Phase transitions in turbulence

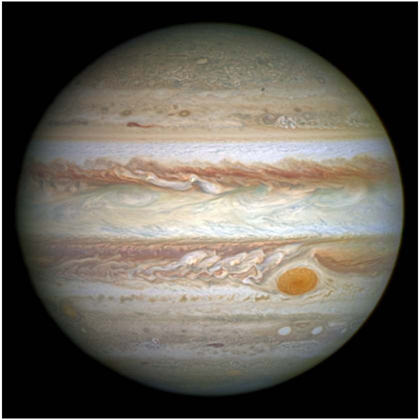
Alexandros Alexakis,

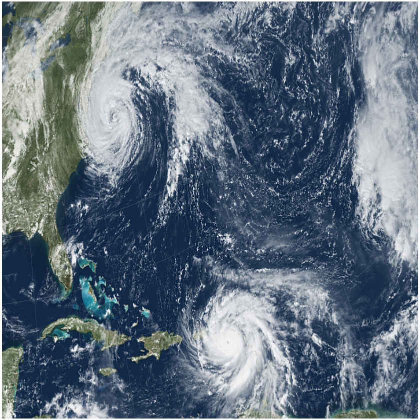
with many thanks for their contributions to:
Santiago Benavides, Kannabiran Seshasayanan,
Adrian van Kan, Takahiro Nemoto, Xander de Wit
Pablo Minini, Raffaele Marino, Francois Petrelis, Basile Gallet
Luca Biferale & Marc Brachet

A. Alexakis Reviews of Modern Plasma Physics, 7, 31 (2023),

A. Alexakis, L. Biferale Physics Reports 767-769, 1-101 (2018)

Turbulence and large scale structures





More often than not turbulent flows in nature deviate from the classical description of homogeneous and isotropic turbulence self-organizing into large scale coherent structures

Upscale and downscale cascades in planets, the lab and in simulations

nature physics LETTERS
PUBLISHED ONLINE: 6 FEBRUARY 2011 | DOI: 10.1038/NPHYS1910

Upscale energy transfer in thick turbulent fluid layers

H. Xia[1], D. Byrne[1], G. Falkovich[2] and M. Shats[1]*

nature physics ARTICLES
https://doi.org/10.1038/s41567-021-01458-y
Check for updates

OPEN

Moist convection drives an upscale energy transfer at Jovian high latitudes

Lia Siegelman[1], Patrice Klein[2,3,4], Andrew P. Ingersoll[2], Shawn P. Ewald[2], William R. Young[1],
…ndro Mura[6], Alberto Adriani[6], Davide Grassi[6], Christina Plainaki[7] and

nature physics ARTICLES
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OPEN

Forward and inverse kinetic energy cascades in Jupiter's turbulent weather layer.

Roland M. B. Young[1] and P.L. Read[2]

Geophysical Research Letters

Height-dependent transition from 3-D to 2-D turbulence in the hurricane boundary layer

David Byrne, Jun A. Zhang

AGU ADVANCING EARTH AND SPACE SCIENCES

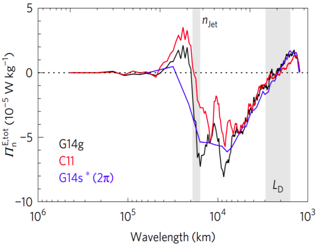
RESEARCH ARTICLE
ATMOSPHERIC DYNAMICS

Science AAAS

Large-scale self-organization in dry turbulent atmospheres

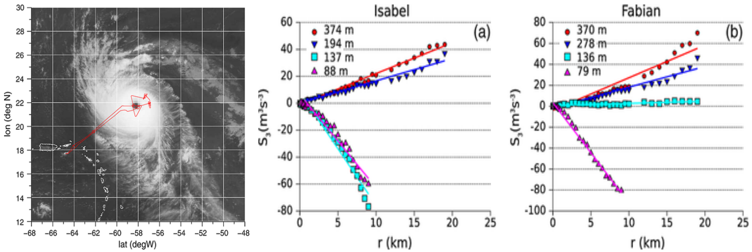
Alexandros Alexakis[1]*, Raffaele Marino[2], Pablo D. Mininni[3], Adrian van Kan[4], Raffaello Foldes[2], Fabio Feraco[2,5]

