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Stochastic growth in time-dependent environments

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We study the Kardar-Parisi-Zhang (KPZ) growth equation in one dimension with a noise variance $c(t)$ depending on time. We find that for $c(t) \propto t^{-\alpha}$ there is a transition at $\alpha = 1/2$. When $\alpha > 1/2$, the solution saturates at large times towards a nonuniversal limiting distribution. When $\alpha < 1/2$ the fluctuation field is governed by scaling exponents depending on α and the limiting statistics are similar to the case when $c(t)$ is constant. We investigate this problem using different methods: (1) Elementary changes of variables mapping the time-dependent case to variants of the KPZ equation with constant variance of the noise but in a deformed potential. (2) An exactly solvable discretization, the log-gamma polymer model. (3) Numerical simulations.

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Introduction. Many growth models in one spatial dimension share universal scaling properties. In the Kardar-Parisi-Zhang universality class, this phenomenon is particularly manifest since not only scaling exponents are universal, but depending on initial data, the limiting distribution of fluctuations is universal as well. A much studied model in this class is the KPZ Eq. [1] in one dimension, where the height field $h(x,t)$ satisfies

$$\partial_t h(x,t) = \partial_x^2 h(x,t) + [\partial_x h(x,t)]^2 + \sqrt{2c}\,\xi(x,t), \qquad (1)$$

with $\overline{\xi(x,t)\xi(x',t')} = \delta(x-x')\delta(t-t')$. Its solution is related, via $h(x,t) = \log Z(x,t)$, to (minus) the free energy of a continuum directed polymer (DP) in the random potential $\xi(x,t)$ at finite temperature. The partition function of the DP with endpoint at x, t, $Z(x,t)$, obeys the stochastic heat equation (SHE). It is known [2–5] that the local height fluctuations grow at large time t as $\delta h \simeq (c^2 t)^{1/3}\chi$ [6] where the probability distribution function (PDF) of the random variable χ is related to random matrix theory and depends on some features of the initial condition (IC), which fall into IC classes. For the class containing the droplet IC, χ follows the the GUE Tracy-Widom (TW) distribution. The integrability of the KPZ equation is related to the integrability of quantum models, e.g., of the one-dimensional (attractive) delta Bose gas with interaction parameter $-c < 0$ [7]. One also defines a spatial correlation scale for the height fluctuations, which grows as $x \propto t^{2/3}$. The transverse wandering of the DP also grows with this length scale.

In this paper we study the case where the amplitude of the noise depends itself on time, i.e., c becomes a function $c(t)$ in (1). One expects that if $c(t)$ decays very fast, some saturation of the fluctuations may occur, if $c(t)$ decays slowly, perhaps the fluctuations are similar as the homogeneous case. Considering a time-dependent interaction parameter $c(t)$ is also of great importance in the related problem of quantum quenches [8–13].

There is to our knowledge only one exact result in the growth context. Before describing it, let us go back to the homogeneous case. In a seminal paper, Johansson [14] obtained an exact and concise formula for the optimal energy for a directed polymer at zero temperature on the square lattice (of coordinate i, j) with random exponential on site energies. He then extended in [15] the solution to an inhomogeneous disorder, where the amplitude of the disorder was chosen $1/(i^a + j^a)$, with $a \geqslant 0$. He discovered a sharp transition at $a = 1/3$: $\delta h \propto t^{1/3-a}$ with universal TW fluctuations if $a > 1/3$ and $\delta h < \infty$ with nonuniversal fluctuations if $a < 1/3$.

A natural question is whether, for the KPZ equation itself, or more generally for other finite temperature models, there is a similar transition, and how does it depend on the profile of $c(t)$? We probe this question here by considering three models, using complementary methods.

(i) The first one is the KPZ equation, for which we use change of variables. Despite the simplicity of the method, it leads to interesting results. We obtain exact solutions for the inhomogeneous KPZ equation

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + V(x,t) + \sqrt{2c(t)}\,\xi(x,t), \qquad (2)$$

with both time-dependent noise amplitude and an external potential $V(x,t) = a(t)x^2 + b(t)x$, for some specific relation between $a(t)$ and $c(t)$. One finds a transition from TW to nonuniversal fluctuations at large time depending on whether $\int_0^{+\infty} dt\, c(t)^2$ is, respectively, divergent or convergent. In the case $c(t) \propto t^{-\alpha}$, the transition is at $\alpha = 1/2$. We obtain the exponents for the height fluctuations $\delta h \propto t^{\beta(\alpha)}$ and for the correlation scale $x \propto t^{\zeta(\alpha)}$ and the PDF of the height for various cases.

