Probability Theory and Related Fields
https://doi.org/10.1007/s00440-022-01122-3

CORRECTION

Correction to: Random-walk in Beta-distributed random environment

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Correction to: Probab. Theory Relat. Fields (2017) 167:1057–1116 https://doi.org/10.1007/s00440-016-0699-z

Since the publication of this paper, we have noticed or been made aware of several minor mistakes. These mistakes do not affect significantly the main results and are all readily corrected below.

1 Expression for the constant *d* in Corollary 6.8

The expression given in Corollary 6.8 for the constant d, controlling the magnitude of fluctuations, was incorrectly calculated. The correct value is

$$d = \frac{2^{1/3} c^{2/3} (1-c)^{2/3}}{\sqrt{1-(1-c)^2}} \tag{1}$$

Indeed, at the bottom of page 1101, it is correctly stated that the constant d depends on the parameter c as $d = \sigma(x_0)/I'(x_0)$. Here $x_0 \in (0, 1)$ is such that $I(x_0) = c$,

The original article can be found online at https://doi.org/10.1007/s00440-016-0699-z.

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Published online: 04 May 2022

where

$$I(x) = 1 - \sqrt{1 - x^2},$$

and where, using [1, Eq. (43)],

$$\sigma(x) = \left(\frac{2I(x)^2}{1 - I(x)} \right)^{1/3}.$$

Since $I(x_0) = c$, we have $x_0 = \sqrt{1 - (1 - c)^2}$, so that

$$\sigma(x_0) = \left(\frac{2c^2}{1 - c} \right)^{1/3} \tag{2}$$

and

$$I'(x_0) = \frac{x_0}{\sqrt{1 - x_0^2}} = \frac{\sqrt{1 - (1 - c)^2}}{1 - c}. \tag{3}$$

Combining (2) and (3) yields (1). The previous formula for d came from a mistake in simplifying the formula for $d = \sigma(x_0)/I'(x_0)$.

2 Sign mistakes

We have identified a few sign mistakes in the paper and remedied them below.

In Proposition 4.6, the determinant inside the integrand of Eq. (27) should be

$$\det \left(\frac{1}{v_i + \lambda_i - v_j} \right)_{i,j=1}^{\ell(\lambda)} \text{ instead of } \det \left(\frac{1}{v_j - v_i - \lambda_i} \right)_{i,j=1}^{\ell(\lambda)}. \tag{4}$$

This was noticed in [7], see footnote 4 page 23 therein. The proof of Proposition 4.6 involves a $q \to 1$ limit of Proposition 3.8 from [5, Section 3.2.1] and the sign mistakes arose in taking this $q \to 1$ limit. A similar $q \to 1$ limit was performed (correctly) in [4, Proposition 5.1] so that our sign mistake becomes apparent, even though the choice of contours there are slightly different than ours.

The sign mistake in Proposition 4.6 implies sign mistakes in several further statements using Proposition 4.6. In Theorem 2.12, the first equation should read

$$\mathbb{E}\left[e^{uZ(t,n)}\right] = \det\left(I - K_u^{\mathrm{BP}}\right)_{\mathbb{L}^2(C_0)} \text{ instead of } \det\left(I + K_u^{\mathrm{BP}}\right)_{\mathbb{L}^2(C_0)}. \tag{5}$$

In Theorem 2.13, the first equation should read

$$\mathbb{E}\left[e^{uP(t,x)}\right] = \det\left(I - K_u^{\mathrm{RW}}\right)_{\mathbb{L}^2(C_0)} \text{ instead of } \det\left(I + K_u^{\mathrm{RW}}\right)_{\mathbb{L}^2(C_0)}. \tag{6}$$

This was already noticed in [3, Remark 1.21]. Similarly, in Theorem 2.18, the first equation should read

$$\mathbb{P}\left(T(n,m) > r\right) = \det\left(I - K_r^{\mathrm{FPP}}\right)_{\mathbb{L}^2(C_0')} \text{ instead of } \det\left(I + K_r^{\mathrm{FPP}}\right)_{\mathbb{L}^2(C_0')}$$

This was noticed in [2, Remark 1.6].

