The Kardar-Parisi-Zhang equation and its universality class

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One dimensional KPZ equation

The [Kardar-Parisi-Zhang 1986] equation is a nonlinear stochastic PDE describing the time evolution of a height h

$$\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}\big(\nabla h\big)^2 + \xi,$$

where ξ is a space-time white noise. In one spatial dimension, the function $h(x,t)$ satisfies

$$\partial_t h(x,t) = \frac{1}{2}\partial_{xx} h(x,t) + \frac{1}{2}\big(\partial_x h(x,t)\big)^2 + \xi(x,t).$$

[Kardar-Parisi-Zhang] postulated that a variety of phenomena modelled by the stochastic growth of a rough interface obey universal laws (same scaling exponents). The KPZ equation was introduced as a toy model.

Initially, it concerned models of deposition of material on a substrate, or phenomena of propagation. Over the years, the subject has grown to include more and more unexpected topics (cf talk of Jacqueline Bloch!).

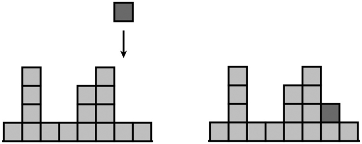
An example of deposition of material

An example of propagation phenomena

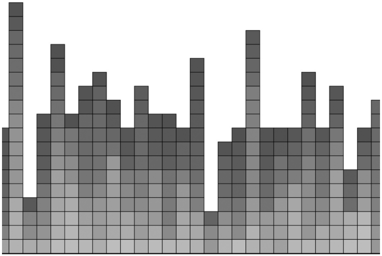
A mathematical model of deposition

Blocks fall on a one dimensional flat substrate:

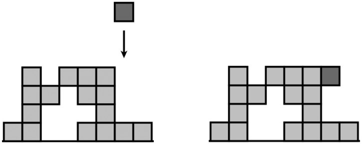
Independently on each column



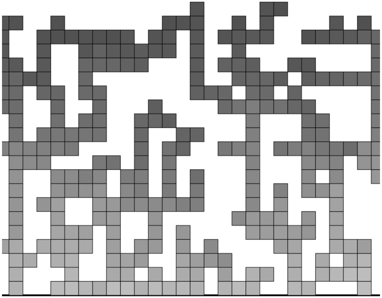
After some time



Blocks have sticky edges



After some time

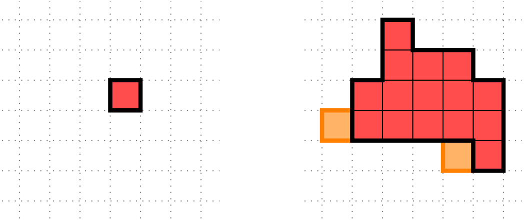


After a long time

Simulation by [T. Halpin-Healy]. The interface becomes quite smooth, with strong spatial correlations.

Mathematical analysis is an open problem.

A mathematical model of propagation



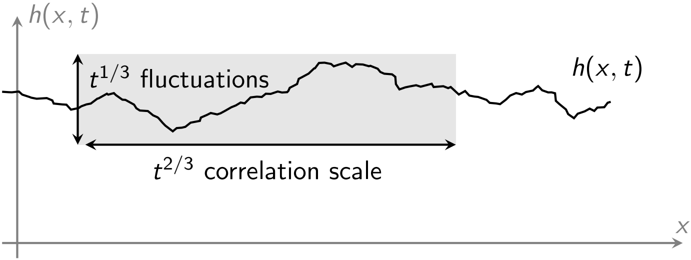
Eden model

- Start with a unit square in the $\mathbb{Z}^2$ lattice
- Add the squares in the border of the cluster randomly at exponential rate 1.

This model is also too complicated!