(ii) Next, we study a discretization of the KPZ equation, a directed polymer on the square lattice at finite temperature, called the log-gamma polymer. This model is known to be integrable in presence of inhomogeneity parameters $\gamma_{i,j} = \theta(i^a + j^a)$ which control the strength of the disorder at location (i, j). As θ goes to zero, after rescaling, one recovers

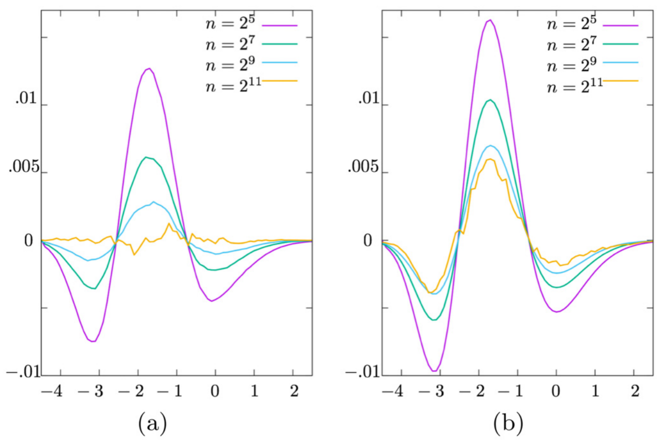

FIG. 1. Model (iii) at zero temperature: Difference between the empirical CDF of the ground state energy and the CDF of the GUE TW distribution (centered and scaled to the same mean and variance). (a) For $a = 0.3$ and various polymer lengths n. (b) For $a = 0.4$ and the same polymer lengths. See also a comparison of the tails in [19].

Johansson's model. For the positive temperature model we find a transition at $a = 1/2$: $\delta h \propto t^{\frac{1-2a}{3}}$ with universal TW fluctuations if $a > 1/2$ and $\delta h < \infty$ with nonuniversal fluctuations if $a < 1/2$. In particular, when the exponent $a \in (1/3, 1/2)$, the free energy fluctuations are universal at positive temperature ($\theta > 0$) and nonuniversal at zero temperature ($\theta \to 0$).

(iii) Finally, we perform numerical simulations of a polymer model on the square lattice with exponentially distributed energies with rates $\tilde{\gamma}_{i,j} = (i + j)^{a'}$, both at zero and finite temperature. In this model the noise is thus purely time dependent (without additional potential). The results at zero temperature are shown in Fig. 1. There is a strong evidence that the fluctuations are TW distributed for $a' = 0.3 < 1/3$ and converge to another limit when $a' = 0.4 > 1/3$, consistent with the same critical value $a' = 1/3$ as in the Johansson model. At positive temperature the numerics indicate that the transition occurs between $a' = 0.2$ and $a' = 0.3$. This is in agreement with a general criterium that we obtain for the occurrence of TW fluctuations in inhomogeneous models, which predicts $a' < 1/4$ for this model, while it correctly predicts $a < 1/2$ for the log-gamma polymer. The resulting phase diagram for models (ii) and (iii) is presented in Fig. 2.

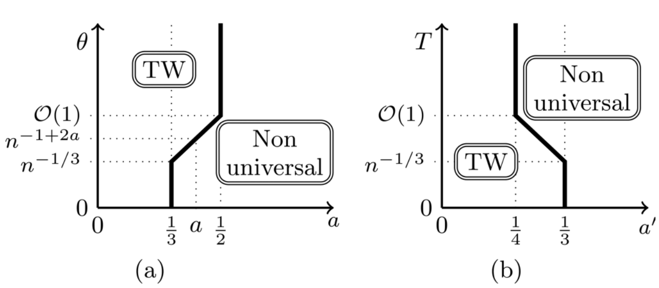

FIG. 2. (a) The two phases depending on "temperature" θ and exponent a, for the log-gamma polymer (ii). The crossover arises when zooming at a point on the oblique thick line. (b) The two phases for the polymer (iii) with Boltzmann weights $e^{E_{i,j}/T}$ with exponentially distributed energies $E_{i,j}$ of parameter $\tilde{\gamma}_{i,j} = (i + j)^{a'}$.