The statement of Theorem 2.15 is correct. Even though it relies on Theorem 2.13 (which had sign mistakes indicated above) there was another sign mistake which cancelled the previous ones. Indeed, the first displayed equation at the top of page 1096, which explains how to recover the standard form of the Fredholm determinant defining the Tracy Widom distribution, is not correct. It should read

$$\det(I + K_y)_{\mathbb{L}^2(\mathcal{C})} = \det(I - K_{\mathrm{Ai}})_{\mathbb{L}^2(x,+\infty)} \text{ instead of}$$
$$\det(I - K_y)_{\mathbb{L}^2(\mathcal{C})} = \det(I - K_{\mathrm{Ai}})_{\mathbb{L}^2(x,+\infty)}.$$

The statement of the convergence towards the Tracy-Widom distribution for the Bernoulli-Exponential FPP model also contained a sign mistake, as noticed in [2, Remark 1.3]. The displayed equation in the statement of Theorem 2.19 should read

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T\big(n,\kappa(\theta)n\big) - \tau(\theta)n}{\rho(\theta)n^{1/3}} \geq -y\right) = F_{\mathrm{TW}}(y).$$

3 Asymptotic analysis of the Fredholm determinant kernels (Propositions 6.7 and 7.6)

In the proof of Proposition 6.7, Eq. (53) reads

$$\left| t\cdot h(v) - \frac{\sigma(\theta)^3}{3}\tilde{v}^3 \right| < Ct(v-\theta)^4.$$

This bound should, instead, be

$$\left| t\cdot h(v) - \frac{\sigma(\theta)^3}{3}\tilde{v}^3 \right| < Ct|v-\theta|^4. \tag{7}$$

The following sentences aimed at proving an estimate on the kernel $K^t_{y,\epsilon}$ given in the displayed equation following (53), where modulus bars are also missing. Due to the factor $\exp(-th(v))$ in the integrand defining the kernel, we used (7) to argue that we may find constants $C', C'' > 0$ such that for $\tilde{v}, \tilde{v}'$ along their contour,

$$|K^t_{y,\epsilon}(\tilde{v},\tilde{v}')| < C'' \exp(-C'|\tilde{v}|^3). \tag{8}$$

However, there is a lack of rigor in the justifications given. Indeed, as pointed out to us by Sergei Korotkikh, the argument of the complex variable $\tilde{v}$ is $\pi/2 + O(\epsilon)$, so

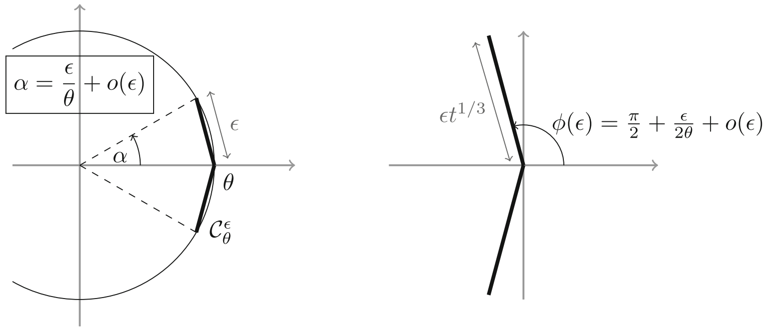

Fig. 1 Left: The contour $\mathcal{C}_\theta^\epsilon$ for variables v, v' is shown (thick black segments). Since the length of each segment is ϵ, the length of the corresponding arc it intercepts is $\epsilon + o(\epsilon)$. Taking into account that the circle has radius θ, it implies that the angle α shown in the figure is $\alpha = \frac{\epsilon}{\theta} + o(\epsilon)$. Right: The contour (thick black segments) for variables $\tilde{v}, \tilde{v}'$ obtained after the change of variables $v = \theta + t^{-1/3}\tilde{v}$. The angle of the segments is the same as on the left, that is $\phi(\epsilon) = \frac{\pi}{2} + \frac{\epsilon}{2\theta} + o(\epsilon)$

that the real part of $\frac{\sigma(\theta)^3}{3}\tilde{v}^3$ may not decay faster than the bound in the R.H.S. of (7), as $|\tilde{v}|$ increases along the contour.