Scaling exponents

By renormalization group calculations from [Nelson-Forster-Stephen 1977], [Kardar-Parisi-Zhang] predicted universal exponents $(1/3, 2/3)$ for the one dimensional KPZ equation (roughness $\chi = 1/2$, dynamical exponent $z = 3/2$)



This suggests to consider the $1 : 2 : 3$ scaling, that is to study

$$\lim_{L \to +\infty} \left\{ \frac{1}{L} h(L^2 x, L^3 t) \right\},$$

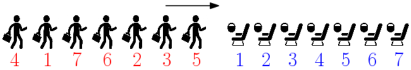
and one may ask about limiting distributions, large deviations, sensitivity to initial conditions, universality...
But the KPZ equation is not a very tractable model

How mathematicians came into the subject?

A seemingly unrelated mathematical problem

How long it takes to board a plane with n passengers?

For simplicity, we assume that the plane has only one seat per row, and that people need 1 minute to place their suitcase in the overhead bin compartment. Passengers $1, 2, \dots n$ queue up in line at the gate in a random order.



The total boarding time is the longest increasing subsequence (LIS) of the permutation

$\mathrm{LIS}(5326714) = 3$, because 5 6 7 or 3 6 7 or 2 6 7 are increasing subsequences.

Random permutations

For a random permutation σ of $\{1, 2, \ldots, n\}$ [Hammersley 1972, Logan-Shepp, Vershik-Kerov 1977]

$$\mathrm{LIS}(\sigma) \sim 2\sqrt{n}.$$

[Baik-Deift-Johansson ($\approx$ 1998)] found that

$$\mathbb{P}\left(\frac{\mathrm{LIS}(\sigma) - 2\sqrt{n}}{n^{1/6}} \leqslant s\right) \xrightarrow[n\to\infty]{} F_2(s).$$

Tracy-Widom distribution

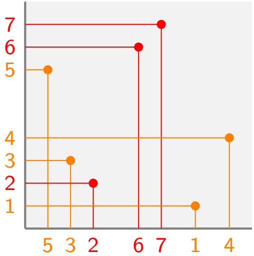
Let M be a $n \times n$ hermitian matrix with independent complex Gaussian entries. Then, the largest eigenvalue λ_1 is such that

$$\mathbb{P}\left(\frac{\lambda_1 - 2\sqrt{n}}{n^{1/6}} \leqslant s\right) \xrightarrow[n\to\infty]{} F_2(s),$$

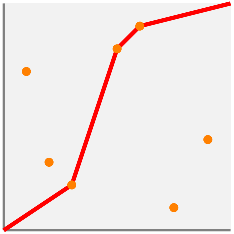
where F_2 is a cdf now called the [Tracy-Widom (1994)] distribution

A discrete analogue of the KPZ equation

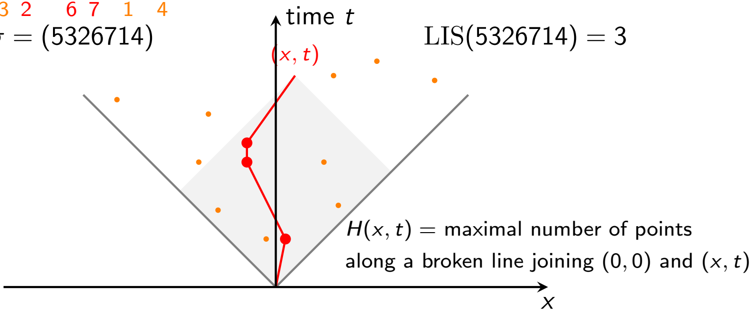
Let n points distributed uniformly on the square. Label them according to the vertical coordinate. The horizontal coordinate defines a random permutation.

