0\cdot\hat{q}_1);\tag{2.9}$$

and

$$\hat{Q}_1(t)=[\hat{Q}_0\sin Q_0\cdot\cos q_1+\hat{q}_1\sin q_1\cdot\cos Q_0 + (\hat{q}_1\times\hat{Q}_0)\sin q_1\cdot\sin Q_0](\sin Q_1)^{-1},$$

or

$$\hat{Q}_1(t)=\hat{Q}(\hat{Q}_0,\hat{q}_1;Q_0,q_1),\tag{2.10}$$

where the quantities Q and $\hat{Q}$ are defined by the first lines of (2.9) and (2.10), respectively.

But the same process can be repeated: instead of the U_0 of (2.1) we now have the U_1 of (2.8), and can define a better approximation $U_2=\exp[i\boldsymbol{\sigma}\cdot\mathbf{Q}_2]$, with

$$\mathbf{E}_2(t)=\mathscr{E}(Q_1,\hat{Q}_1;\mathbf{E}),$$

$$\hat{q}_2(t)=\hat{E}_2(t),\quad q_2(t)=\int_0^t dt'|\mathbf{E}_2(t')|,$$

$$Q_2(t)=Q(Q_1,q_2;\hat{Q}_1\cdot\hat{q}_2),\quad \hat{Q}_2(t)=\hat{Q}(\hat{Q}_1,\hat{q}_2;Q_1,q_2).$$

Clearly, the process can be repeated an infinite number of times; and if it converges, can be represented by the fixed-point equations

$$\begin{aligned}&Q^*=Q(Q^*,q^*;\hat{Q}^*\cdot\hat{q}^*),\\ &\hat{Q}^*=\hat{Q}(\hat{Q}^*,\hat{q}^*;Q^*,q^*),\\ &\hat{q}^*=\hat{\mathscr{E}}(Q^*,\hat{Q}^*;\mathbf{E}),\\ &q^*=\int_0^t dt'|\mathscr{E}(Q^*,\hat{Q}^*;\mathbf{E})|,\end{aligned}\tag{2.11}$$

where Q^*, $\hat{Q}^*$, $\hat{q}^*$, q^*, and $\mathbf{E}$ are functions of t, and the functional forms $\mathscr{E}$, Q, and $\hat{Q}$ are given by (2.3), (2.9), and (2.10).

For an arbitrary input $\mathbf{E}(t)$, there is probably little hope of finding or proving convergence; but for some suitably simple input this might be possible. For our purposes, we note that if $\mathbf{E}(t)$ is chosen to be a vector of constant magnitude E rotating in the (x,y) plane with a constant angular frequency ω, then for $\rho=\omega/E\sim0.1$, U_1 is a better approximation to the exact U than is U_0, as illustrated in Fig. 1 where the first two approximations to $F_0(t)$ are compared with the exact, or numerically integrated result. In fact, U_1 provides somewhat too much modulation, which should be removed by U_2, etc.

Results equivalent to the U_1 correction to U_0 have been discussed, in special contexts, in Refs. 2 and 3. To our knowledge, the algorithm for general U_n, as well as the fixed-point equations (2.11), is new; however, these latter statements are probably too complicated to be of much practical use. Generalization to SU(N) is simple for U_0 (Refs. 5 and 6), and while the general algorithm can be defined for arbitrary N, the more complicated statement of unitarity there will make this task much more tedious.

III. THE STOCHASTIC LIMIT

For $\rho\gg1$ we again choose for $U(t)$ the manifestly unitary form, $U(t)=\exp[i\boldsymbol{\sigma}\cdot\mathbf{G}(t)]$, with $\mathbf{G}=\hat{G}\cdot G$, $G=+\sqrt{\mathbf{G}^2}$, and substitute into (1.1) to obtain

$$\boldsymbol{\sigma}\cdot\mathbf{E}(t)=\int_0^1 d\mu\, e^{i\mu\boldsymbol{\sigma}\cdot\mathbf{G}}\boldsymbol{\sigma}\cdot\frac{d\,\mathbf{G}}{dt}e^{-i\mu\boldsymbol{\sigma}\cdot\mathbf{G}}\tag{3.1}$$

or

$$\mathbf{E}(t)=\hat{G}\frac{dG}{dt}-\frac{1}{2}[1-\cos(2G)]\left(\hat{G}\times\frac{d\hat{G}}{dt}\right)+\frac{1}{2}\sin(2G)\frac{d\hat{G}}{dt},\tag{3.2}$$

which is equivalent to the pair of exact relations

$$\frac{dG}{dt}=\hat{G}(t)\cdot\mathbf{E}(t),\tag{3.3}$$

$$\frac{d\hat{G}}{dt}=E\,[\hat{\mathrm{G}}\times\hat{E}+\cot G(\hat{E}-\hat{G}(\hat{E}\cdot\hat{G}))].\tag{3.4}$$

With the initial conditions $G(0)=0$, $\hat{G}(0)=\hat{E}(0)$, the magnitude $G(t)$ is completely determined by $\hat{G}$. For simplicity, we suppose that E lies in the (x,y) plane; the three-dimensional generalization is treated in Sec. V.

Since $\hat{G}$ is a unit vector, it can be specified by two independent quantities which we choose in the following way. For $\rho\ll1$, we know that $\hat{G}(t)\sim\hat{E}(\int_0^t dt'\,\omega)$, but as $\omega=|\hat{E}\times d\hat{E}/dt|$ increases from zero this cannot be retained; rather, we suppose that the argument of $\hat{E}$ can be specified by a phase change relative to $\int_0^t\omega dt'$: $\hat{E}(\mathrm{t})\to\hat{E}(\int_0^t\omega\,dt'-\delta(t))$. It will be convenient to use a dimensionless time variable, τ, given by $d\tau=E\,dt$, and so write this phase-shifted unit vector as $\hat{E}(\int_0^t d\tau'\rho-\delta(\tau))$. But since $\hat{E}$ lies in the (x,y) plane, and $\hat{G}$ will have a $\hat{z}$ component for arbitrary ρ while remaining a unit vector, we choose the ansatz

$$\hat{G}(\tau)=\cos\phi(\tau)\hat{E}\left(\int_0^\tau d\tau'\rho-\delta(\tau)\right)+\hat{z}\sin\phi(\tau),\tag{3.5}$$

with $\phi(\tau)$ and $\delta(\tau)$ the two independent functions needed to characterize $\hat{G}$. Substitution of (3.5) into (3.4) yields the two independent equations

$$\frac{d\delta}{d\tau}=\rho-\tan\phi\cdot\cos\delta-\frac{\sin\delta}{\cos\phi}\cdot\cot G,\tag{3.6}$$

$$\frac{d\phi}{d\tau}=\sin\delta-\sin\phi\cdot\cos\delta\cdot\cot G,\tag{3.7}$$

which, together with the initial conditions $\delta(0)=\phi(0)=0$ and the relation

$$G(\tau)=\int_0^\tau d\tau'\cos\delta(\tau')\cdot\cos\phi(\tau'),\tag{3.8}$$

completely determine $\hat{G}$. Equation (3.8) is obtained using the assumed variation of $\hat{E}$, for arbitrary $\boldsymbol{\omega}(t)$, $E(t)$:

$$\frac{d\hat{E}}{d\tau}\left(\int_0^\tau\rho\,d\tau'\right)=\boldsymbol{\omega}\times\hat{E}\left(\int_0^\tau\rho\,d\tau'\right),$$

with

$$\hat{E}\left(\int_0^\tau\rho\,d\tau'\right)=\hat{\imath}\cos\left(\int_0^\tau\rho\,d\tau'\right)+\hat{\jmath}\sin\left(\int_0^\tau\rho\,d\tau'\right).$$

It is clear that Eqs. (3.6)–(3.8) are very nonlinear, and it is difficult to have any intuitive feeling about their solutions in the large ρ limit. In order to obtain this intuition, one may machine-integrate these equations—or, equivalently, those that follow by substituting the ansatz $U=F_0+i\boldsymbol{\sigma}\cdot\mathbf{F}$ into (1.1), along with unitarity restriction $F_0^2+\mathbf{F}^2=1$—and find for $\rho\gg1$ a remarkable simplification. For simplicity, we again for the moment consider ω and E, and therefore ρ, as constants, and watch the exact solutions for $F_0 = \cos G$ change as ρ is increased through unity to large values. For $\rho \gg 1$, one finds that a very rapid oscillation, of frequency $\omega/2\pi$ and mangitude ρ^{-1}, is superimposed upon a relatively, slowly varying oscillation, of frequency $\sim E/2\pi\rho$ and magnitude unity. The rapid oscillations should be irrelevant to any physical property described by this system of equations, and it is therefore natural to phrase the question: is it possible to approximate $U(t) = U(t\,|E)$ as a function of E so that, in the large ρ limit, one reproduces only the "averaged," or slowly varying behavior, and not the rapid fluctuations? The answer to this question is, indeed, yes; it is the point of this section, and we now outline the derivation of such "averaged" functions, to be denoted by $\overline{F}_{0,i}$.

