Stationary measures for integrable models on a strip

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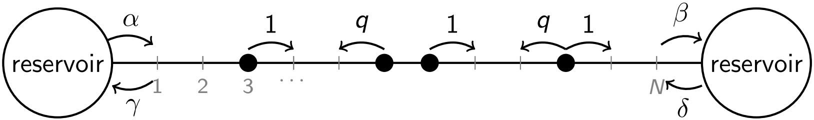
Stationary measures of open boundary systems

The stationary measure of ASEP with boundary parameters $\alpha, \beta, \gamma, \delta$ can be obtained using **Matrix Product Ansatz (MPA)**



The stationary measure $\mathbb{P}(\eta), \eta \in \{0, 1\}^N$ can be written as [Derrida-Evans-Hakim-Pasquier 1993]

$$\mathbb{P}(\eta) = \frac{1}{Z_N} \langle w| \prod_{i=1}^{N} (\eta_i D + (1 - \eta_i) E) |v\rangle$$

provided the matrices E, D and the vectors $\langle w|, |v\rangle$ satisfy

$$DE - qED = D + E$$
$$\langle w| (\alpha E - \gamma D) = \langle w|$$
$$(\beta D - \delta E) |v\rangle = |v\rangle$$

and $Z_N = \langle w| (E + D)^N |v\rangle < \infty$.

Stationary measures of open boundary systems

The MPA applies as well for systems with multiples species and other interacting particle systems.

For quantum systems, it is common to search for ground states in the form of matrix product states, or more general tensor networks.

MPA representations are related to

- Askey-Wilson orthogonal polynomials [Uchiyama-Sasamoto-Wadati 2004],
- Combinatorial structures, e.g. staircase tableaux [Corteel-Williams 2010],
- Algebraic Bethe Ansatz
- Families of symmetric functions [Cantini-De Gier-Wheeler 2015].

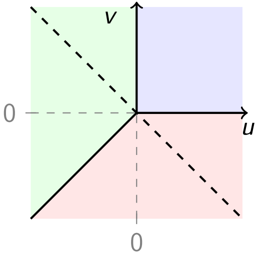
In this talk

Joint work with **Ivan Corwin and Zongrui Yang**

For models related to Schur or Macdonald processes, and integrable stochastic vertex models, stationary measures can be written as a marginal of a Gibbs measure on a larger state space, related to the branching structure of underlying symmetric functions.

This method

- yields a probabilistic description, suitable for large scale asymptotics,
- allows the full range of boundary parameters (shock phase),
- works well for models on infinite state spaces, even continuous.



Plan

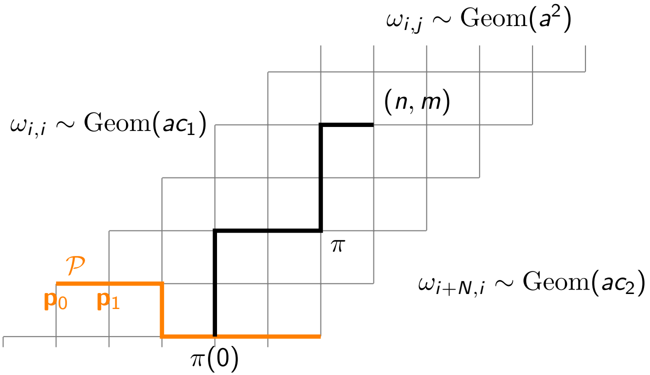
1. Illustrate the method for the simplest model: Last Passage Percolation on a strip (Schur case)

2. Extensions to the log-gamma polymer (Whittaker) and the KPZ equation (scaling limit)

3. Connection with Matrix Product Ansatz.

Geometric LPP on a strip

Consider the strip $\{(x,y) \in \mathbb{Z}^2; y \leqslant x \leqslant y+N\}$.