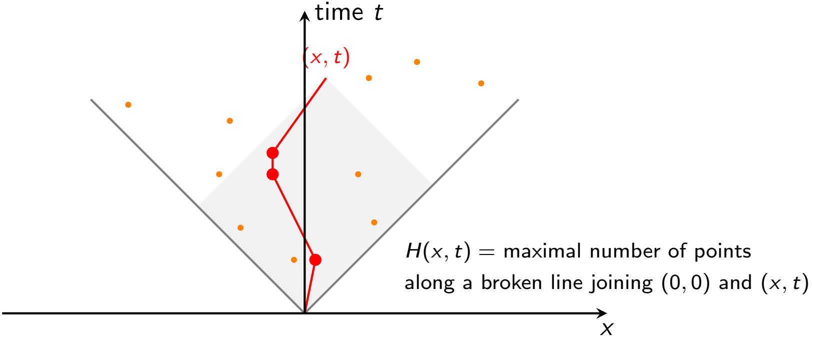


$\sigma = (5326714)$



$\mathrm{LIS}(5326714) = 3$





The result of [Baik-Deift-Johansson] says that

$$\mathbb{P}\left(\frac{H(0,t)-2t}{t^{1/3}} \leqslant s\right) \xrightarrow[n\to\infty]{} F_2(s).$$

The function $H(x,t)$ is a discrete analogue of the KPZ equation $h(x,t)$. A similar result is expected to hold for any model in the KPZ universality class.

What is universal in the KPZ universality class?

KPZ fixed point

For any model described by a height function $H(x,t)$, there should be a universal process $\mathfrak{h}(x,t)$, called the KPZ fixed point, such that

$$\frac{1}{L}\left(H(xL^2, tL^3) - f(L,x,t)\right) \xRightarrow[L\to\infty]{} \mathfrak{h}(x,t).$$

The KPZ fixed point satisfies the scale invariance, for all $\lambda > 0$,

$$\mathfrak{h}(x,t) \overset{(d)}{=} \lambda^{-1}\mathfrak{h}(\lambda^2 x, \lambda^3 t)$$

and was recently constructed [Matetski-Quastel-Remenik 2017] [Dauvergne-Ortmann-Virag 2018]. Note: $\mathfrak{h}(x,t) \neq h(x,t)$ (the KPZ equation is not scale invariant).

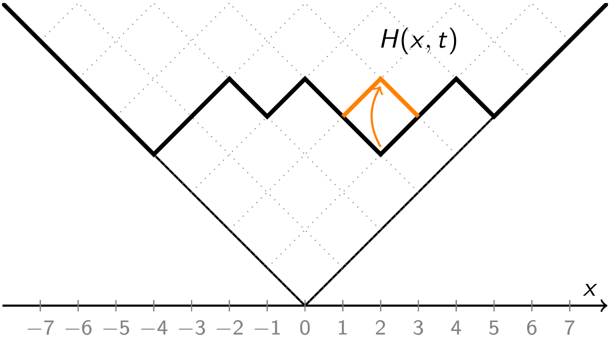
By the result of [Baik-Deift-Johansson], one should expect

$$\mathbb{P}\left(\mathfrak{h}(0,1) \leqslant s\right) = F_2(s),$$

as well as many other results about correlations, dependence on the initial data, regularity, obtained in the past decades.

An integrable model: corner growth model

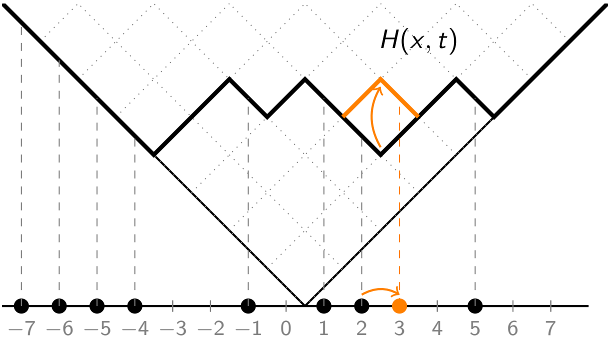
- Consider an interface $H(x,t)$, (where $x \in \mathbb{Z}$) starting from $H(x,0) = |x|$, and add unit boxes at rate 1 in every valley.



- The interface is mapped to an interacting particle system on $\mathbb{Z}$ called TASEP.