Instead, we need to consider a higher order Taylor approximation, as it was kindly suggested to us by Sergei Korotkikh (see also [6] for an alternative approach). Using Eq. (48), we may compute the fourth derivative of the function h and find that

$$h^{(4)}(\theta) = -\frac{6(1+2\theta)}{\theta^2(1+\theta)^2(1+2\theta+2\theta^2)}. \tag{9}$$

By Taylor expansion, there exist a constant $C > 0$ such that for $|v - \theta| < \epsilon$ with ϵ chosen small enough,

$$\left| th(v) - \frac{\sigma(\theta)^3}{3}\tilde{v}^3 - \frac{t^{-1/3}h^{(4)}(\theta)}{4!}\tilde{v}^4 \right| < Ct|v-\theta|^5 < C\epsilon^2|\tilde{v}|^3. \tag{10}$$

According to the choice of contours made just before the statement of Proposition 6.7, the variable $\tilde{v}$ belongs to a contour formed by two segments leaving 0 with angle $\pm\phi(\epsilon)$ where $\phi(\epsilon) = \frac{\pi}{2} + \frac{\epsilon}{2\theta} + o(\epsilon)$, see Fig. 1.

Thus, using that $t^{-1/3}|\tilde{v}| < \epsilon$ and $h^{(4)}(\theta) < 0$,

$$\begin{aligned} \mathrm{Re}\left[\frac{-\sigma(\theta)^3}{3}\tilde{v}^3 - \frac{t^{-1/3}h^{(4)}(\theta)}{4!}\tilde{v}^4\right] &= -\sin\left(\frac{3\epsilon}{2\theta} + o(\epsilon)\right)\frac{\sigma(\theta)^3}{3}|\tilde{v}|^3 \\ &\quad - \cos\left(\frac{2\epsilon}{\theta} + o(\epsilon)\right)\frac{t^{-1/3}h^{(4)}(\theta)}{4!}|\tilde{v}|^4 \\ &< -\epsilon|\tilde{v}|^3\left(\frac{\sigma(\theta)^3}{2\theta} + \frac{h^{(4)}(\theta)}{4!}\right) + o(\epsilon) \end{aligned} \tag{11}$$

Using (9) and the expression of $\sigma(\theta)$ in Eq. (43), we find that

$$\frac{\sigma(\theta)^3}{2\theta} + \frac{h^{(4)}(\theta)}{4!} = \frac{1}{4\theta^2(1+\theta)^2(1+2\theta+2\theta^2)} > 0,$$

Combining (10) and (11), there exist a constant $c > 0$ such that

$$\mathrm{Re}\,[-th(v)] < -c\epsilon|\tilde{v}|^3 + C\epsilon^2|\tilde{v}|^3.$$

Hence, choosing ϵ small enough, we may find constants $C', C'' > 0$ such that

$$|K^t_{y,\epsilon}(\tilde{v}, \tilde{v}')| < C'' \exp(-C'\epsilon|\tilde{v}|^3),$$

which allows to apply dominated convergence on the Fredholm determinant expansion as explained in the proof of Proposition 6.7.

In the asymptotic analysis of the Bernoulli-Exponential FPP model in Section 7, the proof of Proposition 7.6 was claimed to be identical to the proof of Proposition 6.7. Thus, one might expect that a similar issue needs to be addressed there as well. However, in Section 7, the contour for the variables u, u' (i.e. the variables of the kernel $K^{\mathrm{FPP}}_{y,\epsilon}(u, u')$ in Proposition 7.6) departs θ with angle $\pm\phi$, where $\phi \in (\pi/2, 5\pi/6)$ does not depend on ϵ. Hence, using a Taylor expansion of the function H to the third order as in (7) suffices to show that

$$\mathrm{Re}\,[-tH(u)] < -c|\tilde{u}|^3,$$

for some constant $c > 0$, which allows to bound the kernel appropriately.