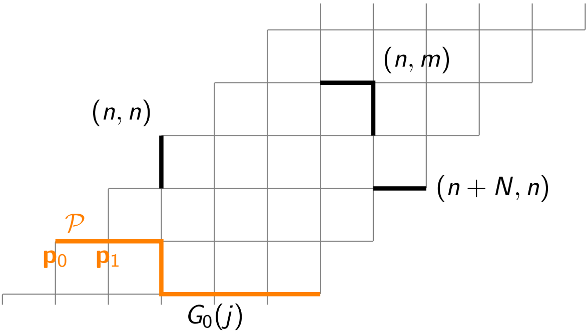


Fix a reference down-right path (orange) $\mathcal{P} = (\mathbf{p}_0, \dots, \mathbf{p}_N)$, and some initial condition $G(\mathbf{p}_j) = G_0(j)$.

Define the Last Passage Percolation time

$$G(n,m) = \max_{\pi:\mathcal{P}\to(n,m)} \left\{ G(\pi(0)) + \sum_{(i,j\in\pi)} w_{i,j} \right\}$$

Recurrence relation



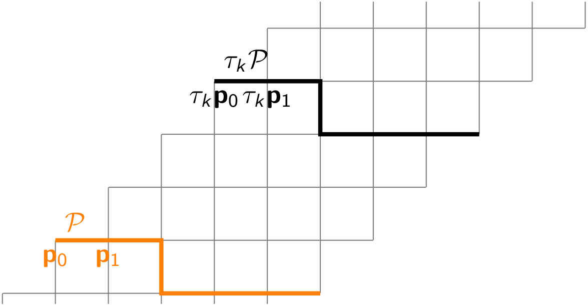
It satisfies the recurrence

$$G(n,m) = \omega_{n,m} + \begin{cases} \max(G(n-1,m), G(n,m-1)) & \text{if } 0 < m < n < m+N, \\ G(n,m-1) & \text{if } n = m, \\ G(n-1,m) & \text{if } n = m+N, \end{cases}$$

On the path $\mathcal{P} = (\mathbf{p}_0, \ldots, \mathbf{p}_N)$, $G(\mathbf{p}_j) = G_0(j)$.

Stationary measures

Denote by τ_k the translation by (k, k).



Consider the **increments** process

$$G_k(j) := \big(G(\tau_k \mathbf{p}_j) - G(\tau_k \mathbf{p}_0)\big)_{1 \leqslant j \leqslant N}$$

This defines a discrete time Markov process on $\mathbb{Z}^N$ (infinite state space).

Definition

The law of G_0 is stationary if the law of G_k is the same for all k.

Stationary measure for Geometric LPP

Assume that $\mathcal{P}$ is horizontal (for simplicity).

Let $\big(L_1(j)\big)_{0\leqslant j\leqslant N}$ and $\big(L_2(j)\big)_{0\leqslant j\leqslant N}$ be random walks starting from $L_1(0) = L_2(0) = 0$ with i.i.d increments

$$L_i(j) - L_i(j-1) \sim \mathrm{Geom}(a).$$

Denote by $\mathbb{P}_{\mathrm{RW}}$ the associated probability measure.

Define the **reweighted random walk** measure

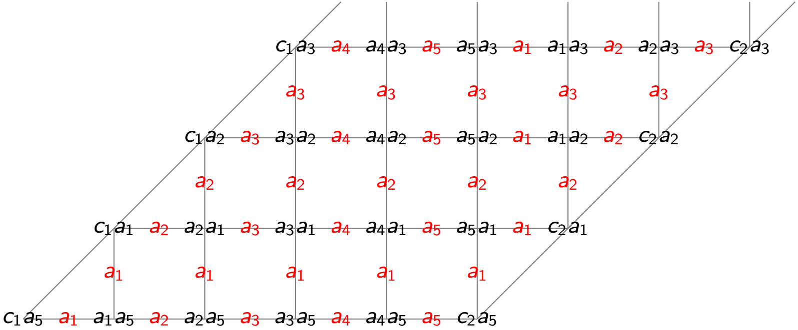
$$\mathbb{P}^{a,c_1,c_2}_{\mathrm{Geo}}(\mathbf{L}_1, \mathbf{L}_2) := \frac{1}{\mathcal{Z}^{a,c_1,c_2}_{\mathrm{Geo}}} \, (c_1c_2)^{\max_{1\leqslant j\leqslant N}\big(L_2(j)-L_1(j-1)\big)} c_2^{L_1(N)-L_2(N)} \mathbb{P}_{\mathrm{RW}}(\mathbf{L}_1, \mathbf{L}_2),$$