Corner growth model $\Longleftrightarrow$ TASEP

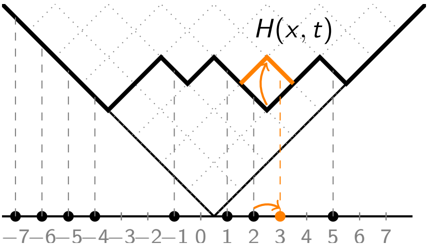
- Consider an interface $H(x,t)$, starting from $H(x,0) = |x|$, and add unit boxes at rate 1 in every valley.



- The interface is mapped to an interacting particle system on $\mathbb{Z}$ called totally asymmetric simple exclusion process.
- KPZ universality class also describes interacting particle systems, traffic models...

Construction of the KPZ fixed point

[Matetski-Quastel-Remenik 2017] defined the KPZ fixed point $\mathfrak{h}(x,t)$ as a Markov process on the space of upper semi-continuous functions, based on a scaling limit of TASEP



They showed that under the 1:2:3 scaling, for arbitrary functions $G_1(x), G_2(x)$ that rescale to some function $\mathfrak{g}_1(x), \mathfrak{g}_2(x)$,

$$\mathbb{P}\left(H(\cdot,t) \leqslant G_2 | H(\cdot,s) = G_1\right) \xrightarrow[\text{1:2:3 scaling limit}]{} \mathbb{P}\left(\mathfrak{h}(\cdot,t) \leqslant \mathfrak{g}_2 | \mathfrak{h}(\cdot,s) = \mathfrak{g}_1\right)$$

and proved that there exist a Markov process corresponding to the RHS.

Back to the KPZ equation

$$\partial_t h(x,t) = \tfrac{1}{2}\partial_{xx} h(x,t) + \tfrac{1}{2}\big(\partial_x h(x,t)\big)^2 + \xi(x,t)$$

Rigorous solution theories

When ξ is a space-time white noise, $\partial_x h(x,t)$ is not a function, it can only be understood as a distribution, thus $\big(\partial_x h(x,t)\big)^2$ is ill-defined.

Thus one mollifies the noise as $\xi^\epsilon = \xi * \phi$ for some $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, consider

$$\partial_t h^\varepsilon = \tfrac{1}{2}\partial_{xx} h^\varepsilon + \tfrac{1}{2}\big(\partial_x h^\varepsilon\big)^2 + \xi^\varepsilon,$$

and show that the solution h^ϵ converges as $\epsilon \to 0$, cf [Hairer 2011] [Gubinelli-Imkeller-Perkowski 2012]

It is more convenient to define a solution h as $h(x,t) = \log Z(x,t)$ where

$$\partial_t Z(x,t) = \tfrac{1}{2}\partial_{xx} Z(x,t) + Z(x,t)\xi(x,t).$$

that is

$$Z(x,t) = \int_{\mathbb{R}} dy Z_0(y) p_t(y,x) + \int_0^t ds \int_{\mathbb{R}} d_y p_{t-s}(y,x) Z(y,s)\xi(y,s)$$

where $p_t(y,x)$ is the standard heat kernel.

Directed polymer interpretation

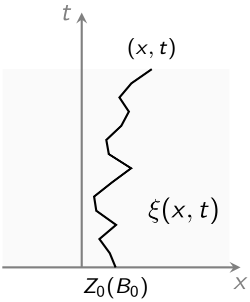
Via the Feynman-Kac formula, a solution of

$$\partial_t Z(x,t) = \tfrac{1}{2}\partial_{xx} Z(x,t) + Z(x,t)\xi(x,t).$$

with initial condition Z_0 can be written as

$$Z(x,t) = \mathbb{E}\left[Z_0(B_0) :\!\exp\!: \left(\int_0^t \xi(B_s,s)ds \right) \right]$$

where the expectation is taken over a Brownian motion B from B_0 to $B_t = x$.



The distribution of the solution

The probability distribution of $Z(0,t)$ is characterized by the Laplace transform formula

Theorem

$$\mathbb{E}\left[e^{-uZ(0,2t)e^{t/12}}\right] = \mathbb{E}\left[\prod_{i=1}^{k} \frac{1}{1+ue^{t^{1/3}\mathfrak{a}_i}}\right]$$

where $\mathfrak{a}_1 > \mathfrak{a}_2 > \dots$ *is the Airy point process.*

Airy point process

It describes the scaling limit of largest eigenvalues of hermitian random matrices.