4 Proof of the steep-descent property (Lemma 6.4)

The proof of Lemma 6.4 contained a mistake that we remedy here. Note that we only used this statement in the special case of $\alpha = \beta = 1$, where all formulas simplify and the statement of the lemma is quite easy to check. Nevertheless, let us explain how the statement of Lemma 6.4 can be proved in the general α, β case. We will follow the proof of [3, Lemma 2.6], which is a statement very similar as Lemma 6.4 (it corresponds to the $\alpha, \beta \to 0$ limit). In the proof of Lemma 6.4, we reduced the statement to proving that for $0 < \theta_1 < \theta_2$

$$\frac{\Psi_2(\theta_1)}{\Phi'(\theta_1)} > \frac{\Psi_2(\theta_2)}{\Phi'(\theta_2)}, \qquad (12)$$

where Ψ_n is the nth derivative of the digamma function, and

$$\Phi(\theta) = \sum_{n\geq 0} \frac{1}{(n+\theta)^2 + y^2}, \quad \text{with} \quad y > 0.$$

There are several mistakes in Eq. (63), which should read

$$
\begin{aligned}
&\Psi_2(\theta_1)\Phi'(\theta_2) > \Psi_2(\theta_2)\Phi'(\theta_1)\\
&\quad\Leftrightarrow \sum_{n=0}^{\infty} \frac{2}{(n+\theta_1)^3} \sum_{m=0}^{\infty} \frac{2(m+\theta_2)}{\left((m+\theta_2)^2+y^2\right)^2}\\
&\qquad > \sum_{n=0}^{\infty} \frac{2}{(n+\theta_2)^3} \sum_{m=0}^{\infty} \frac{2(m+\theta_1)}{\left((m+\theta_1)^2+y^2\right)^2}\\
&\quad\Leftrightarrow \sum_{n,m=0}^{\infty} \frac{1}{(n+\theta_1)^3(m+\theta_2)^3} \frac{1}{\left(1+\frac{y^2}{(m+\theta_2)^2}\right)^2}\\
&\qquad > \sum_{n,m=0}^{\infty} \frac{1}{(n+\theta_2)^3(m+\theta_1)^3} \frac{1}{\left(1+\frac{y^2}{(m+\theta_1)^2}\right)^2},
\end{aligned}
\tag{13}
$$

and at this point, the inequality is not obvious to prove.

We will instead use a different method from [3, Lemma 2.6]. We need to show that the function $\Psi_2(\theta)/\Phi'(\theta)$ is decreasing. Taking the derivative, this amount to showing that for all $\theta > 0$,

$$\Psi_3(\theta)\Phi'(\theta) - \Psi_2(\theta)\Phi''(\theta) < 0.$$

Using, the series representations for the digamma and Φ functions, this is equivalent to showing that

$$\sum_{n,m=0}^{\infty} T_{n,m} > 0 \quad \text{where} \quad T_{n,m} = a(n+\theta)B(m+\theta) - A(n+\theta)b(m+\theta), \tag{14}$$

with

$$
\begin{aligned}
a(x) &= \frac{1}{x^4}, \quad b(x) = \frac{1-\frac{y^2}{3x^2}}{x^4\left(1+\frac{y^2}{x^2}\right)^3},\\
A(x) &= \frac{1}{x^3}, \quad B(x) = \frac{1}{x^3\left(1+\frac{y^2}{x^2}\right)^2}.
\end{aligned}
$$

Note that

$$T_{n,m} > \tilde{T}_{n,m} := a(n+\theta)B(m+\theta) - A(n+\theta)\tilde{b}(m+\theta),$$

where $\tilde{b}(x) = \frac{1}{x^4\left(1+\frac{y^2}{x^2}\right)^3} \geq b(x)$.

In order to prove (14), we will show that $\tilde{T}_{n,m} + \tilde{T}_{m,n} \geq 0$, and for that we will show that

(1) For all $0 \leq n \leq m$, $\tilde{T}_{n,m} \geq 0$.
(2) For all $0 \leq n \leq m$, either $\tilde{T}_{n,m}/\tilde{T}_{m,n} \geq 0$ or $|\tilde{T}_{n,m}/\tilde{T}_{m,n}| \geq 1$.