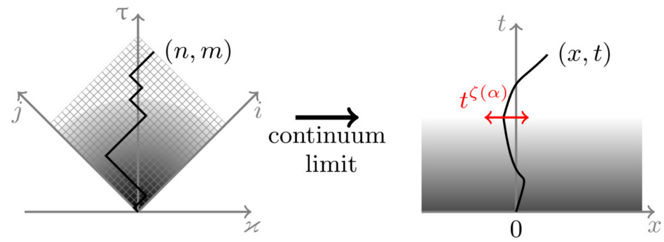

FIG. 3. Directed polymer paths in the discrete lattice (left panel) and in the continuous limit (right panel). We indicated the wandering exponent in red, which is the lateral extension of typical paths.

A property of the positive temperature models (ii) and (iii), not shared by the Johansson model, is that, in the discrete to continuous scaling limit at high temperature (see Fig. 3), their partition functions converge to a solution of the inhomogeneous SHE

$$\partial_t Z = \partial_x^2 Z + [V(x,t) + \sqrt{2c(t)}\xi(x,t)]Z \tag{3}$$

so that $h(x, t) = \log Z(x, t)$ solves the KPZ equation (2). In that limit, their transition points $a = 1/2, a' = 1/4$ correspond to the inhomogeneous KPZ equation (2) where $c(t) \propto t^{-a}$, $c(t) \propto t^{-2a'}$ (i.e., $\alpha = a = 2a'$). Note that in the case of the integrable model (ii), the quadratic term $V(x, t)$, dictated by the form of inhomogeneity parameters $\gamma_{i,j}$ that preserve the integrability, turns out to be exactly the one that we found when performing changes of variables directly on the KPZ equation.

Inhomogeneous KPZ equation. We start with studying the KPZ equation in the presence of a time-dependent noise of amplitude $c(t)$, i.e., Eq. (2). It is convenient to also include a quadratic external potential $V(x, t) = a(t)\frac{x^2}{2}$ with a time-dependent curvature $a(t)$. The standard KPZ problem is recovered for $a(t) = 0$, $c(t) = c(0)$. One can then ask what are the space and time change of coordinates in Eq. (2) which retain its general form and lead to time-independent noise. The answer is that the space transformation must be linear and one arrives at

$$h(x,t) = H[y, \tau(t)] + \frac{c'(t)}{4c(t)}x^2 + \frac{1}{2}\log\frac{c(t)}{c(0)}, \tag{4}$$

$$y = c(t)x, \quad \tau(t) = \int_0^t c(s)^2 ds. \tag{5}$$

Under the transformations (4) and (5) Eq. (2) is mapped onto the following equation for $H(y, \tau)$,

$$\partial_\tau H = \partial_y^2 H + (\partial_y H)^2 - A(\tau)\frac{y^2}{2} + \sqrt{2}\,\hat{\xi}(y,\tau), \tag{6}$$

where $\hat{\xi}$ is again a standard white noise in the coordinates y, τ, and

$$A[\tau(t)] := \frac{a_c(t) - a(t)}{c(t)^4}, \tag{7}$$

$$a_c(t) := \frac{c(t)c''(t) - 2c'(t)^2}{2c(t)^2} = \frac{-c(t)}{2}\left(\frac{1}{c(t)}\right)''. \tag{8}$$

Note that similar transformations have been considered for 1D quantum systems [8,16,17] and for the Burgers equation [18]. Here the mapping works because the white noise is invariant by linear transformations. We assume $0 < c(0) < +\infty$. The correspondence between the initial conditions at $t = \tau = 0$ is then

$$H(y,0) = h[y/c(0),0] - \frac{c'(0)}{4c(0)^3}y^2. \tag{9}$$

Let us first consider the case where the functions $a(t)$ and $c(t)$ are related by the condition $a(t) = a_c(t)$. In that case $A(\tau) = 0$. Hence the full solution of (2) for $h(x,t)$ is given by (4) and (5) where $H(y,\tau)$ is the solution of the standard KPZ equation (1) with initial condition (9). Since a lot is known about the statistics of the standard KPZ equation, a wealth of information can thus be obtained for the case $a(t) = a_c(t)$. Regarding the large time asymptotics, there is clearly a transition depending on whether $\tau(t)$ diverges or remains finite when $t \to +\infty$. In the first case large t maps onto large τ and one can use the universal results for the KPZ equation at large time (which are common to the full KPZ class). Then the one point height fluctuations grow for large t as $\delta h \propto \tau(t)^{1/3}$ with a $O(1)$ PDF depending on the initial condition, as discussed below. The correlation scale grows as $x \propto \tau(t)^{2/3}/c(t)$. In the second case, $\tau(+\infty) < +\infty$, the growth saturates and is described by finite time KPZ. Some results are available, but they are not universal [unless $\tau(+\infty) \gg 1$]. The mapping (4) and (5) extends to several space-time points correlations.