Just the "experimental" knowledge that, for $\rho \gg 1$, the output for, e.g., F_0 consists of rapid oscillations superimposed on a slowly varying function of form $\sim\cos(Et/\rho)$ is enough to suggest an argument that can be followed. For, from (3.8), this means that as far as the "averaged" behavior is concerned, the quantity $J \equiv \cos\delta(\tau)\cdot\cos\phi(\tau)$ can be treated as a constant. (This statement will be refined, in Sec. VI, when we discuss "fine tuning.") It will be useful to define the associated quantity $H \equiv \cos\phi\cdot\sin\delta$, so that $\cos^2\phi = J^2 + H^2$, and the exact equations (3.6)–(3.8) can be expressed as

$$J' = -\rho H + [1 - J^2]\cot G, \tag{3.9}$$

$$H' = -\sin\phi + \rho J - HJ\cot G, \tag{3.10}$$

and

$$G = \int_0^\tau d\tau'\, J(\tau'). \tag{3.11}$$

For the "averaged" behavior, $J \sim \text{const} \equiv \xi(\rho)$, (3.9) may be replaced by

$$H \simeq ((1 - \xi^2)/\rho)\cot G, \tag{3.12}$$

with $G \simeq \tau\xi$. Just as G depends on the slowly varying time dependence, so must the "averaged" H of (3.12). Substituting the latter into (3.10), with $G = \tau\xi$, yields an equation for an "averaged" $\sin\phi$,

$$\sin\phi = \rho\xi + \xi(1 - \xi^2)/\rho. \tag{3.13}$$

The form of (3.13) will be more complicated if ρ depends upon t, or τ, but for $\rho \gg 1$ this extra dependence need not be important; this will be discussed in Sec. V. For the remainder of this derivation we shall continue to assume that ρ is essentially constant; but we shall not hesitate to state our results for time-dependent ρ, where our final formulas continue to work in a satisfactory way.

If our analysis leading to (3.13) is correct, $\sin\phi$ should display an "averaged" behavior, with rapid oscillations superimposed on a constant background; and this is true, experimentally, as one can see in Fig. 3. One may note that there is a change of procedure used here, in the following sense. An exact (numerical) integration of (3.6)–(3.8) yields a value of G that never increases past π, while $\sin\phi$ and J are positive when the average G is increasing and negative when it decreases (so that $\hat{G}$ can cover all points on the unit sphere). In contrast, our "averaged" G will increase

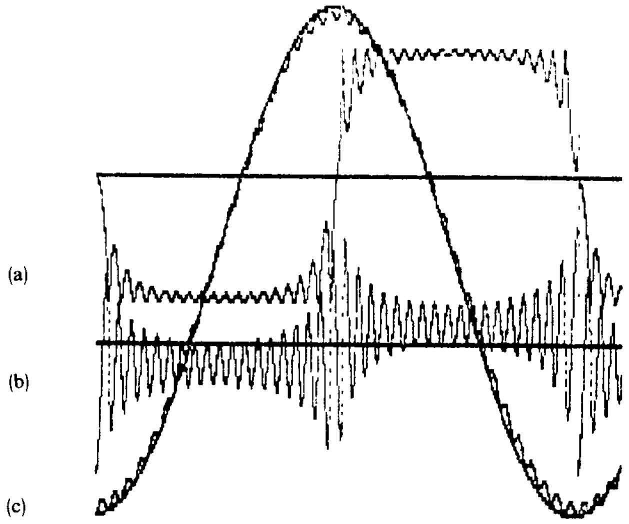

FIG. 3. Graphs of (a) $\sin\phi$, (b) $J = \cos\phi\cdot\cos\delta$, and (c) a superposition of F_0 and $\overline{F}_0$; for $\rho = 6$, $E = 10$.

without limit, so that $\sin G$ may become negative (just when the exact G was decreasing), while the averaged $\sin\phi$ and J are replaced by positive constants. In this way we are able to represent the correct signs of all the (F_0, F_i). This same feature of always positive $\sin\phi$ and J can occur in numerically integrated solutions of the exact equations (3.6)–(3.8) depending on the accuracy of the computation and the passage through the singular regions of $\cot G$. For our purposes, both $\sin\phi$ and J can be thought of as having an "averaged," constant value, even though in reality they oscillate about that value, and oscillate wildly near the regions $G \sim n\pi$. In contrast, a plot of $\sin\delta$ displays an almost uniform density of points spread over the same intervals.

We now use the "averaged" constancy of $\sin\phi$, or of $\cos^2\phi = J^2 + H^2$, to determine the dependence of ξ on ρ. For, if the "averaged" value of $(d/d\tau)$ $(\frac{1}{2}\cos^2\phi)$ is to vanish, from (3.9) and (3.10) one finds another expression for the "averaged" H,

$$0 \simeq -H\sin\phi + J[1 - (J^2 + H^2)]\cot G,$$

or

$$H = \xi\sin\phi\cot(\xi\tau). \tag{3.14}$$

Comparing with (3.12) we obtain

$$\xi\sin\phi = (1 - \xi^2)/\rho, \tag{3.15}$$

and finally, comparing (3.15) with (3.13) yields

$$(1 - \xi^2)/\rho = \rho\xi^2[1 + (1 - \xi^2)/\rho^2],$$

or

$$\xi(\rho) = \sqrt{1 + \rho^2/4} - \rho/2. \tag{3.16}$$

In obtaining (3.16) it has been supposed that $\xi > 0$ and $1 - \xi^2 > 0$. The slightly more complicated form of $\xi(\rho)$ used in Ref. 1 is exactly equivalent to (3.16). Limiting forms are

$$\xi(\rho)_{\rho\gg 1} \sim 1/\rho - 1/\rho^2 + \cdots$$

and

$$\xi(\rho)_{\rho\ll 1} \sim 1 - (\rho/2) + \cdots.$$

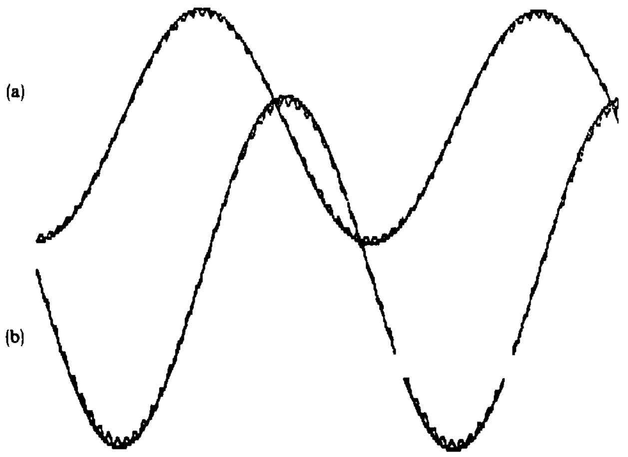

FIG. 4. Superpositions of (a) F_0 and $\overline{F}_0$, and (b) F_3 and $\overline{F}_3$; for $\rho = 6$, $E = 10$.

From (3.16) and (3.15), it follows that our "averaged" $\sin\phi$ is given by unity, which is certainly compatible with the curve of $\sin\phi$ illustrated in Fig. 3.