To prove (1), observe that using the shorthand notation $n_\theta = n + \theta$, $m_\theta = m + \theta$, we have

$$\tilde{T}_{n,m} = \frac{m_\theta \left(m_\theta(m_\theta - n_\theta) + y^2\right)}{n_\theta^4 \left(m_\theta^2 + y^2\right)^3}$$

which is clearly non-negative for all $0 \leq n \leq m$. Now we turn to proving (2). We have

$$\frac{\tilde{T}_{n,m}}{\tilde{T}_{m,n}} = \frac{n_\theta}{m_\theta} \frac{\left(1 + \frac{y^2}{n_\theta^2}\right)^3}{\left(1 + \frac{y^2}{m_\theta^2}\right)^3} \frac{m_\theta(m_\theta - n_\theta) + y^2}{n_\theta(n_\theta - m_\theta) + y^2}$$

For $n \leq m$, the numerator is always positive. Regarding the denominator, there are two cases to consider. Either it is nonnegative, which implies $T_{n,m}/T_{m,n} > 0$, or the denominator is negative. In the latter case, we have that $y^2 < n_\theta(m_\theta - n_\theta)$ and

$$\left|\frac{\tilde{T}_{n,m}}{\tilde{T}_{m,n}}\right| = \frac{\left(1 + \frac{y^2}{n_\theta^2}\right)^3}{\left(1 + \frac{y^2}{m_\theta^2}\right)^3} \frac{n_\theta}{m_\theta} \frac{m_\theta(m_\theta - n_\theta) + y^2}{n_\theta(m_\theta - n_\theta) - y^2}.$$

For $n \leq m$, we clearly have that $\frac{\left(1+\frac{y^2}{n_\theta^2}\right)^3}{\left(1+\frac{y^2}{m_\theta^2}\right)^3} \geq 1$ and

$$\frac{n_\theta}{m_\theta} \frac{m_\theta(m_\theta - n_\theta) + y^2}{n_\theta(m_\theta - n_\theta) - y^2} \geq \frac{n_\theta}{m_\theta} \frac{m_\theta(m_\theta - n_\theta)}{n_\theta(m_\theta - n_\theta)} = 1,$$

so that $\left|\tilde{T}_{n,m}/\tilde{T}_{m,n}\right| \geq 1$. Therefore we have proved (2) and this concludes the proof of Lemma 6.4.

Acknowledgements We are grateful to Mark Rychnovsky for noticing the mistake in the proof of Lemma 6.4 and to Sergei Korotkikh for pointing out to us the mistake in Proposition 6.7. We also thank Thimothée Thiery and Pierre Le Doussal for telling us about the sign mistake in Proposition 4.6.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Barraquand, G., Corwin, I.: Random-walk in Beta-distributed random environment. Probab. Theory Relat. Fields **167**(3–4), 1057–1116 (2017)
2. Barraquand, G., Rychnovsky, M.: Tracy–Widom asymptotics for a river delta model. In: International Workshop on Stochastic Dynamics out of Equilibrium, pp. 483–522. Springer (2017)
3. Barraquand, G., Rychnovsky, M.: Large deviations for sticky Brownian motions. Electron. J. Probab. **25**, 1–52 (2020)
4. Borodin, A., Bufetov, A., Corwin, I.: Directed random polymers via nested contour integrals. Ann. Phys. **368**, 191–247 (2016)
5. Borodin, A., Corwin, I.: Macdonald processes. Probab. Theory Relat. Fields **158**(1), 225–400 (2014)
6. Korotkikh, S.: Hidden diagonal integrability of q-Hahn vertex model and Beta polymer model. arXiv:2105.05058 (2021)
7. Thiery, T., Le Doussal, P.: Exact solution for a random walk in a time-dependent 1D random environment: the point-to-point Beta polymer. J. Phys. A: Math. Theor. **50**(4), 045001 (2016)