Let M be a $n \times n$ Hermitian matrix with independent complex Gaussian entries with eigenvalues $\lambda_1 > \lambda_2 > \dots > \lambda_n$. Then,

$$\left(\frac{\lambda_i - 2\sqrt{n}}{n^{1/6}}\right)_{i \geqslant 1} \xRightarrow[n\to\infty]{} (\mathfrak{a}_i)_{i\geqslant 1}.$$

Detailed knowledge of eigenvalue statistics transfers to the KPZ equation

Equivalently

$$\mathbb{E}\left[e^{-uZ(0,2t)e^{t/12}}\right] = \det(I - \sigma_u K)_{\mathbb{L}^2(\mathbb{R}_+)}$$

where $\sigma_u(x) = \frac{1}{1+ue^{t^{1/3}x}}$ and K is an operator on $\mathbb{L}^2(\mathbb{R}_+)$ with kernel

$$K(x,y) = \int_0^\infty \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)dz,$$

the so-called Airy kernel.

Proof ($\approx$ 2008):

- In Physics: [Calabrese-Le Doussal-Rosso][Dotsenko] via Replica method + Bethe ansatz
- In Maths: [Amir-Corwin-Quastel][Sasamoto-Spohn] via [Tracy-Widom] Bethe ansatz solution of ASEP

Corollary (by the same groups of authors $\approx$ 2008)

Recalling that $h(x,t) = \log Z(x,t)$, one can deduce

$$\mathbb{P}\left(\frac{h(0,2t) - t/12}{t^{1/3}} \leqslant s\right) \xrightarrow[n\to\infty]{} F_2(s).$$

Stationary measures of the KPZ equation

The KPZ equation is modeling out of equilibrium systems. So, it should not have admit true stationary measure. Actually, we saw that $h(0,t) \sim \frac{-t}{24}$, which clearly diverges.

Definition (Non-equilibrium steady-state)

We say that the law of a process $h^{\mathrm{stat}}(x)$ is stationary for the KPZ equation when the following holds:
If $h(x,0) = h^{\mathrm{stat}}(x)$, then for all $t > 0$,

$$h(t,x) - h(t,0) \overset{(d)}{=} h^{\mathrm{stat}}(x) - h^{\mathrm{stat}}(0).$$

For the KPZ equation on $\mathbb{R}$, the Brownian motion with drift μ ($\mu \in \mathbb{R}$ can be arbitrary) is stationary for the KPZ equation [Forster-Nelson-Stephen 1977, Bertini-Giacomin 1997, Funaki-Quastel 2014].

Stationary measures of stochastic PDEs

This stationarity of the Brownian motion is far from obvious!

- (Linear case) For stochastic PDEs of the form

$$\partial_t u = Lu + \xi$$

  where L is a linear differential operator, stationary measures are Gaussian and there exists a general theory.

- (Equilibrium case) The path integral measure

$$e^{-S[\varrho]}\mathcal{D}\varrho$$

  is the stationary measure for the equation

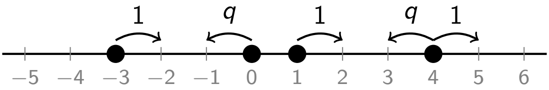
$$\partial_t u = -\frac{\delta S[u]}{\delta u} + \sqrt{2}\xi.$$

  [Nelson 1966, Parisi-Wu 1981]

- The KPZ equation is non linear and out of equilibrium.

ASEP

ASEP (asymmetric simple exclusion process) is a continuous Markov process on $\{0,1\}^{\mathbb{Z}}$, whose transition rates depend on an asymmetry parameter q.