To be specific consider now a noise amplitude decaying as $c(t) \propto t^{-\alpha}$, of the form

$$c(t) = \left(\frac{t_0}{t+t_0}\right)^{\alpha}, \tag{10}$$

with $t_0 > 0$ a constant. Then the amplitude of the quadratic potential decays as t^{-2}, i.e., $a(t) = a_c(t) = \frac{\alpha(1-\alpha)}{2}(t+t_0)^{-2}$. There is thus one particular case, $\alpha = 1$, where $a(t) = 0$ and the present solution is the full solution of the model (2) without external potential $V(x,t) = 0$. For $\alpha > 1/2$, the rescaled time is

$$\tau(t) = \frac{t_0}{1-2\alpha}\left[\left(1+\frac{t}{t_0}\right)^{1-2\alpha} - 1\right], \tag{11}$$

with $\tau(t) = t_0 t/(t+t_0)$ for $\alpha = 1$. The transition thus occurs at $\alpha = 1/2$. For $\alpha < 1/2$ the growth is unbounded, with $\delta h \propto t^{\beta(\alpha)}$ and the spatial scale grows as $x \propto t^{\zeta(\alpha)}$ with exponents

$$\beta(\alpha) = \frac{1-2\alpha}{3}, \quad \zeta(\alpha) = \frac{2-\alpha}{3}. \tag{12}$$

At the transition, for $\alpha = 1/2$, $\tau(t) = t_0 \log(1+\frac{t}{t_0})$, hence $\delta h \propto (\log t)^{1/3}$, and the spatial scale is $x \propto t^{1/2}(\log t)^{2/3}$, barely superdiffusive. For $\alpha > 1/2$ we see from (11) that $\tau(+\infty) = t_0/(2\alpha - 1)$. The heuristics is that the KPZ noise/DP disorder acts only for some finite time $\propto t_0$ and is absent beyond. One finds that the transverse wandering of the polymer is diffusive $x \propto t^{1/2}$. However there is a distinct and fast growing scale $x \propto 1/c(t) \propto t^{\alpha}$ which measures the spatial extent of regions which are correlated by what happened at the earlier times $t < t_0$ (see Fig. 4 in [19]).

As is well known for the standard KPZ equation, the precise distribution of the large time distribution can be classified according to initial condition. We now address this problem in the time-dependent case and stress that the mapping (4) and (5) may map initial data for h and H from different IC classes. In terms of the DP partition functions $Z(x,t) = e^{h(x,t)}$, $\hat{Z}(y,\tau) = e^{\hat{H}(y,\tau)}$, the mapping between initial conditions (9), with the choice (10), reads

$$\hat{Z}(y,0) = Z(y,0)e^{\frac{\alpha}{4t_0}y^2}. \tag{13}$$

The positive sign in the exponential makes the mapping of the IC classes a bit delicate. The droplet IC for h, $Z(x,0) = \delta(x)$, clearly maps to $\hat{Z}(y,0) = \delta(y)$, i.e., to the droplet IC for H, from (13), leading to the GUE-TW distribution for the scaled fluctuations of $h(0,t)$. This remains true for initial conditions $Z(x,t) = e^{-Bz^2}$ with $B > \frac{\alpha}{4t_0}$. Indeed, from (13), $\hat{Z}(y,0)$ decays fast enough so that H still belongs to the droplet IC class. However, for $B = \frac{\alpha}{4t_0}$, H now belongs to the flat IC class, since $\hat{Z}(y,0) = 1$. It leads now to the GOE-TW distribution. Hence we see that many IC which belong to the droplet class when $c(t) = c(0)$, such as the wedge $h(x,0) = -w|x|$, are actually not in that class in the time-dependent problem.

When $B < \frac{\alpha}{4t_0}$ a blow up of the solution can occur. Let us study here and below the flat IC $h(x,0) = 0$, i.e., $B = 0$. One finds that for $0 < \alpha < 1$ the solution blows up at finite time t^*. It can be estimated as $\tau(t^*) \simeq t_0/\alpha$, i.e., $t^* = [(\frac{1}{\alpha} - 1)^{\frac{1}{1-2\alpha}} - 1]t_0$. For $\alpha \geqslant 1$ there is no blow up, since $\tau(+\infty) < t_0/\alpha$. This change of behavior appears to be related to the sign of the quadratic term, $a_c(t) > 0$ for $\alpha < 1$ and $a_c(t) < 0$ for $\alpha > 1$. Let us focus on the case $\alpha = 1$, i.e., $c(t) \propto 1/t$, where there is no external potential, $a_c(t) = 0$, and the blow-up occurs at infinite time $t^* = +\infty$. Since in this case the growth saturates at timescales $t \propto t_0$, with $\tau(+\infty) = t_0$, we now study the universal limit in which both t_0 and t are large with a fixed ratio t/t_0. One finds for the flat IC