With these relations, our "averaged" solutions for F_0, F_3 are given by

$$\overline{F}_0 = \cos G, \tag{3.17}$$

$$\overline{F}_3 = \sin G, \tag{3.18}$$

where

$$G = \tau\xi(\rho) \to \int_0^\tau d\tau'\, \xi(\rho(\tau')) \tag{3.19}$$

is appropriate as a first generalization to time-dependent E and ω. The accuracy of these expressions is quite good for $\rho > 5$, where errors, or deviations from the numerically integrated F_0, F_3 are rarely worse than a few percent, and frequently much less. Even for $\rho \sim 1$, where the analysis is certainly not valid, one finds that these expressions for F_0 and F_3 do tend to average out the then, nonrapid fluctuations of the machine integrated F_0, F_3. Some typical outputs may be seen in Figs. 4–7, including several examples of t-dependent E and ω. One finds, generally, that even if ρ has some oscillation superimposed on a constant value$\gg 1$, the "averages"

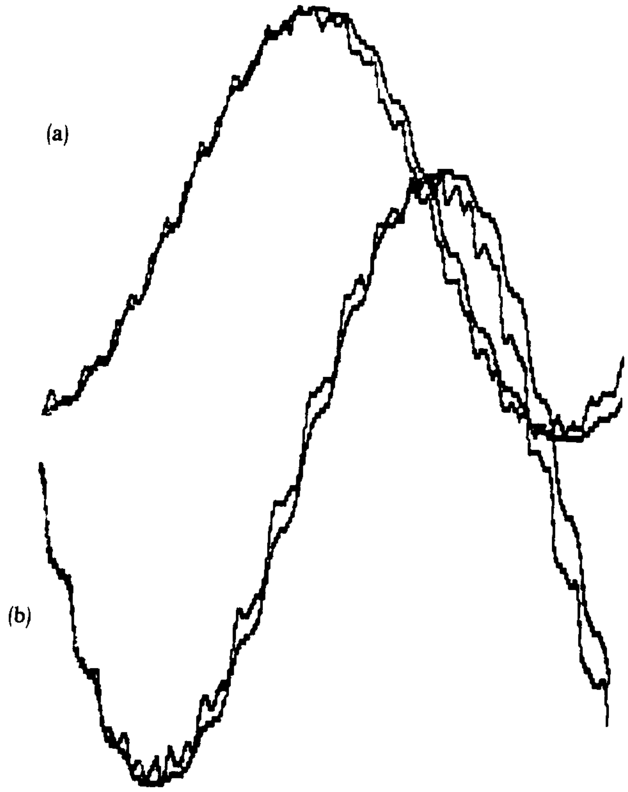

FIG. 6. Superpositions of (a) F_0 and $\overline{F}_0$, and (b) F_3 and $\overline{F}_3$; for $\omega = 60$, $E(t) = 10 + 5\sin(30t)$.

given by (3.17)–(3.19) continue to be reasonably accurate. One must, of course, be careful about the errors that accumulate in numerical integrations; a typical such effect will be the appearance of a phase lag between F_0 and $\overline{F}_0$, and between F_3 and $\overline{F}_3$, which is a "Δt-effect," and may be decreased by choosing a smaller integration step or a more accurate method of integration.

Analogous approximate expressions for $F_{1,2}$ are easily written. One has, exactly,

$$F_1 = \sin G\,[J\cos L + H\sin L\,], \tag{3.20}$$

$$F_1 = \sin G\,[J\sin L - H\cos L\,], \tag{3.21}$$

with $L = \int_0^t dt'\,\omega(t')$. Inserting the same "averaged" ap-

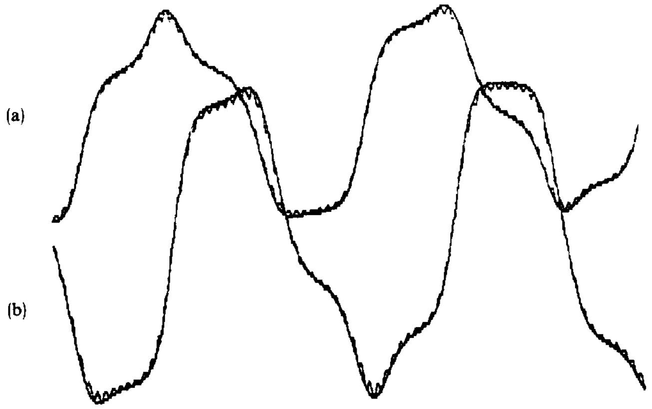

FIG. 5. Superpositions of (a) F_0 and $\overline{F}_0$, and (b) F_3 and $\overline{F}_3$; for $\omega = 60$, $E(t) = 10 + 5\sin(5t)$.

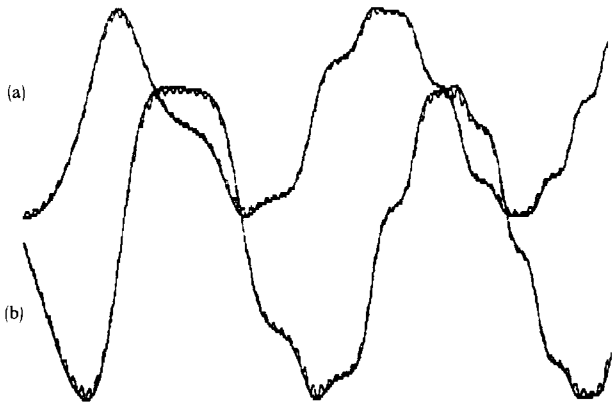

FIG. 7. Superpositions of (a) F_0 and $\overline{F}_0$, and (b) F_3 and $\overline{F}_3$; for $\omega = 60$, $E(t) = 10 + \cos(t^2)$.

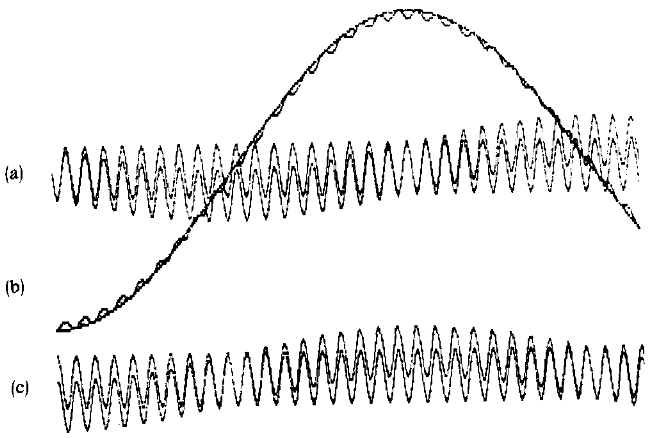

FIG. 8. Superpositions of (a) F_1 and $\overline{F}_1$, and (b) F_0 and $\overline{F}_0$, and (c) F_2 and $\overline{F}_2$; for $\rho = 6$, $E = 10$.

proximations for J, H, G as before, one finds

$$\overline{F}_1 = \xi \sin(G + L), \tag{3.22}$$

$$\overline{F}_2 = -\xi \cos(G + L), \tag{3.23}$$

where G is again given by (3.19). For large ρ, $\xi \sim 1/\rho$, and these $\overline{F}_{1,2} \sim O(1/\rho)$, in contrast to our $\overline{F}_{0,3} \sim O(1)$. These $\overline{F}_{1,2}$ are therefore small, and oscillate rapidly, and should have little physical importance in any specific problem. However, as seen in Fig. 8, Eqs. (3.22) and (3.23) do miss some of the slowly varying dependence of the exact $F_{1,2}$, even if the dependence is itself on the order of $1/\rho$. In Sec. VI we will give a simple argument to correct the $\overline{F}_{1,2}$ above, so that they will be correct to $O(1/\rho)$; and in the process, use the requirement of unitarity to "fine tune" our $\overline{F}_{0,3}$, giving them a rapid oscillation superimposed on their "averaged" values which is correct to $O(1/\rho^2)$. Unitarity is, of course, approximately satisfied by the $F_{0,1,2,3}$ above,

$$(\overline{F}_0^2 + \overline{F}_3^2) + (\overline{F}_1^2 + \overline{F}_2^2) = 1 + \xi^2 \simeq 1 + (1/\rho^2) + \cdots .$$

In Sec. VI we shall arrange to have this unitarity sum given by $1 + O(1/\rho^4)$, and hence infer that the "fine-tuned" $\overline{F}_{0,i}$ are themselves correct to at least $O(1/\rho)$.

There is one qualification to the remarks of this section that must be noted, and that will be relevant to some of the functional integrals performed over our stochastic averaged forms. If the input $\mathbf{E}(t)$ can be split into two nonparallel parts of radically different magnitudes, then it is the large-magnitude input that controls the final output of U. For example, motion corresponding to the input $\mathbf{E} = \hat{i}E_1 + \hat{j}E_2 \cos(\omega t)$, with $E_1 \gg E_2$, is essentially adiabatic, regardless of the value of ω.