- For any $\varrho \in [0,1]$, i.i.d. Bernoulli(ϱ) is a stationary measure.
- Define a height function $H(x,t)$ so that

$$H(x,t) - H(x-1,t) = \begin{cases} 1 & \text{if site } x \text{ is occupied.} \\ -1 & \text{if site } x \text{ is empty.} \end{cases}$$

and $H(0,t)$ is the number of particles which have crossed the origin.

Convergence ASEP $\to$ KPZ

Theorem ([Bertini-Giacomin 1997])

Let $\mathcal{Z}_t(x) = q^{\frac{1}{2}H(x,t)-\nu t}$*, where* $\nu = (1-\sqrt{q})^2$*. For* $q = e^{-\varepsilon}$*, when* $\varepsilon \to 0$

$$\mathcal{Z}_{\epsilon^{-4}t}(\epsilon^{-2}x) \Longrightarrow Z(x,t),$$

the solution of

$$\partial_t Z = \tfrac{1}{2}\Delta Z + Z\,\xi.$$

ASEP height function converges to a solution of KPZ equation.

When occupation variables are i.i.d. Bernoulli, ASEP's height function converges to a Brownian motion (with drift), up to a global shift.

Corollary ([Bertini-Giacomin 1997])

For any drift $\mu \in \mathbb{R}$*, the Brownian motion* $B_x^{(\mu)}$ *is stationary*

KPZ equation on a segment

The one dimensional KPZ equation can be considered on $\mathbb{R}$, but also on $\mathbb{R}/\mathbb{Z}$, $[0, L]$, $\mathbb{R}_+$...

Consider the KPZ equation on the segment $[0, L]$,

$$\partial_t h(x,t) = \tfrac{1}{2}\partial_{xx} h(x,t) + \tfrac{1}{2}\big(\partial_x h(x,t)\big)^2 + \xi(x,t), \;\; x \in [0, L].$$

For the solution to be unique, one needs to impose boundary conditions. It is natural to impose a Newman type condition

$$\partial_x h(0,t) = u, \quad \partial_x h(L,t) = -v,$$

where $u, v \in \mathbb{R}$ are two real parameters.

Physically, $\partial_x h$ corresponds to the density in ASEP but $h(x,t)$ is not differentiable, so some care is needed to define the model.

Stationary measures on $[0, L]$

On a segment, the KPZ equation stationary measures are not simply Brownian:

Theorem

For any $u, v \in \mathbb{R}$, there exists a unique stationary process $h^L_{u,v}(x)$ with law

$$h^L_{u,v}(x) \overset{(d)}{=} W(x) + X(x)$$

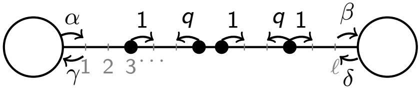
where W is a Brownian motion on $[0, L]$ and X is a reweighted Brownian motion

$$\frac{d\mathbb{P}(X)}{d\mathrm{Brownian}} = e^{-vX(L)} \left(\int_0^L e^{-X(s)} ds \right)^{-u-v}.$$

- When $u + v = 0$, $h^L_{u,v}$ is a Brownian motion with drift $-v$.
- When $u, v \to +\infty$, $h^L_{u,v}$ is the sum of a Brownian motion and a Brownian excursion, similar to [Derrida-Enaud-Lebowitz 2004].
- Letting $L \to \infty$, one obtains stationary measures for the KPZ equation on $\mathbb{R}_+$ [B.-Le Doussal 2021][B.-Corwin 2022].

Proof

- The first proof was restricted to the case $u + v \geqslant 0$ and involved
  - The characterization of open ASEP stationary measure via the matrix product ansatz [Derrida-Evans-Hakim-Pasquier 1993]