Theorem ([B.-Corwin-Yang 2023])

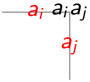
For any parameters a, c_1, c_2, the law of $\mathbf{L}_1$ under $\mathbb{P}^{a,c_1,c_2}_{\mathrm{Geo}}$ is the unique stationary measure.

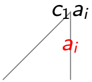
Inhomogeneity parameters

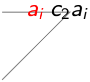
Let $a_1, a_2, \ldots, a_N$ be inhomogeneity parameters, assigned to the horizontal and vertical edges in a periodic way



We assume now that the weights are

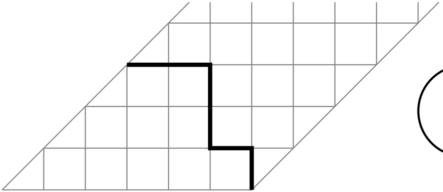


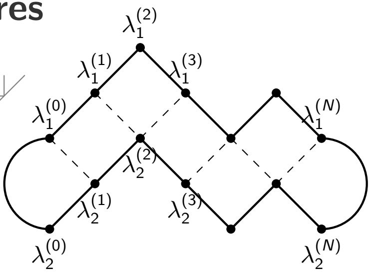




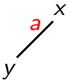
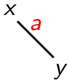
$w_{i,j} \sim \mathrm{Geom}(a_i a_j)$ $\quad$ $w_{i,i} \sim \mathrm{Geom}(c_1 a_i)$ $\quad$ $w_{i+N,i} \sim \mathrm{Geom}(c_2 a_i)$

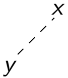
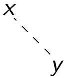
Two-layer Gibbs measures

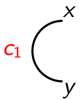
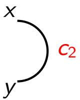




We assign a weight $\mathrm{wt}(\boldsymbol{\lambda})$ taking the product of Boltzmann weights

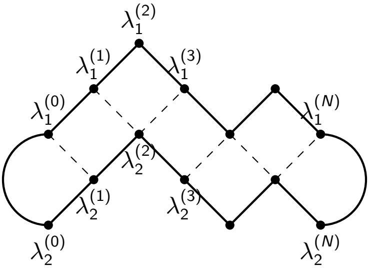
wt () = wt () $= a^{x-y}\mathbb{1}_{x\geqslant y}$

wt () = wt () $= \mathbb{1}_{x\geqslant y}.$

wt () $= c_1^{x-y}$, wt () $= c_2^{x-y}$,

Two-layer Gibbs measure

$\mathrm{wt}(\boldsymbol{\lambda})$ defines an **infinite measure** on $\mathbb{Z}^{2N+2}$ (because $\mathrm{wt}(\boldsymbol{\lambda})$ is invariant by translation).



When $c_1 c_2 < 1$, for $\lambda_1^{(0)}$ fixed,

$$Z = \sum_{\boldsymbol{\lambda} \setminus \{\lambda_1^{(0)}\} \in \mathbb{Z}^{2N+1}} \mathrm{wt}(\boldsymbol{\lambda}) < \infty$$

so that $\mathrm{wt}(\boldsymbol{\lambda})$ defines a probability measure on differences

$$\lambda_i^{(x)} - \lambda_1^{(0)}.$$

Infinite Schur measure

$$\mathrm{Sign}_n := \{\lambda \in \mathbb{Z}^n, \lambda_1 \geqslant \ldots \geqslant \lambda_n\}.$$