More generally if $\mathbf{E}(t)$ is given as the sum of two nonparallel pieces, $\mathbf{E} = \mathbf{E}_1(t) + \mathbf{E}_2(t)$, with arbitrary time dependence but where the magnitude of one is much larger than the other, say $E_1 \gg E_2$, then the prudent way to set up the calculation is to separate all the E_1 dependence into a unitary V, leaving a rotated E_2 dependence, say E_2' in W: $U = VW$, where

$$V = \left(\exp\left[i \int_0^t dt' \, \boldsymbol{\sigma}\cdot\mathbf{E}_1(t') \right] \right)_+ .$$

We assume that all the components of E_1 are of the same order of magnitude; anything much smaller is put into E_2. Then

$$W = \left(\exp\left[i \int_0^t dt' \, \boldsymbol{\sigma}\cdot\mathbf{E}_2'(t') \right] \right)_+ ,$$

where

$$\boldsymbol{\sigma}\cdot\mathbf{E}_2' = V^+ \boldsymbol{\sigma}\cdot\mathbf{E}_2 V, \quad \mathbf{E}_2'^2 = \mathbf{E}_2^2 .$$

When we calculate the ρ-value of $\mathbf{E}_2'$ we will find it of order (ω_2/E_2) and/or $(E_1 E_2)$; explicitly

$$\rho^2(\mathbf{E}_2') = \left[\left(\frac{d\mathbf{E}_2}{dt} + [\mathbf{E}_1 \times \mathbf{E}_2] \right)^2 - \left(\frac{dE_2}{dt} \right)^2 \right] (E_2^4)^{-1} .$$

As long as $\mathbf{E}_1 \times \mathbf{E}_2 \neq 0$—and this was assumed—there will be a piece of this ρ proportional to $(E_1/E_2) \gg 1$. It is then appropriate to use the stochastic form for W, leading to the contribution

$$G_W \sim \int_0^t dt' \frac{E_2}{\rho(E_2')} \sim \int_0^t dt' \, O\left(\frac{E_2^2}{E_1} \right),$$

which will be a small correction to the G_V if G_V is adiabatic, or a contribution of the same form as G_V if G_V itself is stochastic.

IV. STOCHASTIC FUNCTIONAL INTEGRATION

One very nice check on our approximations is their ability to reproduce the result of the one, nontrivial functional integration over an ordered exponential which can be performed analytically, that of WNG integration over the $U(t|E)$ of (1.2). Indeed, one type of application of our results should be to stochastic functional integration over weightings more complicated than Gaussian. In this section, we first show why the stochastic limit is appropriate to WNG integration, and then just how our approximate forms can reproduce the known, exact result

$$N \int d[E] \exp\left[-\frac{1}{2c} \int_0^t dt' \, \mathbf{E}^2(t') \right] U(t|\mathbf{E}) = e^{-tc}, \tag{4.1}$$

where N is a normalization constant defined by

$$N^{-1} = \int d[E] \exp\left[-\frac{1}{2c} \int_0^t dt' \, \mathbf{E}^2(t') \right].$$

In (4.1) we denote by c a real, positive constant; and continue to suppose that $\mathbf{E}$ lies in the (x, y) plane.

We first remaind the reader of the derivation of (4.1). Imagine the interval $(0,t)$ broken up into n subintervals, of width $\Delta t = t/n$ and labeled by an index i, so that the $\mathbf{E}(t')$ field in each subinterval is denoted by $\mathbf{E}_i$. Then, one definition of the functional integral is

$$\lim_{n \to \infty} \prod_{i=1}^n N_i \int d^2 E_i \, e^{-\Delta t \, \mathbf{E}_i^2 / 2c} (e^{i \Delta t \boldsymbol{\sigma}\cdot\mathbf{E}_i}), \tag{4.2}$$

and the ordering of the brackets is such that terms with the larger value of i stand to the left. But each integral yields a result independent of i—that is, independent of $\boldsymbol{\sigma}$—by the following argument.

Because of the Gaussian weighting, each E_i scales as $(\Delta t)^{-1/2}$; that is, in (4.2) replace each $\mathbf{E}_i$ by $\mathbf{F}_i/\sqrt{\Delta t}$ (including the normalization, $N_i \to N_i'/\Delta t$), and for small Δt

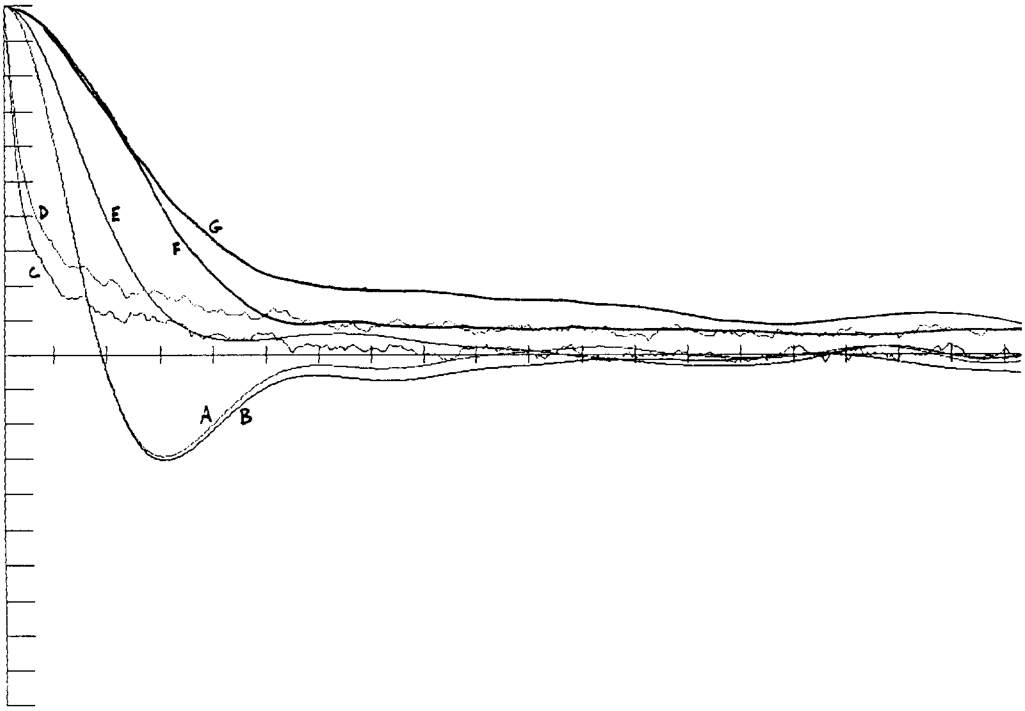

FIG. 9. A comparison of the result of functional integration over the exact $U(t|E)$ with several approximations, for $\tau^{-1} = 100$, $E_m = 10$, $\Delta t = 0.005$. The labeling of the curves is A = exact, numerical; B = renormalized $(1/\rho)$; $C = (1/\rho)$; D = full $\xi(\rho)$; E = renormalized, full $\xi(\rho)$; F = renormalized adiabatic; G = adiabatic.

expand each $\exp(i\sqrt{\Delta t}\,\boldsymbol{\sigma}\cdot\mathbf{F}_i)$ so that (4.2) becomes

$$\lim_{n\to\infty}\prod_{i=1}^{n} N_i' \int d^2F_i\, e^{-\mathbf{F}_i^2/2c}\Big(1 + i\sqrt{\Delta t}\,\boldsymbol{\sigma}\cdot\mathbf{F}_i - \frac{\Delta t}{2}\mathbf{F}_i^2 + \cdots\Big), \tag{4.3}$$

in which we retain only the leading, nonzero dependence proportional to Δt (the coefficient of $\sqrt{\Delta t}$ vanishes by symmetry). Each ith integral is the same; and it is trivial, yielding

$$\lim_{n\to\infty}(1 - c\Delta t)^n = e^{-ct}, \tag{4.4}$$

with $\Delta t = t/n$.

The essential part of this computation has been the observation that, for WNG integration, each $\mathbf{E}_i$ scales as $(\Delta t)^{-1/2}$. We now consider the same functional integration over our "averaged" forms. The first point to be settled is whether the stochastic limit is valid, and for this we must estimate the size of $\rho^2 = (d\hat{E}/dt)^2/E^2$. But, upon breaking up the interval $(0,t)$ into subintervals, any $\rho^2(t)$ would be replaced by

$$\rho_i^2 = (\hat{E}_i - \hat{E}_{i+1})^2/E_i^2(\Delta t)^2.$$

The $\hat{E}_i$ dependence is of $O(1)$; but because E_i scales as $(\Delta t)^{-1/2}$, $\rho_i^2 \sim O(1/\Delta t)$ and is *large*. Hence the stochastic limit most certainly is relevant, and we consider the functional integrals of our "averaged" forms in the limit of very large ρ, $\overline{U} \to \overline{F}_0 + i\sigma_3\overline{F}_3$, $\overline{F}_0 = \cos G$, $\overline{F}_3 = \sin G$. One then requires

$$N\int d[E]\exp\Big[-\frac{1}{2c}\int_0^t dt'\,\mathbf{E}^2(t')\Big]e^{\pm iG}, \tag{4.5}$$

which upon writing $G \simeq \int_0^t dt'(E/\rho)$ and breaking up the integration region into subintervals, generates

$$\lim_{n\to\infty}\prod_{i=1}^{n} N_i \int d^2E_i\, e^{-\Delta t\,\mathbf{E}_i^2/2c}e^{\pm i\Delta t E_i/\rho_i}, \tag{4.6}$$

where $\rho_i = +[(\hat{E}_i - \hat{E}_{i+1})^2]^{1/2}/E_i\Delta t$. Again rescaling $\mathbf{E}_i$, we now find in each subinterval both an integral over the magnitude F_i and a nontrivial angular dependence. Integration over each magnitude is trivial, leaving