Coexistence of forward and inverse cascades



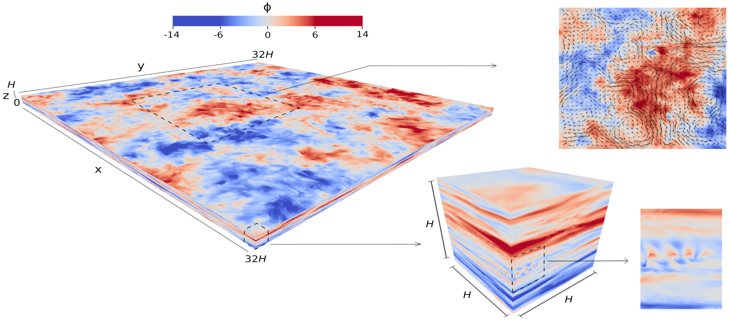
- **Cassini observations in Jupiter's atmosphere**
  Roland M. B. Young & Peter L. Read, Nature Physics (2017)
  "Forward and inverse kinetic energy cascades in Jupiter's turbulent weather layer"

Coexistence of forward and inverse cascades



- **Airplane flights in a hurricane's boundary layer**
  David Byrne & Jun A. Zhang - Geophys.R.L (2013)
  "*Height dependent transition from 3D to 2D turbulence in the hurricane boundary layer*"

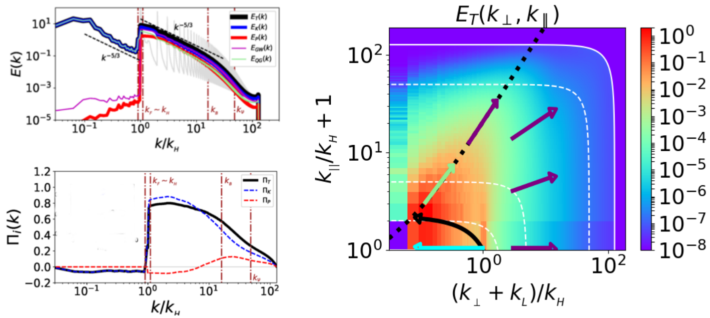
Coexistence of forward and inverse cascades



- **Numerical Simulations** $12288^2 \times 384$
  A. Alexakis, R. Marino, P.D. Mininni, A. van Kan, R. Foldes & F. Feraco - Science (2024)
  "*Large-scale self-organization in dry turbulent atmospheres*"

Coexistence of forward and inverse cascades

Energy spectra $E(k_{\|}, k_{\perp})$, $E(k)$ and Energy flux $\Pi_E(k)$



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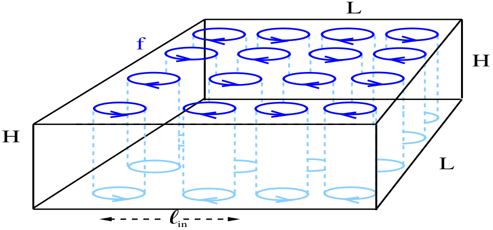
  A. Alexakis, R. Marino, P.D. Mininni, A. van Kan, R. Foldes & F. Feraco - Science (2024)
  "*Large-scale self-organization in dry turbulent atmospheres*"

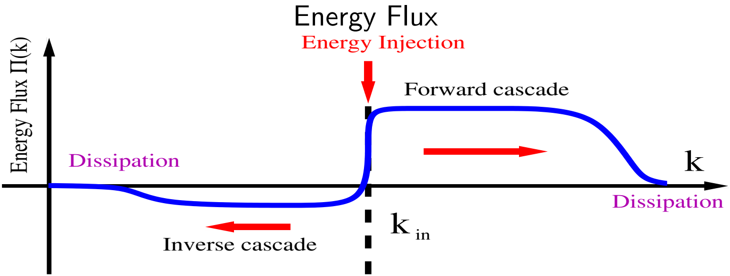
A simplified set up



$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P' + \nu \nabla^2 \mathbf{u} - \alpha \overline{\mathbf{u}} + \mathbf{f}$$

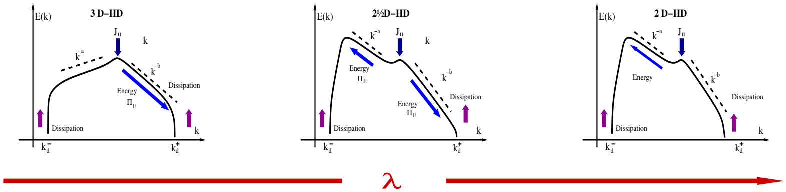
"*Everything should be made as simple as possible, but not simpler*"
Albert Einstein