$$h(0,t) \simeq \left(\frac{t_0 t}{t+t_0}\right)^{1/3} \max_{\hat{z}\in\mathbb{R}} \left\{\mathcal{A}_2(\hat{z}) - \tfrac{t_0}{t+t_0}\hat{z}^2\right\}. \tag{14}$$

where $\mathcal{A}_2(\hat{z})$ is the so-called Airy$_2$ process (see, e.g., [20,21]). We can now use the results of [22, Example 1.25] and conclude that the CDF of the fluctuating part $h(0,t) \propto (\frac{t_0 t}{t+t_0})^{1/3}s$, is given by the universal "parabolic IC" function $F^{\beta,\beta}_{\text{parbl}}(s)$, with $\beta = -\frac{t}{t+t_0}$. It interpolates between the GOE-TW ($\frac{t}{t_0} \ll 1$) and Gumbel ($\frac{t}{t_0} \gg 1$) distributions. It can also be related to ingrowing circular interfaces [23] the blow-up time $\tau = \tau(+\infty) = t_0$ being the time at which the circular droplet collapses.

An important question is how do the models with $a(t) = 0$ and $a(t) = a_c(t)$ compare, i.e., how does the presence of the (time-dependent) quadratic potential $\sim x^2/t^2$ change the results. We can safely surmise that it does not change the scaling exponents. However it is probable that it changes the PDF, as it also has some effect on the classification of the IC.

We now study case (10), i.e., $c(t) \propto t^{-\alpha}$, with no external potential $a(t) = 0$. From (7) it maps onto the usual KPZ equation plus a quadratic potential of curvature $A(\tau) = \frac{\alpha(1-\alpha)}{2[t_0+(1-2\alpha)\tau]^2}$. For $\alpha < 1/2$ is it again a potential of the form

y^2/τ^2. For $\alpha > 1/2$, $A(\tau)$ diverges at $\tau = t_0/(2\alpha - 1)$ which corresponds to $t = \infty$.

Let us study the marginal case $\alpha = 1/2$,

$$c(t) = \frac{1}{\sqrt{1+\frac{t}{t_0}}}, \quad \tau(t) = t_0 \log\left(1+\frac{t}{t_0}\right). \tag{15}$$

Then one has $A(\tau) = A = \frac{1}{8t_0^2}$, hence the initial problem maps to a DP in a static $-\frac{1}{2}Ay^2$ confining potential. Although no exact result is known, heuristics is easy.

Let us consider the fixed endpoint DP (i.e., droplet IC) and $t_0 \gg 1$ where universal results can be obtained. For $1 < t \ll t_0$, $\tau \simeq t$, the quadratic potential can be neglected and the fluctuations are the standard TW ones for KPZ. For $t \gg t_0$ the quadratic well confines the DP, i.e., the variance of the endpoint distribution saturates as $\langle y^2\rangle \propto t_0^{4/3}$, and segments of length $\tau \propto t_0$ become uncorrelated. For the initial model it implies that the variance of the endpoint distribution behaves as

$$\langle x^2\rangle \propto c(t)^{-2}\langle y^2\rangle \propto t_0^{4/3}\left(1+\frac{t}{t_0}\right) \simeq t_0^{1/3}t, \tag{16}$$

i.e., diffusion. The free energy fluctuations scale as

$$\delta h \propto t_0^{1/3}\sqrt{\log\left(1+\frac{t}{t_0}\right)} \propto t_0^{1/3}\sqrt{\log\left(\frac{t}{t_0}\right)} \tag{17}$$

but now they have a Gaussian distribution. There is thus, for large t_0, a crossover from TW to Gaussian. For $t_0 = O(1)$ the above scaling still holds but the intermediate time distribution and prefactors are nonuniversal.