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta_i\Big(1 \mp \frac{ic\Delta t}{|\sin|(\theta_i/2)||}\Big)^{-1}. \tag{4.7}$$

The integral of (4.7) can be done exactly; with $q = \pm c\Delta t$, it is

$$1 + \frac{2iq}{\pi(1+q^2)^{1/2}}\ln\Big[\Big(\frac{1-(1+q^2)^{1/2}-iq}{1+(1+q^2)^{1/2}-iq}\Big)\times\Big(\frac{1+(1+q^2)^{1/2}}{1-(1+q^2)^{1/2}}\Big)\Big].$$

As $\Delta t \to 0$, the argument of the log becomes $\pm\, 2i/c\Delta t$, generating for the complete functional integral

$$\lim_{n\to\infty}\prod_{i=1}^{n}\Big(1 - c\Delta t \pm \frac{2ic\Delta t}{\pi}\ln\Big(\frac{2}{c\Delta t}\Big)\Big), \tag{4.8}$$

which can be written as

$$e^{-ct}e^{\pm(2ict/\pi)\ln(2/c\Delta t)}|_{\Delta t\to 0}. \tag{4.9}$$

Comparison with (4.1) shows that a spurious phase has appeared; but one that can be understood, and removed, by the following argument. In every subinterval's integration, our "averaged" forms have made a small error, which is (fortunately) imaginary, and which must be removed "by hand." Instead of calculating (4.5) as we have done, we must add the proviso that we keep only the real part of every subinterval's contribution; and in this way, by not retaining and compounding the small error generated by our "averaged" forms, we can reproduce (4.1). We expect this tendency towards a spurious phase factor to show up in more complicated functional integrals, or in functional integrals that are Gaussian but not precisely in the white-noise limit, and it will be necessary to remove such spurious dependence.

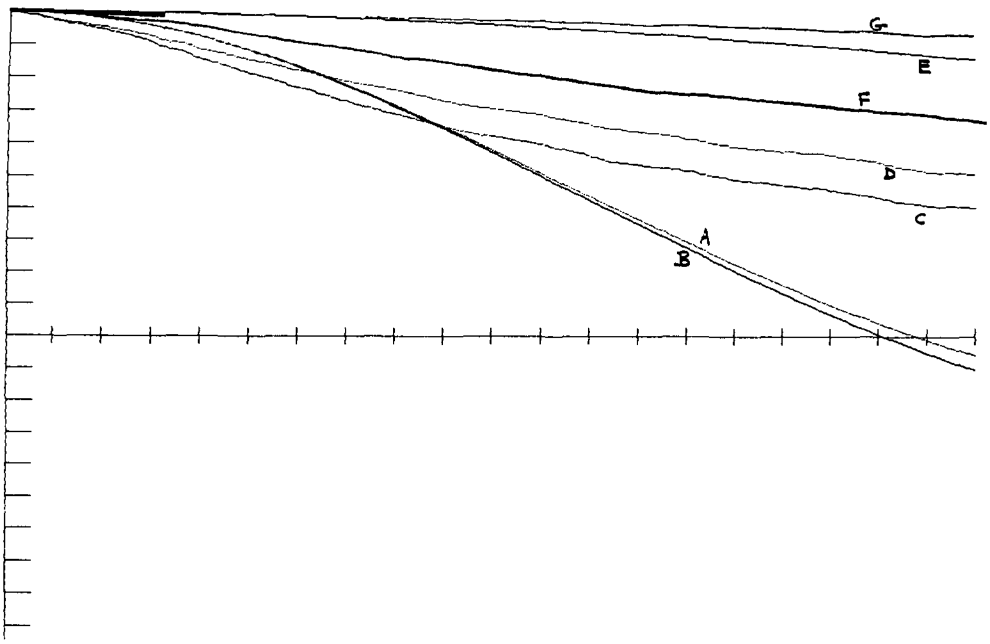

FIG. 10. The same comparisons, with the same labeling as in Fig. 9, using $\tau^{-1} = 100$, $E_m = 1$, $\Delta t = 0.005$.

This can be done most simply by replacing the functional integral over $e^{\pm iG}$, which we call $\langle e^{\pm iG}\rangle$, by the quantity $[|\langle e^{\pm iG}\rangle|^2]^{1/2}$, a computation we henceforth label "renormalized."

More general, non-WNG weightings may be treated by calculating Gaussian fluctuations with correlation function given by

$$\Delta_{ij}(t_1 - t_2) = \langle E_i(t_1)E_j(t_2)\rangle = \delta_{ij}(E_m/2\tau)e^{-|t_1 - t_2|/\tau},$$

where τ is a correlation time, and E_m an appropriate magnitude. The limit $\tau^{-1}\to\infty$ for $E_m = 1$ is the WNG case, $\Delta_{ij}\to\delta_{ij}\delta(t_1 - t_2)$, while the opposite limit, $\tau^{-1}\to 0$ is effectively the adiabatic limit. (This last remark would be strictly true if ρ were defined as $|d\mathbf{E}/dt|/E^2$ rather than as $|d\hat{E}/dt|/E$; in practice there seems to be little difference.)

We illustrate in Figs. 9 and 10 calculations in the WNG region ($\tau^{-1} = 100$) over a variety of different possible approximations, and note that here the best agreement with the exact functional integration is given by first performing the large-ρ approximation of ξ, $\xi(\rho)\sim 1/\rho$, and then performing the functional integration. Why this is true—rather than using the exact $\xi(\rho)$ and letting the natural, large-ρ fluctuations automatically induce the effective large-ρ form of ξ—is a reflection of the comments made at the end of Sec. III. In the numerical computations there are many successive choices of E_i that correspond to large variations of $\hat{E}$, but of small magnitude, superimposed on a perpendicular component of large magnitude and slow variation; and these fluctuations are to be interpreted as adiabatic contributions of small, effective ρ. When the full $\xi(\rho)$ is used, such small-ρ contributions are incorrectly taken into account. However, with the large-ρ form of ξ, $\xi\sim 1/\rho$, the corresponding contributions to $\langle \mathrm{tr}\, U\rangle$ are small for small ρ, since such exponentiated terms are rapidly damped away. Using the large-ρ form of $\xi(\rho)$ suppresses such incorrect, effectively small-ρ behavior; and, as one can see from Figs. 9 and 10, provides fairly reasonable approximations to the exact result.

V. THREE-DIMENSIONAL INPUT

We here consider the generalization of the material of Sec. III to three-dimensional input $\mathbf{E}(t)$, which requires a generalization to time-dependent ρ. It will be appropriate to comment, firstly, on the derivation given in that section for a time-dependent ρ, and then to extend the analysis to three dimensions.

The passage from the exact equations (3.9)–(3.11) to our approximate, "averaged" forms was performed assuming a constant ρ, and using the "experimental" properties that J and $\sin\phi$ are given by rapid oscillations superimposed upon a constant background. If $\rho = \rho(t)$, one must first determine if the same properties of "averaged" constancy of J and $\sin\phi$ still exist, before an analysis of the same type can be given. The experimental answer, obtained for a variety of choices of the t dependence of ρ (but always insisting on $\rho\gg 1$) is that the angular integrations represented, e.g., by Fig. 11, are only slightly modified; experimentally, J and $\sin\phi$ may still be represented as constant quantities on which are superimposed rapid oscillations. This being the case, it does make sense to apply the same form of argument as was used to arrive at (3.12); but the form of (3.13) will now be complicated by the appearance of an extra term proportional to

$$\frac{d\rho}{dt}\left[\frac{(1-\xi^2)}{\rho^2} + \frac{2\xi}{\rho}\frac{\partial\xi}{\partial\rho}\right]\cot G.$$

The result is that (3.13) and (3.15) no longer yield an algebraic equation for $\xi(\rho)$, but rather, with specific input $d\rho/dt$, a differential equation for $\xi(\rho)$. The complication is decidedly nontrivial. Fortunately, if ξ still falls off as ρ increases, for $\rho\gg 1$ these terms should not have any important effect. More precisely, even if a time-dependent ρ (but, always, $\rho\gg 1$) adds small and rapid oscillations to our "averaged" forms, which need not agree with the small and rapid oscillations of the numerically integrated functions, the slowly varying behavior of the "averaged" forms still repro-