Define skew Schur functions by

$$s_{\lambda/\mu}(a) = \mathbb{1}_{\mu \prec \lambda} a^{|\lambda|-|\mu|},$$

where $|\lambda| = \sum_{i=1}^{n} \lambda_i$, slightly extending the usual definition. Then,

$$\mathrm{wt}(\boldsymbol{\lambda}) = c_1^{\lambda_1^{(0)}-\lambda_2^{(0)}} s_{\lambda^{(1)}/\lambda^{(0)}}(a_{i_1}) \times s_{\lambda^{(2)}/\lambda^{(1)}}(a_{i_2}) \times \cdots \times c_2^{\lambda_1^{(N)}-\lambda_2^{(N)}}$$

The structure is similar with the (Pfaffian) Schur measure [Borodin-Rains 2005] on partitions and the free boundary Schur measure [Betea-Bouttier-Nejjar-Vuletic 2017], except that

- The measure is infinite!
- The signatures are not necessarily nonnegative and have all length 2.

We will need two summation identities for these Schur functions indexed by signatures.

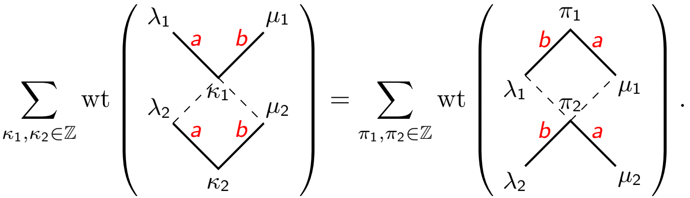
(I) (skew) Cauchy identity

One may extend the definition of Schur functions through the branching rule.
For any $\mathbf{a} = \{a_1, \dots, a_k\}$, $\mathbf{b} = \{b_1, \dots, b_k\}$ with $|a_i b_j| < 1$,

$$\sum_{\kappa \in \mathsf{Sign}_n} s_{\lambda/\kappa}(\mathbf{a}) s_{\mu/\kappa}(\mathbf{b}) = \sum_{\pi \in \mathsf{Sign}_n} s_{\pi/\lambda}(\mathbf{b}) s_{\pi/\mu}(\mathbf{a}),$$

This is different from the usual Cauchy identity. There is no normalization $\prod \frac{1}{1-a_i b_j}$.

Pictorially,



Other Cauchy identities

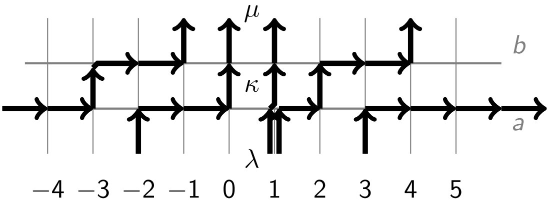
Identities of the form

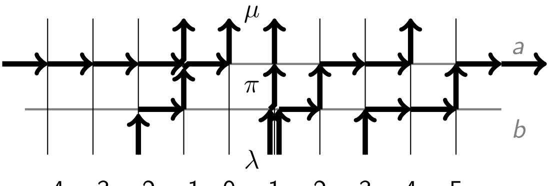
$$\sum_{\kappa \in \mathrm{Sign}_n} f_{\lambda/\kappa}(\mathbf{a}) g_{\mu/\kappa}(\mathbf{b}) = \sum_{\pi \in \mathrm{Sign}_n} g_{\pi/\lambda}(\mathbf{b}) f_{\pi/\mu}(\mathbf{a}),$$

were proved for

- Hall-Littlewood polynomials [Bufetov-Matveev 2018]
- Spin Hall-Littlewood functions [Bufetov-Petrov 2019]

Such identity can be proved using a Yang-Baxter zipper argument.