  - A representation of the matrix product ansatz [Uchiyama-Sasamoto-Wadati 2004] and its relation to Askey-Wilson processes [Bryc-Wesolowski 2015]
  - [Corwin-Knizel 2021] Proved existence, and characterized the distribution through (complicated) formula for the distribution
  - [Bryc-Kuznetsov-Wang-Wesołowski 2022] and [B.- Le Doussal 2022] worked out inversion.
  - [Matetski-Knizel 2023, Parekh 2023] Uniqueness of the stationary process using ideas of [Hairer-Mattingly 2015]
- There exists a simpler derivation, still restricted to $u + v \geqslant 0$ [B.-Le Doussal 2022], using ideas from [Derrida-Enaud-Lebowitz 2004].
- And more recently, for any u, v, yet another method, inspired by symmetric functions theory rather than the matrix product ansatz [B.-Corwin 2022], [B.-Corwin-Yang 2023]

Liouville quantum mechanics

The reason for the restriction to $u + v \geqslant 0$ in most works is that the process $X(x)$ is written as

$$X(x) = Y(x) - Y(0),$$

where Y is a reweighted Brownian motion

$$\frac{d\mathbb{P}(Y)}{d\mathrm{Brownian}} = \exp\left(uY(0) - vY(L) - \int_0^L e^{-Y(s)} ds\right).$$

The process Y can only be defined for $u + v > 0$. In terms of path integral,

$$\mathbb{P}(Y) = \exp\left(-uY(0) - vY(L) - \int_0^L e^{-Y(s)} ds - \int_0^L \left(\frac{dY(s)}{ds}\right)^2 ds\right) \mathcal{D}(Y)$$

This is a one dimensional analogue of Liouville field theory. The initial proof of the theorem came from recognizing Liouville quantum mechanics in exact formulas...

Another type of models in the KPZ class

Random walks in random environment

Let X_t be a random walk on $\mathbb{Z}$, starting from 0, such that when $X_t = x$,

$$X_{t+1} = \begin{cases} x+1 \text{ with probability } p_{x,t}, \\ x-1 \text{ with probability } 1-p_{x,t}. \end{cases}$$

If $p_{x,t} \equiv 1/2$, the model is well-understood. If $p_{x,t}$ are disordered, say independent and uniform in $(0,1)$, then

Theorem (B.-Corwin 2015)

Consider n independent walks in the same environment $X_t^{(1)}, \dots, X_t^{(n)}$, then for $n = e^{ct}$, $c \in (0,1)$,

$$\mathbb{P}\left(\frac{\max_{i\in\{1,\dots,n\}} - t\sqrt{c(2-c)}}{\sigma(c) t^{1/3}} \leqslant s \right) \xrightarrow[t\to\infty]{} F_2(s).$$

The statement can be rephrased in terms of large deviations as

$$-\log\left(\mathbb{P}\left(X_t > xt \middle| \{p_{y,s}\}\right)\right) \approx I(x)\cdot t + c'' \cdot t^{1/3} \cdot \chi$$

where $\mathbb{P}(\chi \leqslant s) = F_2(s)$.

Random walks in random environment

Consider n independent walks in the same environment $X_t^{(1)}, \dots, X_t^{(n)}$. If one lets $t \to \infty$ and then take n large, the maximum would behave as the maximum of Gaussian variables, following well known extreme value statistics (and have much smaller fluctuations).

$$\lim_{t\to\infty} \max_{i\in\{1,\dots,n\}} \left\{ \frac{X_t^{(i)}}{\sqrt{t}} \right\} \approx \sqrt{\log(n)} + \frac{1}{\sqrt{2\log(n)}} \left(G - \frac{1}{2} \log\log(n) \right),$$

where G follows the Gumbel distribution.

Theorem ([B.-Le Doussal 2019])

Consider n independent walks in the same environment $X_t^{(1)}, \dots, X_t^{(n)}$ and scale $n = e^{\sqrt{t\tau}}$,

$$\max_{i\in\{1,\dots,n\}} \left\{ \frac{X_t^{(i)}}{\sqrt{t}} \right\} \approx \sqrt{\log(n)} + \frac{1}{\sqrt{2\log(n)}} \left(G + h(0,\tau) - \frac{1}{2} \log\log(n) \right)$$

where G follows the Gumbel distribution and $h(0,\tau)$ is distributed as in the KPZ equation.

Thank you