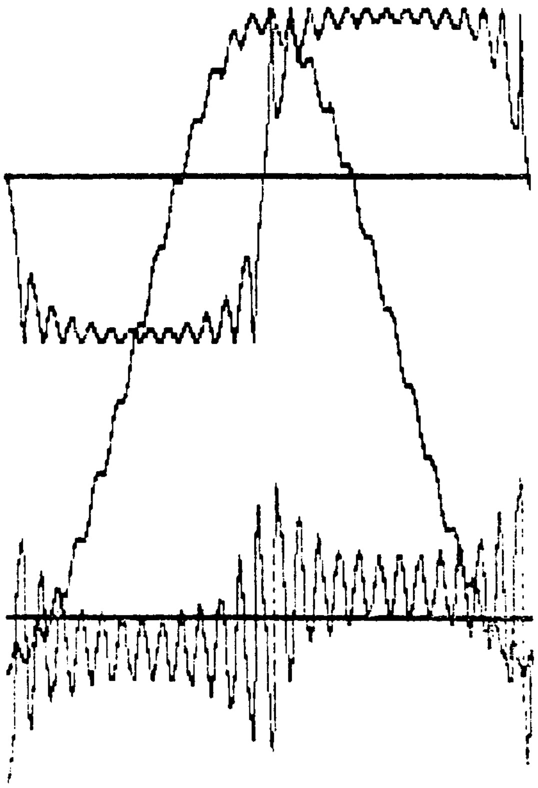

FIG. 11. Graphs of (a) $\sin\phi$, (b) F_0, and (c) $J=\cos\phi\cos\delta$; for $E=10$, $\rho(t)=30+30|\cos(60t)|$.

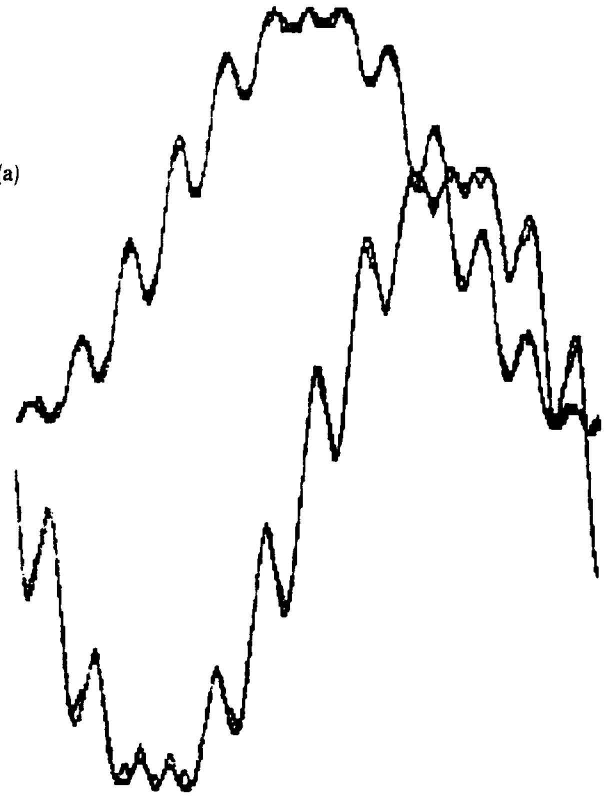

FIG. 12. Superpositions of (a) F_0 and $\overline{F}_0$, and (b) F_3 and $\overline{F}_3$; for $\omega=60$, $\nu=20$, $E_T=10$, and $E_L=5$.

duces that of the exact solutions. This can be clearly seen, even after the "fine-tuning" of Sec. VI, in Fig. 12, where an ω containing rapid oscillations inserted into our essentially constant (or slowly varyinbg) ω-analysis produces a curve whose small and rapid oscillations do not match the exact ones, but whose "averaged" shape continues to reproduce that of the exact curves.

We emphasize that we have not attempted a careful study of this quite complicated point; but we are convinced that, for $\rho \gg 1$, the specifically time-dependent effects of ρ are not important in developing the "averaged" forms in any way other than the elementary generalizations we have made,

$$\omega t \to \int_0^t dt'\,\omega(t'),\quad G=\tau\xi(\rho)\to\int_0^t dt'\,E(t')\xi(\rho(t')),$$

in writing our final formulas (3.17)–(3.24). To substantiate this claim, we point to the superimposed curves of F_0 and $\overline{F}_0$ of Figs. 5, 6, 7, and 12 made for a variety of choices of $\rho(t)$, and using only the $\overline{F}_0$ of (3.17).

In treating the problem of three-dimensional input $\mathbf{E}(t)$, it is always possible to perform a transformation on the basic equation (1.1) to yield a similar equation for a related quantity in which there appears a two-dimensional input $\mathscr{E}(t)$. For, if one defines another unitary quantity $V=e^{-(i/2)\theta(t)\sigma_3}\cdot U$, where $\theta(t)$ is a function to be determined, then the matrix V will satisfy

$$\frac{\partial V}{\partial t}=i\left(\sigma_1\mathscr{E}_1+\sigma_2\mathscr{E}_2+\sigma_3\left[E_3-\frac{\dot\theta}{2}\right]\right)V, \tag{5.1}$$

with $\theta(0)=0$, $V(0)=1$, and

$$\mathscr{E}_1\equiv E_1\cos\theta+E_2\sin\theta, \tag{5.2a}$$

$$\mathscr{E}_2\equiv E_2\cos\theta-E_1\sin\theta. \tag{5.2b}$$

If choose $\frac{1}{2}\theta(t)=\int_0^t dt'\,E_3(t')$, then the problem has been reduced to one of two-dimensional input.

Writing the exact solutions for V in the form $V=\mathscr{F}_0+i\boldsymbol{\sigma}\cdot\boldsymbol{\mathscr{F}}$, and comparing $U=F_0+i\boldsymbol{\sigma}\cdot\mathbf{F}$ with the solution obtained from $U=\exp[i\sigma_3(\theta/2)]\ V$, one has the exact statements

$$F_0=\mathscr{F}_0\cos(\theta/2)-\mathscr{F}_3\sin(\theta/2), \tag{5.3a}$$

$$F_3=\mathscr{F}_0\sin(\theta/2)+\mathscr{F}_3\cos(\theta/2), \tag{5.3b}$$

$$F_1=\mathscr{F}_1\cos(\theta/2)+\mathscr{F}_2\sin(\theta/2), \tag{5.3c}$$

$$F_2=\mathscr{F}_2\cos(\theta/2)-\mathscr{F}_1\sin(\theta/2). \tag{5.3d}$$

In order to write approximate, "averaged" expressions for the lhs of equations (5.3), we now apply the technique of Sec. III, writing, e.g.,

$$\overline{F}_0=\overline{\mathscr{F}}_0\cos(\theta/2)-\overline{\mathscr{F}}_3\sin(\theta/2), \tag{5.4}$$

and similarly for the other lines of (5.3). Here the $\overline{\mathscr{F}}$ are constructed in terms of a $\rho(\mathscr{E}_{1,2})$ of the two-dimensional problem. Clearly, $\mathscr{E}=(\mathscr{E}_1^2+\mathscr{E}_2^2)^{1/2}=(E_1^2+E_2^2)^{1/2}$, and $\hat{\mathscr{E}}_1=\mathscr{E}_1/\mathscr{E}$, $\hat{\mathscr{E}}_2=\mathscr{E}_2/\mathscr{E}$.