(II) (skew) Littlewood identity

Let

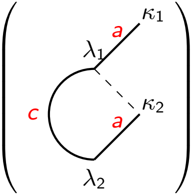
$$\tau_\lambda(c) := c^{\sum_{j=1}^{n}(-1)^{j-1}\lambda_j} = c^{\lambda_1-\lambda_2+\lambda_3-\lambda_4+\dots}.$$

For even n, and parameters such that $|a_i a_j| < 1$ and $|a_i c| < 1$,

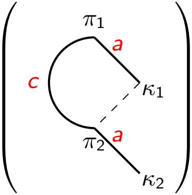
$$\sum_{\lambda \in \mathrm{Sign}_n} \tau_\lambda(c) s_{\kappa/\lambda}(\mathbf{a}) = \sum_{\pi \in \mathrm{Sign}_n} \tau_\pi(c) s_{\pi/\kappa}(\mathbf{a}),$$

Pictorially, for any fixed κ

$$\sum_{\lambda_1,\lambda_2 \in \mathbb{Z}} \mathrm{wt}$$



$$= \sum_{\pi_1,\pi_2 \in \mathbb{Z}} \mathrm{wt}$$



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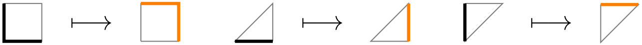
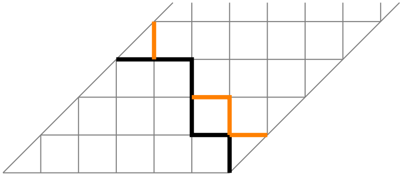
Dynamics

We will define dynamics of $\boldsymbol{\lambda}$ corresponding to evolving paths by the elementary moves





One needs to show that

- There exists Markov dynamics $\boldsymbol{\lambda} \to \boldsymbol{\lambda}'$ such that

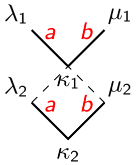
$$\sum_{\boldsymbol{\lambda}} \mathrm{wt}^{\mathcal{P}}(\boldsymbol{\lambda}) \mathbb{P}(\boldsymbol{\lambda} \to \boldsymbol{\lambda}') = \mathrm{wt}^{\mathcal{P}'}(\boldsymbol{\lambda}')$$

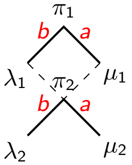
- The marginal on $\boldsymbol{\lambda}_1$ of those dynamics corresponds to last passage percolation as one evolves the down-right path from $\mathcal{P}$ to $\mathcal{P}'$.

Push-block dynamics

For bulk moves, we want transition probabilities $\mathcal{U}^{\sqcup}(\pi|\lambda,\kappa,\mu)$ such that

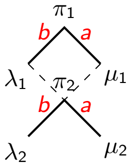
$$\sum_{\kappa\in\mathsf{Sign}_2} \mathcal{U}^{\sqcup}(\pi|\lambda,\kappa,\mu)\,\mathrm{wt}\left(\right) = \mathrm{wt}\left(\right)$$

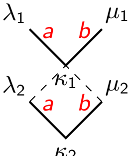




Assuming $\mathcal{U}^{\sqcup}(\pi|\lambda,\kappa,\mu; a, b)$ does not depend on κ, there is a unique solution (similar to push-block dynamics [Borodin-Ferrari 2008])

$$\mathcal{U}^{\sqcup}(\pi|\lambda,\mu) = \frac{\mathrm{wt}\left(\right)}{\sum_{\kappa\in\mathsf{Sign}_2} \mathrm{wt}\left(\right)}$$





First layer marginals

$\mathcal{U}^{\sqcup}(\pi|\lambda,\mu)$ is a valid transition matrix thanks to the Cauchy identity.

Similarly, one defines transition matrices $\mathcal{U}^{\ulcorner}(\pi|\kappa)$ and $\mathcal{U}^{\lrcorner}(\pi|\kappa)$ using the Littlewood identity.

The marginal dynamics of the first layer are Markov, and the Schur weights and boundary weights are chosen precisely so that

$$\mathcal{U}^{\sqcup}(\pi_1|\lambda_1,\mu_1) \propto (ab)^{\pi_1-\max\{\lambda_1,\mu_1\}}\mathbb{1}_{\pi_1\geqslant\max\{\lambda_1,\mu_1\}},$$

$$\mathcal{U}^{\lrcorner}(\pi_1|\kappa_1) \propto (ac_1)^{\pi_1-\kappa_1}\mathbb{1}_{\pi_1\geqslant\kappa_1}$$

$$\mathcal{U}^{\ulcorner}(\pi_1|\kappa_1) \propto (ac_2)^{\pi_1-\kappa_1}\mathbb{1}_{\pi_1\geqslant\kappa_1}$$

first layer marginal dynamics correspond to geometric LPP

Final trick

Let

$$\mathbf{L}_i(x) = \lambda_i^{(x)} - \lambda_i^{(0)},$$

and $\Delta = \lambda_1^{(0)} - \lambda_2^{(0)}$.