Inhomogeneous discrete model. We consider now an integrable discretization of the KPZ equation, the so-called log-gamma directed polymer on the square lattice $\mathbb{Z}^2_{>0}$. The (point-to-point) partition function of the model is defined by

$$\mathcal{Z}(n,m) = \sum_{\pi}\prod_{(i,j)\in\pi} w_{i,j}, \tag{18}$$

where the sum is over all up-right directed paths from (1,1) to (n,m) in the square lattice. The model is integrable [24] when the random weights $w_{i,j}$ are independent and distributed according to the inverse of a gamma random variable with parameter γ, i.e., its PDF $P(w)$ is $P(w) = \frac{1}{\Gamma(\gamma)}w^{-\gamma-1}\exp(-1/w)$. It was noted in [25] that the model remains exactly solvable when the parameter γ depends on the position i, j as $\gamma_{i,j} = \alpha_i + \beta_j$, where α_i and β_j are any sequences of real numbers such that the $\gamma_{i,j}$ are positive. In order to emulate the case of a disorder whose amplitude decays with time as a power law, we will consider the case where $\gamma_{i,j} = \theta(i^a + j^a)$ for some parameter $\theta > 0$.

The Laplace transform of the partition function of a polymer of length n can be written as a Fredholm determinant [25,26]

$$\mathbb{E}[e^{-u\mathcal{Z}(n,n)}] = \det(I+K)_{\mathbb{L}^2(\mathcal{C})}, \tag{19}$$

where the operator K is defined by its integral kernel as

$$K(v,v') = \int_{-\mathbf{i}\infty}^{\mathbf{i}\infty}\frac{dz}{2\mathbf{i}\pi}\frac{\pi}{\sin[\pi(v-z)]}\frac{1}{z-v'}\frac{e^{G(z)}}{e^{G(v)}}, \tag{20}$$

with

$$G(z) = z\log(u) + \sum_{i=1}^{n}\log\frac{\Gamma(\theta i^a - z)}{\Gamma(\theta i^a + z)}. \tag{21}$$

The kernel is acting on a contour $\mathcal{C}$ of the complex plane enclosing all singularities of the kernel at the points $-\theta i^a$ for all $i \geqslant 1$. We now analyze the large n asymptotics of (20) using a saddle point method. It is easy to notice that $G''(0) = 0$, so that by Taylor expansion,

$$G(z) = z\log(u) + zf_n/\theta + \sigma_n^3 z^3/(3\theta^3) + O(z^5), \tag{22}$$

where $f_n = -2\theta\sum_{i=1}^n \psi(\theta i^a)$, $\psi(x) = \frac{d}{dx}\log[\Gamma(x)]$, and

$$\sigma_n^3 = \sum_{i=1}^{n} -\theta^3\psi''(\theta i^a). \tag{23}$$

The quantity f_n is the leading order of (minus) the free energy $\mathcal{F}_n = \theta\log\mathcal{Z}(n,n)$, and σ_n should be understood as the amplitude of free energy fluctuations. The asymptotic behavior of $\mathcal{F}_n$ will depend on whether σ_n stays bounded or diverges as $n \to \infty$. In the zero temperature limit $\theta \to 0$, the threshold found in [15] was for $a = 1/3$. Interestingly, the result is different for positive θ. For $\theta > 0$, a careful analysis of (23) shows that it diverges for $a \leqslant 1/2$ and converges for $a > 1/2$ [this is due to the fact that $\psi''(x) \simeq -1/x^2$ as $x \to +\infty$].

When $a > 1/2$, the weights decay rapidly away from the origin. The parameter σ_n converges to a constant, i.e., the weights which contribute significantly to free energy fluctuations is a finite set near the origin. The one point distribution of those fluctuations can be computed explicitly (see [19] Eq. (136)).

When $a \leqslant 1/2$, the magnitude of free energy fluctuations diverge as

$$\sigma_n \simeq \begin{cases}[\theta n^{1-2a}/(1-2a)]^{1/3} & \text{for} \quad a < 1/2,\\ (\theta\log n)^{1/3} & \text{for} \quad a = 1/2.\end{cases} \tag{24}$$

To analyze the limit distribution, let us choose $u = e^{-f_n - r\sigma_n}$. Since $\sigma_n \to +\infty$ at large n one has

$$\mathbb{E}[e^{-u\mathcal{Z}(n,n)}] \simeq \mathbb{P}\left(\frac{\mathcal{F}_n - f_n}{\sigma_n} \leqslant r\right). \tag{25}$$

The Fredholm determinant $\det(I+K)_{\mathbb{L}^2(\mathcal{C})}$ converges to $\det(I+K^{\mathrm{GUE}})_{\mathbb{L}^2(-1+\mathbf{i}\mathbb{R})}$ where

$$K^{\mathrm{GUE}}(v,v') = \int_{1+\mathbf{i}\mathbb{R}}\frac{dz}{2\mathbf{i}\pi}\frac{1}{v-z}\frac{1}{z-v'}\frac{e^{\frac{z^3}{3}-rz}}{e^{\frac{v^3}{3}-rv}}. \tag{26}$$

This comes from (22) upon rescaling z by θ/σ_n, implying

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{\mathcal{F}_n - f_n}{\sigma_n} \leqslant r\right) = F_{\mathrm{GUE}}(r), \tag{27}$$

where F_{GUE} is the CDF of the TW GUE distribution.