The description is simplest using cylindrical coordinates; if we choose $E_1=E_T\cos(\omega t)$, $E_2=E_T\sin(\omega t)$, $E_3=E_L\cos(\nu t)$, then $\mathscr{E}_1=E_T\cos(\omega t-\theta)$, $\mathscr{E}_2=E_T\sin(wt-\theta)$. For simplicity, suppose again that E_L and E_T, as well as ω and ν, are all constants; then one immediately calculates

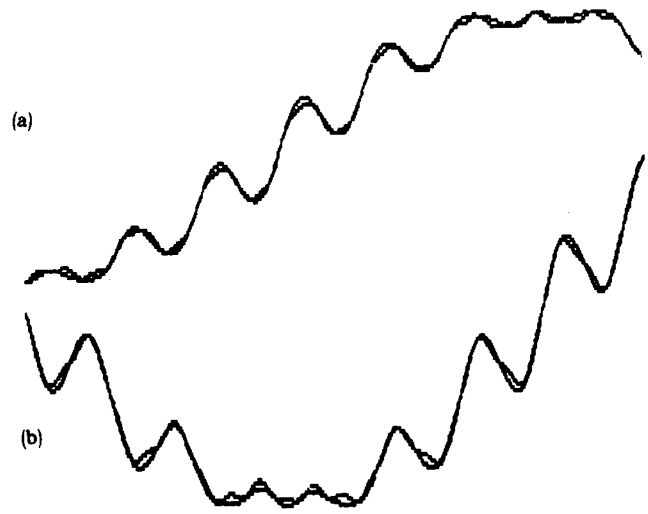

FIG. 13. Fine detail of the curves of Fig. 12, starting from $t = 0$.

$$\rho = |[\omega - 2E_L \cos(\nu t)]/E_T|, \tag{5.5}$$

exhibiting an explicitly time-dependent ρ, which will be used to calculate the $\mathscr{F}_{0,i}$. We again insist on the requirement

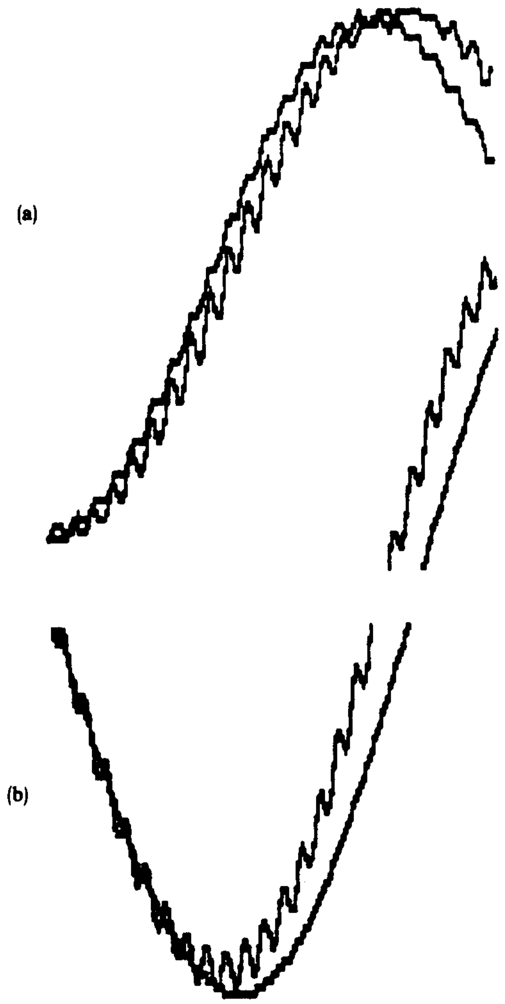

FIG. 14. Superpositions of (a) F_0 and $\overline{F}_0$, and (b) F_3 and $\overline{F}_3$; for $\omega = 60$, $\nu = 60$, $E_T = 10$, and $E_L = 5$.

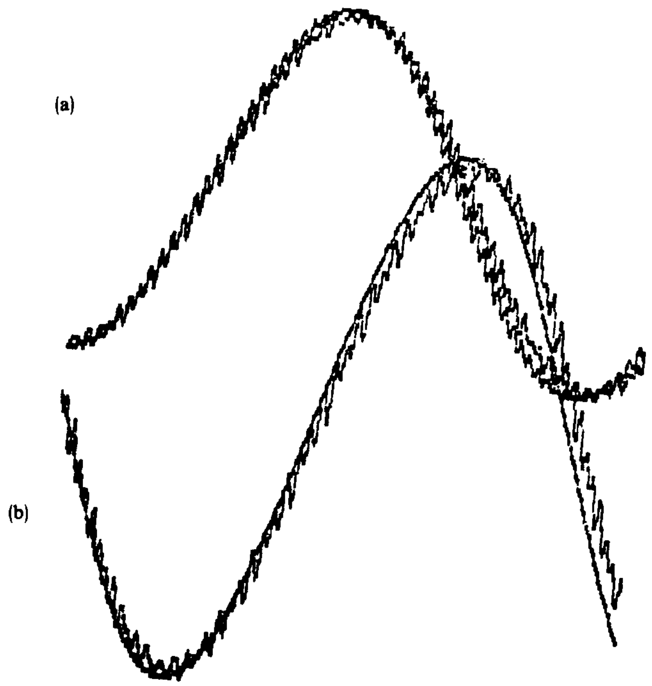

FIG. 15. Superpositions of (a) F_0 and $\overline{F}_0$, and (b) F_3 and $\overline{F}_3$; for $\omega = 60$, $\nu = 90$, $E_T = 10$, and $E_L = 5$.

$\rho \gg 1$, which condition governs the possible choices of ω, E_L, E_T.

Just how well the $\overline{F}_0$ and $\overline{F}_3$ reproduce the numerically integrated exact solutions can be seen from the examples of Figs. 12–15. For ω significantly larger or smaller than ν, the agreement is superb. For $\omega \sim \nu$ the agreement is less pleasing; but much of the discrepancy here seems to be tied up with the "in phase" errors made during the numerical computations. For example, a slowly forming phase lag gradually appearing between F_0 and $\overline{F}_0$, for $\omega = \nu$, is definitely diminished by using a finer time step in the numerical equations; however, we have not succeeded in completely removing this phase lag. This difficulty aside, which we believe is tied up with the deails of the numerical integration, it is difficult to be anything but enthusiatic over the quality of the results given by these "averaged" forms, using a three-dimensional input. Again, one finds that generalizations to time-dependent E_L, E_T, continue to be well represented by Eqs. (5.4), using in the computation of $\overline{\mathscr{F}}_{0,i}$ the elementary generalizations of Sec. III for time-dependent E,ω.

VI. FINE TUNING

Of all the qualitative agreements between the exact solutions and our "averaged" functions, only the agreement between $F_{1,2}$ and $\overline{F}_{1,2}$ is less than satisfactory, because the $\overline{F}_{1,2}$ of Sec. III miss the low-frequency behavior clearly visible in the $F_{1,2}$; this is illustrated in Fig. 8. As a practical matter, it is not important because the $F_{1,2}$ are of order $1/\rho$ and are small; but as a matter of principle one would like to be able to extract all the correct, slowly varying behavior.

The trouble resides in our neglect of the small, rapid oscillations of J and H, in Sec. III, because those neglected fast oscillations could themselves be combined with similar

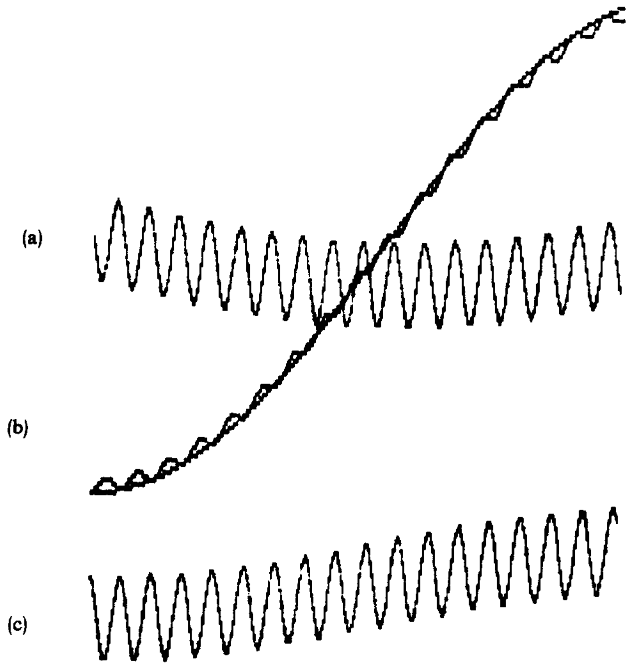

FIG. 16. Superpositions of (a) F_1 and $\overline{F}'_1$, and (b) F_0 and $\overline{F}_0$; (c) F_2 and $\overline{F}'_2$; for $\omega = 60$, $E = 10$.