Split/Bidirectional Cascade



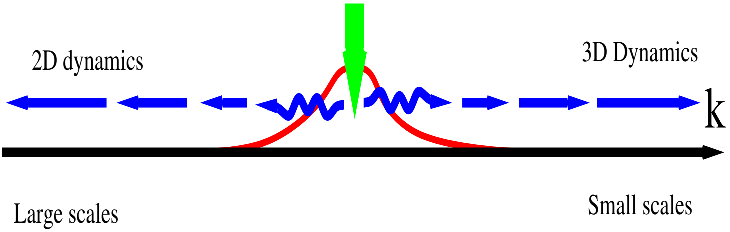
- Energy is injected at a scale $\ell_{in} = k_{in}^{-1}$
- Part of it is transferred to small scales and part of it to large scales
- The ratio of forward to inverse cascade depends on the system parameters

Split/Bidirectional Cascade



- the system transitions from one turbulent state (forward cascading) to an other (inverse cascading) varying a parameter $\lambda = \ell_{in}/H$.
- the transition occurs in the presence of turbulence ($\lambda \neq Re$).
- through a state that cascades energy both forward and inversely: **Split Cascade!**

Can split Cascades exist?

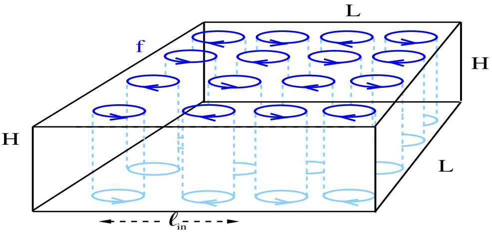


If the system is such that

- Large scales support an inverse cascade
- Small scales support a forward cascade
- The system is forced at intermediate scales

then a split cascade can possibly build up by what ever energy reaches large enough and small enough scales by *spectrally diffusive* processes.

Control and order Parameters



$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P' + \nu \nabla^2 \mathbf{u} - \alpha \bar{\mathbf{u}} + \mathbf{f}$$

- H: Box Height,
- ℓ_{in}: Forcing length
- ν = viscosity
- L: Box Length
- Fixed energy injection ϵ_{in}
- α = Large scale drag

L.M. Smith, J.R. Chasnov, F. Waleffe, Phys. Rev. Lett. **77** 2467 (1996),
A. Celani, S. Musacchio, D. Vincenzi, Phys. Rev. Lett. **104**, 184506 (2010)

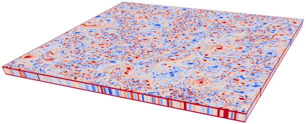
Control and order Parameters

Forcing scale control parameters:

$$\boxed{\lambda = \frac{\ell_{in}}{H}}$$

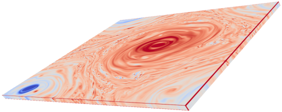
Viscous & Domain size parameters

$$Re = \frac{\epsilon_{in}^{1/3}\ell_{in}^{4/3}}{\nu} \to \infty, \quad R_\alpha = \frac{\epsilon_{in}^{1/3}\ell_{in}^{-2/3}}{\alpha} \to \infty, \quad \Lambda = \frac{L}{\ell_{in}} \to \infty$$



Inverse Cascade

$$1 \ll R_\alpha \ll \Lambda$$



Condensate

$$1 \ll \Lambda \ll R_\alpha$$

Control and order Parameters

Forcing scale control parameters:

$$\boxed{\lambda = \frac{\ell_{in}}{H}}$$

Viscous & Domain size parameters

$$Re = \frac{\epsilon_{in}^{1/3}\ell_{in}^{4/3}}{\nu} \to \infty, \quad R_\alpha = \frac{\epsilon_{in}^{1/3}\ell_{in}^{-2/3}}{\alpha} \to \infty, \quad \Lambda = \frac{L}{\ell_{in}} \to \infty$$

Order parameters (inverse cascade $1 \ll R_\alpha \ll \Lambda$)

$$Q_\alpha = \frac{\epsilon_\alpha}{\epsilon_{in}}, \quad Q_\nu = \frac{\epsilon_\nu}{\epsilon_{in}} \quad \text{with} \quad Q_\alpha + Q_\nu = 1$$