When the parameter θ goes to 0, one recovers the zero-temperature model studied by Johansson [15] (in the sense that $\mathcal{F}_n$ goes to the zero-temperature free energy defined in [15]). Using the asymptotics $\psi''(x) \underset{x\to 0^+}{\simeq} \frac{-1}{x^3}$, one readily sees, taking $\theta \to 0$ in (23), that

$$\sigma_n^3 \simeq \sum_{i=1}^{n} \frac{1}{i^{3a}}, \quad n \to \infty, \quad \theta \ll n^{-a}, \tag{28}$$

hence the transition at $a = 1/3$ at zero temperature. Considering the limit $\theta \to 0$ of (20) an asymptotic analysis recovers the one-point distribution results from [15].

We may also let θ go to zero simultaneously as n goes to infinity and study the crossover between zero and finite temperature. Let $a \in (1/3, 1/2)$. The relevant scale to see a crossover is $\theta = An^{2a-1}$, where A is a free parameter. For higher or lower values of θ, the free energy fluctuations will be either of TW type or nonuniversal according to the two phases depicted in Fig. 2.

At the crossover scale we obtain

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{F}_n - f_n \leqslant s) = \det(I + K^{\text{cross}})$$
$$K^{\text{cross}}(v, v') = \int_{-\mathbf{i}\infty}^{\mathbf{i}\infty} \frac{dz}{2\mathbf{i}\pi} \frac{1}{v - z} \frac{1}{z - v'} \frac{e^{F^{\text{cross}}(z) - sz}}{e^{F^{\text{cross}}(v) - sv}}. \tag{29}$$

The function F_{cross} interpolates between the zero temperature case and a cubic behavior as in the Airy kernel. It depends on A as

$$F_{\text{cross}}(z) = \frac{Az^3}{3(1-2a)} + F_{\theta\to 0}(z), \tag{30}$$

$$F_{\theta \to 0}(z) = \sum_{k=1}^{\infty} \log\left(1 + \frac{z}{k^a}\right) - \log\left(1 - \frac{z}{k^a}\right) - \frac{2z}{k^a}, \tag{31}$$

where $F_{\theta\to 0}$ is the function arising in the zero-temperature kernel as in [15].

From discrete to continuous. It was shown in [27] that the free energy of the (homogeneous) log-gamma polymer model converges to the solution to the KPZ equation. We use the convenient new coordinates $\tau = n + m$, $\varkappa = n - m$, and denote $\mathcal{Z}(n, m) = Z_d(\varkappa, \tau)$ [28]. The subscript d means that Z_d satisfies a discrete version of the stochastic heat equation

$$Z_d(\varkappa, \tau) = w_{\varkappa,\tau}[Z_d(\varkappa - 1, \tau - 1) + Z_d(\varkappa + 1, \tau - 1)], \tag{32}$$

where $w_{\varkappa,\tau}$ is an inverse gamma random variable with parameter $\gamma_{\varkappa,\tau}$ (independent for each $\varkappa, \tau$). Let us rescale Z_d and denote $Z_r(\varkappa, \tau) = Z_d(\varkappa, \tau)(\prod_{s=1}^{\tau} C_s)^{-1}$. A natural choice for the function C_τ would be $\mathbb{E}[w_{\varkappa,\tau}]$ (when it does not depend on $\varkappa$). We may rewrite (32) as

$$\nabla_\tau Z_r(\varkappa, \tau) = \tfrac{1+\eta_{\varkappa,\tau}}{2} \Delta_\varkappa Z_r(\varkappa, \tau - 1) + \eta_{\varkappa,\tau} Z_r(\varkappa, \tau - 1), \tag{33}$$

where $\eta_{\varkappa,\tau} = \frac{2w_{\varkappa,\tau}}{C_\tau} - 1$, ∇_τ is the discrete time derivative, and $\Delta_\varkappa$ is the discrete Laplacian. Let us use the scalings

$$\tau = 2nt, \quad \varkappa = \sqrt{n}x. \tag{34}$$

In order to obtain a time-inhomogeneous variance of the noise, let us scale the parameter γ as $\gamma_{\varkappa,\tau} = \sqrt{n}/c(t)$. In this case, one takes $C_\tau = 2\mathbb{E}[w_{\varkappa,\tau}]$ and the family of random variables $w_{\varkappa,\tau}$ rescales to a white noise in the sense that $n\,\eta_{\varkappa,\tau} = \sqrt{c(t)/2}\xi(x, t)$. Multiplying Eq. (34) by n and taking the continuum limit, we obtain the SHE (3) with $V(x, t) = 0$.