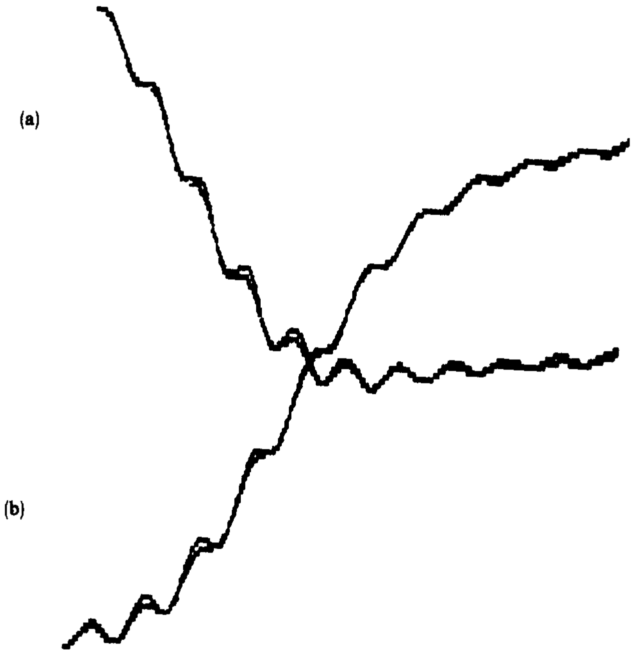

FIG. 18. Detail of the first shoulder for the superpositions of (a) F_0 and $\overline{F}'_0$, and (b) F_3 and $\overline{F}'_3$; for $\omega = 60$, $E(t) = 10 + 5\sin(5t)$.

oscillations appearing in the definition of the $F_{1,2}$, in (3.20) and (3.21), to generate terms independent of the rapid oscillations. To see this, denote by ξ and H_0 our previous choices of the constant and slowly varying J and H dependence, respectively; and then suppose that J and H shall each have a rapidly oscillating part of form

$$J = \xi + \xi\,[\cos L + \sin L \cot G\,], \tag{6.1a}$$

$$H = H_0 + \xi\,[\sin L - \cos L \cot G\,]. \tag{6.1b}$$

Imagine that there are constants, or slowly varying functions, α, β, γ, δ multiplying each of the $\sin L$, $\cos L$ terms in (6.1); and then imagine substituting (6.1) into the defining equations for $F_{1,2}$, to reproduce the $\overline{F}_{1,2}$ of (3.22) and (3.23) plus a part that has *only* a slowly varying time dependence. We denote by $\overline{F}'_{1,2}$ these new, improved functions, and find that we must choose $\alpha = \beta = \gamma = \delta = 1$, and then obtain

$$\overline{F}'_1 = \xi\,[\sin(G + L) + \sin G\,], \tag{6.2a}$$

$$\overline{F}'_2 = -\xi\,[\cos(G + L) - \cos G\,]. \tag{6.2b}$$

The agreement between (6.2) and the exact $F_{1,2}$ is so good that on the scale used in Fig. 16 there is no visible difference at all between them. Only when the scale is enlarged to show effects of order $1/\rho^2$ can one see superpositions of two curves.

These new values of J and H, given by (6.1), can now be used to define new $\overline{F}'_{0,3}$, which themselves are correct to order $1/\rho$. But it is much easier to use an argument suggested by unitarity, which requires

$$\overline{F}'^2_0 + \overline{F}'^2_3 + \overline{F}'^2_1 + \overline{F}'^2_2 = 1 + O(1/\rho^4) \tag{6.3}$$

if the new, "averaged" functions are to be correct to order $(1/\rho)$. For if we write

$$\overline{F}'_0 = \overline{F}_0 + \delta\overline{F}_0, \quad \overline{F}'_3 = \overline{F}_3 + \delta\overline{F}_3, \tag{6.4}$$

and substitute into (6.3), using the $\overline{F}'_{1,2}$ of (6.2) one obtains the relation

$$\delta\overline{F}_0 \cos G + \delta\overline{F}_3 \sin G + \xi^2[1 + \sin G \sin(G + L) - \cos G \cos(G + L)] = 0. \tag{6.5}$$

Rewriting the "1" coefficient of ξ, in (6.5) as $\sin^2 G + \cos^2 G$, and equating the coefficients of sin G and cos G, generates

$$\delta\overline{F}_0 = +\xi^2(\cos(G + L) - \cos G), \tag{6.6a}$$

$$\delta\overline{F}_3 = -\xi^2(\sin(G + L) + \sin G). \tag{6.6b}$$

The agreement between the $\overline{F}'_{0,3}$, given by (6.6) and (6.4) is extremely good, as displayed in Figs. 17 and 18.

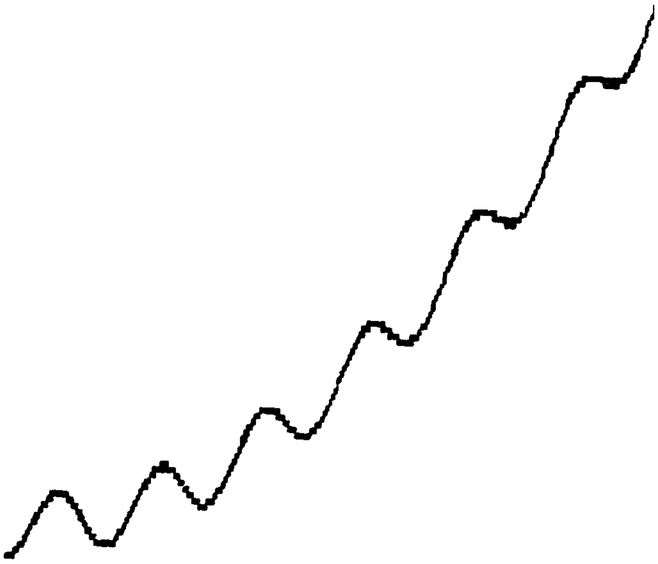

FIG. 17. Detail of the first shoulder of the superposition of F_0 and $\overline{F}'_0$; for $\omega = 60$, $E = 10$.

From this construction we infer that these $\bar{F}'_{0,3}$ are correct to order $1/\rho^2$, while the $\bar{F}'_{1,2}$ are correct to order $1/\rho$. Again, this "fine tuning" is probably irrelevant in any given physical application, but it is pleasing to be able to improve the accuracy of our "averaged" curves in such a simple way.

VII. SUMMARY AND FURTHER QUESTIONS

In this paper we have suggested some methods for the approximate estimation of SU(2) ordered exponentials in the SC limits, adiabatic and stochastic, and have compared the results to exact or machine statements when certain functional integrals are carried out using our approximate forms. Our derivations have been mainly intuitive; but there can be no argument raised against the results which those derivations provide, which nicely match the numerically integrated functions representing the exact ordered exponential in both the adiabatic and stochastic limits. As such, we expect that these approximations will be immediately useful in a variety of physical problems, whose dynamical content can be expressed, approximated, or modeled in terms of SU(2) ordered exponentials.

There are three main areas in which the analyses of this paper raise questions that are surely deserving of futher attention.

(1) Are there possible choices of $\mathbf{E}(t)$ for which the fixed-point equations (3.1) have a nontrivial solution?

(2) A thorough analysis should be made of the generalization to time-dependent $\rho(t)$. Would the result of this investigation show that the $\bar{F}_{0,1}$ are insensitive to the time dependence of ρ, as suggested by all of our examples; or will there be certain situations, certain forms for $\rho(t) \gg 1$, for which our results are invalid?

(3) Can our results be extended to $SU(N)$, $N > 2$?

It is not difficult to write the leading term of the adiabatic approximation for the case of $SU(N)$, but its corrections will surely be more complicated because of the more cumbersome statement of unitarity.[5] In the stochastic limit, on the other hand, the situation seems less well-defined, and the methods of Sec. III would appear to be hazardous and uncertain. In principle, the same techniques can be used; in practice, the greater number of functions $F_{0,i}$, $1 \leqslant i \leqslant N^2 - 1$, makes for a certain amount of confusion. Surely, the much greater number of physical problems that involve $SU(N)$, rather than SU(2), would make this a study of paramount interest.

ACKNOWLEDGMENTS

One of us (H.M.F) would like to express his warm thanks to U. Frisch and his colleagues, at the Observatoire de Nice where this work was begun, for useful criticism and the opportunity to use the necessary numerical facilities; and to colleagues at Physique Théorique, Université de Nice, where this paper was completed, for their very kind hospitality.

Much of this work was made possible by an NSF–CNRS Cooperative Research travel grant. H.M.F. would also like to acknowledge partial support by the U.S. Department of Energy under Contract No. DE-AC02-76A03130. A009—Task A.

[1] M. E. Brachet and H. M. Fried, Phys. Lett. A **103**, 309 (1984).
[2] L. Aspinall and J. C. Percival, Proc. R. Soc. **90**, 315 (1967); and references quoted therein.
[3] A. Dykhne, Sov. Phys. JETP **11**, 411 (1960); JETP Lett. **14**, 941 (1962).
[4] H. M. Fried, Phys. Rev. D **16**, 1916 (1977).
[5] H. M. Fried and J. Tessendorf, J. Math. Phys. **25**, 1144 (1984).
[6] H. M. Fried, "Strong coupling & the infrared structure of QCD," *Proceedings of the US–France Seminar "Theoretical Aspects of QCD,"* in Marseille, June, 1981.