All the differences $\lambda_i^{(x)} - \lambda_1^{(0)}$ are encoded by $\mathbf{L}_1$, $\mathbf{L}_2$ and Δ but we are interested in the marginal law of $\mathbf{L}_1$.

An simple calculation shows that

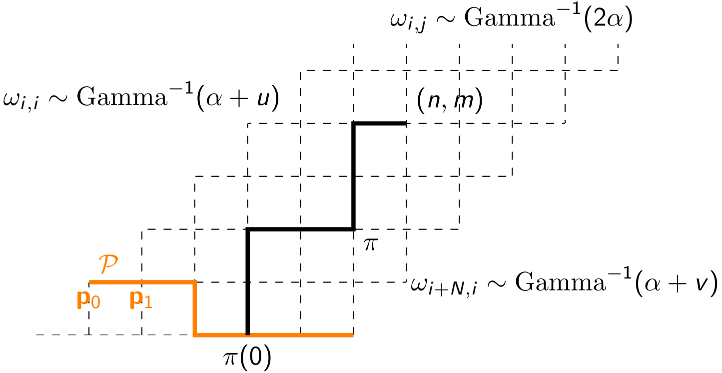
$$\sum_{\Delta} \mathrm{wt}(\boldsymbol{\lambda}) \propto (c_1 c_2)^{\max_{1\leqslant j\leqslant N}\left(L_2(j)-L_1(j-1)\right)} c_2^{L_1(N)-L_2(N)} \mathbb{P}_{\mathrm{RW}}(\mathbf{L}_1, \mathbf{L}_2)$$

where $\mathbb{P}_{\mathrm{RW}}(\mathbf{L}_1, \mathbf{L}_2)$ denotes geometric random walk probabilities.
This proves the theorem for $c_1 c_2 < 1$.

For $c_1 c_2 > 1$, the two-layer Gibbs measure is useless, but the probability measure on $\mathbf{L}_1$ still makes sense. The stationarity can be obtained by analytic continuation.

Other models

Log-gamma polymer on a strip

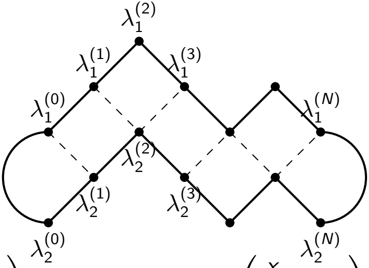


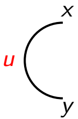
Fix some initial condition $h(\mathbf{p}_j) = h_0(j)$. Define the free energy

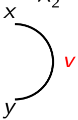
$$h(n, m) = \log \sum_{\pi:\mathcal{P}\to(n,m)} \left\{ e^{h(\pi(0))} \prod_{(i,j\in\pi)} w_{i,j} \right\}$$

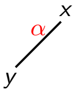
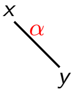
It can also be defined by a recurrence relation as for LPP where $(\max, +)$ becomes $(+, \times)$.