However, in the inhomogeneous log-gamma polymer, one cannot exactly take inhomogeneity parameters depending only on τ. Recall that $\gamma_{\varkappa,\tau} = \alpha_i + \beta_j$ where $\tau = i + j$ and $\varkappa = i - j$. Let us consider the case where

$$\gamma_{\varkappa,\tau} = \frac{\sqrt{n}}{2c(i/n)} + \frac{\sqrt{n}}{2c(j/n)}. \tag{35}$$

We set now $C_\tau = \frac{2}{\sqrt{n}/c(t) - 1} \neq 2\mathbb{E}w_{\varkappa,\tau}$, and the noise converges to a white noise with an extra potential $a_c(t)x^2/2$. In the continuum limit, we obtain the SHE (3) with $V(x, t) = a_c(t)\frac{x^2}{2}$. In particular, choosing γ as

$$\text{Model I:}\;\; \gamma_{i,j} = n^{\frac{1}{2}-a}\left(\frac{i+j}{2} + t_0 n\right)^a,$$
$$\text{Model II:}\;\; \gamma_{i,j} = \tfrac{1}{2}n^{\frac{1}{2}-a}[(i + t_0 n)^a + (j + t_0 n)^a], \tag{36}$$

one obtains for large n the continuous SHE (3) for $c(t) = (t + t_0)^{-\alpha}$, $\alpha = a$. In model I $a(t) = 0$, in model II $a(t) = a_c(t)$. The latter, when $a = 1/2$ and $t_0 = 0$, corresponds to the discrete model analyzed above.

Furthermore, the criterium $\int_0^{+\infty} dt\, c(t)^2 = \infty$ that we found for TW fluctuations in the continuous model (2) becomes equivalent to the criterium $\lim_{n\to+\infty} \sigma_n = \infty$ that we have used in the study of the discrete model. In addition, the two critical models $\alpha = \frac{1}{2}$ and $a = \frac{1}{2}$ match in the double limit $n \to \infty$, $t_0 \to 0$.

We now discuss when a time-inhomogeneous discrete model at finite temperature leads to TW fluctuations. We find that these arise if and only if the sum along the polymer of $(\mathrm{Var} \log w)^2$ [i.e., the discrete analog of $\int_0^t c^2(s)ds$] diverges as the length of the polymer goes to infinity [19]. This criterium predicts a transition at $a' = 1/4$ for the model with exponentially distributed energies E_{ij} and rates $\tilde{\gamma}_{i,j} = (i + j)^{a'}$ (with Boltzmann weights $w_{i,j} = e^{E_{i,j}/T}$), also supported by our numerics [19]. It predicts a transition at $a = 1/2$ for the log-gamma polymer model with $\gamma_{ij} = (i + j)^a$. While both models are identical at zero temperature with $a = a'$, their critical values at finite temperature are distinct. Indeed, the log-gamma distribution of Boltzmann weights induces a temperature dependence on the distribution of energies.

Linear potential. Finally, the KPZ equation (2) with $c(t) = 1$ and a linear potential $V(x, t) = bx$ is solved as

$$h(x, t) = H(x + bt^2, t) + bxt + \tfrac{1}{3}b^2t^3, \tag{37}$$

where $H(y, t)$ is the solution of the standard KPZ equation with the same IC $H(x, 0) = h(x, 0)$. For the droplet IC, the one point height distribution is $h(x, t) \equiv -\frac{(x-bt^2)^2}{4t} + \frac{1}{3}b^2t^3 + H_{\text{droplet}}$ where H_{droplet} is the one-point droplet KPZ height, and the profile has a maximum at $x \simeq bt^2$. This holds for sufficiently localized IC.

Outlook. Using three complementary methods, we have obtained results for growth in presence of time-dependent noise and investigated how KPZ universality extends to this setting. This may be of interest for experiments where the variance of the noise can be controlled, see, e.g., Fig. 20 in [29].

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