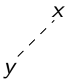
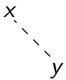
Two-layer Gibbs measure: Whittaker case



wt () $= e^{-u(x-y)}$,

wt () $= e^{-v(x-y)}$,

wt () = wt () $= e^{-\alpha(x-y)-e^{-(x-y)}}$

wt () = wt () $= e^{-e^{-(x-y)}}$

Log-gamma polymer stationary measure

Assume now that $\mathbf{L}_1, \mathbf{L}_2$ are random walks with

$$L_i(j) - L_i(j-1) \sim \log \mathrm{Gamma}^{-1}(\alpha).$$

Define the **reweighted random walk** measure $\mathbb{P}^{\alpha,u,v}_{\mathrm{LG}}$ as

$$\mathbb{P}^{\alpha,u,v}_{\mathrm{LG}}(\mathbf{L}_1, \mathbf{L}_2) := \frac{1}{\mathcal{Z}^{\alpha,u,v}_{\mathrm{LG}}} \left(\sum_{j=1}^{N} e^{L_2(j) - L_1(j-1)} \right)^{-(u+v)} e^{-v(L_1(N) - L_2(N))} \mathbb{P}^{\alpha,\alpha}_{\mathrm{RW}}(\mathbf{L}_1, \mathbf{L}_2),$$

Theorem ([B.-Corwin-Yang 2023])

For any parameters α, u, v, the law of $\mathbf{L}_1$ under $\mathbb{P}^{\alpha,u,v}_{\mathrm{LG}}$ is the unique stationary measure.

Open KPZ equation

Consider the KPZ equation

$$\partial_t h(t,x) = \tfrac{1}{2}\partial_{xx} h(t,x) + \tfrac{1}{2}\big(\partial_x h(t,x)\big)^2 + \xi(t,x), \quad t \geqslant 0, \quad x \in [0,L]$$

with boundary conditions [Corwin-Shen 2016]

$$\partial_x h(t,x)\Big|_{x=0} = u, \quad \partial_x h(t,x)\Big|_{x=L} = -v.$$

and initial condition $h(t,x) = h_0(x)$.

We say that the law of h_0 is stationary if for any $t > 0$ the law of

$$h(t,x) - h(t,0)$$

is the same as h_0.

Open KPZ equation stationary measure

Let $\mathbb{P}_{\mathrm{Brownian}}$ be the probability measure of two independent standard Brownian motions $L_1(x), L_2(x)$ on $[0, L]$. Defined the reweighted Brownian measure $\mathbb{P}^{u,v}_{\mathrm{KPZ}}$ on $C([0, L], \mathbb{R})$ as

$$\frac{d\mathbb{P}^{u,v}_{\mathrm{KPZ}}}{d\mathbb{P}_{\mathrm{Brownian}}}(\mathbf{L}_1, \mathbf{L}_2) = \frac{1}{\mathcal{Z}^{u,v}_{\mathrm{KPZ}}} \left(\int_0^L ds e^{-(\mathbf{L}_1(s)-\mathbf{L}_2(s))} \right)^{-u-v} e^{-v(\mathbf{L}_1(L)-\mathbf{L}_2(L))}.$$

Theorem ([B.-Corwin-Yang 2023])

Assume the convergence of the log-gamma free energy to the open KPZ equation.
For any $u, v \in \mathbb{R}$, the law of $\mathbf{L}_1$ under $\mathbb{P}^{u,v}_{\mathrm{KPZ}}$ is the unique stationary measure.

Uniqueness: [Knizel-Matetski, Parekh 2023]
The result was conjectured in [B.-Le Doussal 2021].

Liouville quantum mechanics

When $u + v \geqslant 0$, the result was already known.

Using exact formulas of [Corwin-Knizel 2021] coming from Askey-Wilson representations of the MPA, the open KPZ stationary measure can be written as [Bryc-Kuznetsov-Wang-Wesolowski, B.-Le Doussal 2021]

$$\mathbf{L}_1(x) = \Lambda_1(x) - \Lambda_1(0)$$

where Λ_1, Λ_2 are reweighted Brownian motions with starting points distributed as Lebesgue measure,

$$\frac{d\mathbb{P}^{u,v}_{\mathrm{KPZ}}}{d\mathbb{P}_{\mathrm{Brownian}}}(\Lambda_1, \Lambda_2) \propto \exp\left(-\int_0^L ds e^{-(\Lambda_1(s)-\Lambda_2(s))}\right) e^{-u(\Lambda_1(0)-\Lambda_2(0))} e^{-v(\Lambda_1(L)-\Lambda_2(L))}$$

This is the continuous version of the two-layer Gibbs measure.

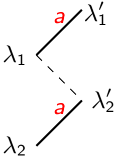
Integrating over $\Delta = \Lambda_1(0) - \Lambda_2(0)$, yields the previous description [B.-Le Doussal 2021].

Back to Matrix Product Ansatz

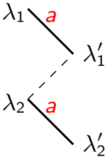
From 2-layer Gibbs measure to MPA

The two-layer Gibbs weight $\mathrm{wt}(\boldsymbol{\lambda})$ can be written in matrix product form. Let

$$\mathbf{M}_x^{\rightarrow}[a](n,n') = \mathrm{wt}\left(\cdots \right)\Bigg|_{\substack{\lambda_1-\lambda_2=n\\ \lambda_1'-\lambda_2'=n'\\ \lambda_1'-\lambda_1=x}}$$



$$\mathbf{M}_x^{\downarrow}[a](n,n') = \mathrm{wt}\left(\cdots \right)\Bigg|_{\substack{\lambda_1-\lambda_2=n\\ \lambda_1'-\lambda_2'=n'\\ \lambda_1-\lambda_1'=x}}.$$



Then the stationary probability can be written as

$$\frac{1}{\mathcal{Z}}\mathbf{w}^t\left(\prod_{i=1}^{N}\mathbf{M}_{x_i}^{\rightarrow,\downarrow}[b_i]\right)\mathbf{v}$$

with $\mathbf{w}^t=(1,c_1,c_1^2,\dots)$ and $\mathbf{v}^t=(1,c_2,c_2^2,\dots)$.

This yields a representation of the MPA relations

$$\mathbf{M}_x^{\rightarrow}[a]\mathbf{M}_y^{\downarrow}[b] = (ab)^{\min\{x,y\}}(1-ab)\sum_{z\geqslant \max\{0,y-x\}} \mathbf{M}_z^{\downarrow}[b]\mathbf{M}_{x-y+z}^{\rightarrow}[a],$$

$$\mathbf{w}^t\mathbf{M}_x^{\downarrow}[a] = (ac_1)^x(1-ac_1)\sum_{y\geqslant 0}\mathbf{w}^t\mathbf{M}_y^{\rightarrow}[a],$$

$$\mathbf{M}_x^{\rightarrow}[a]\mathbf{v} = (ac_2)^x(1-ac_2)\sum_{y\geqslant 0}\mathbf{M}_y^{\downarrow}[a]\mathbf{v}.$$

Similarly, in the log-gamma case, the matrices become operators and the two-layer Gibbs measure yields a representation of the required algebra.

MPA for ASEP

In the case of ASEP, the 2-layer **Hall-Littlewood Gibbs measures** [Borodin-Bufetov-Wheeler 2016, Corwin-Dimitrov 2016] yields representations of

$$DE - qED = D + E$$
$$\langle w| (\alpha E - \gamma D) = \langle w|$$
$$(\beta D - \delta E) |v\rangle = |v\rangle$$

given by

$$D = \frac{1}{1-q}\begin{pmatrix} 1 & 1 & 0 & \\ 0 & 1 & 1 & \ddots \\ 0 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix} \quad E = \frac{1}{1-q}\begin{pmatrix} 1 & 0 & 0 & \dots \\ 1-q^2 & 1 & 0 & \\ 0 & 1-q^3 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

Boundary vectors $\langle w|$ and $|v\rangle$ are given by three term recurrence relations, that are solved by Rogers-Szegö polynomials.
The same polynomials arise as boundary weights of the associated two layer Gibbs measure (based on the results that [Jimmy He] will present on Thursday).

Conclusion

1. Two-layer Gibbs measures yield probabilistic descriptions of stationary measures for open boundary KPZ integrable models;

2. The method is based on Cauchy and Littlewood type identities and should extend to other Yang-Baxter integrable models;

3. Schur/Macdonald/etc. processes yield representations of the matrix product ansatz.