Where at steady state :

$$\epsilon_{in} = \langle \mathbf{f} \cdot \mathbf{u} \rangle, \quad \epsilon_\nu = \nu \langle |\nabla \mathbf{u}|^2 \rangle \quad \epsilon_\alpha = \alpha \langle |\overline{\mathbf{u}}|^2 \rangle$$

Control and order Parameters

Forcing scale control parameters:

$$\lambda = \frac{\ell_{in}}{H}$$

Viscous & Domain size parameters

$$Re = \frac{\epsilon_{in}^{1/3}\ell_{in}^{4/3}}{\nu} \to \infty, \quad R_\alpha = \frac{\epsilon_{in}^{1/3}\ell_{in}^{-2/3}}{\alpha} \to \infty, \quad \Lambda = \frac{L}{\ell_{in}} \to \infty$$

Order parameters (Condensate $1 \ll \Lambda \ll R_\alpha$)

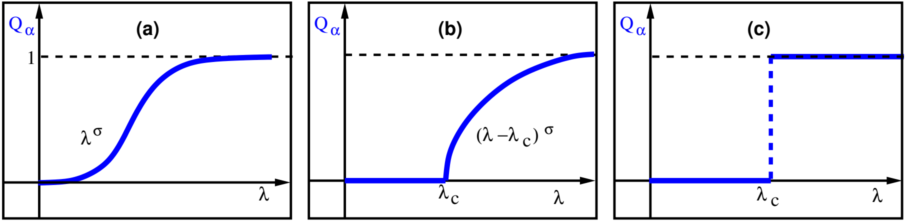
$$Q_\alpha = \frac{\langle |\mathbf{u_{LS}}|^2 \rangle}{(\epsilon_{in}\ell_f)^{2/3}}$$

Where at steady state :

$$\epsilon_{in} = \langle \mathbf{f} \cdot \mathbf{u} \rangle, \qquad \mathbf{u}_{LS} = \sum_{|\mathbf{k}| \leq q} \tilde{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \quad \text{(Large scale velocity)}$$

Classification: Smooth, 2nd order and 1st order transitions

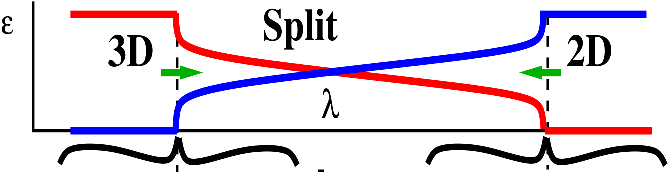
$$Q_\alpha = \frac{Inverse\ Energy\ flux}{Energy\ injection} \quad \text{or} \quad Q_\alpha = \frac{Large\ scale\ energy}{Forcing\ scale\ energy}$$

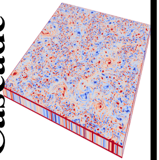
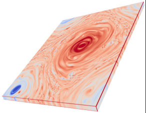


Smooth, 2nd order 1st order

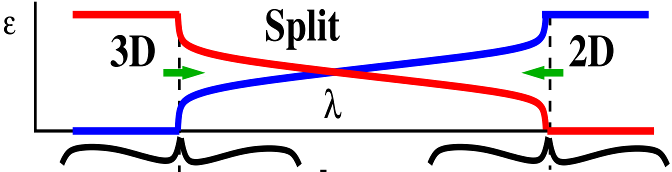
In case b and c we can talk about different **phases** & **phase transitions**

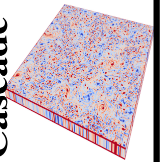
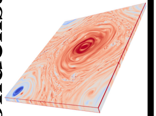
A. Alexakis, L. Biferale Physics Reports **767-769**, 1-101 (2018)



Cascade 	**Case I** From 3D turbulence to split cascade	**Case II** From 2D turbulence to split cascade
Condensate 	**Case III** From 3D turbulence to a condensate	**Case IV** From 2D condensate to split cascade

Transitions and Critical points



	3D → Split	Split ← 2D
Cascade 	**Case I** From 3D turbulence to split cascade	**Case II** From 2D turbulence to split cascade
Condensate 	**Case III** From 3D turbulence to a condensate	**Case IV** From 2D condensate to split cascade

Case I: thin layer $H < \ell_{in}$, $(\lambda = \ell_{in}/H)$

PRL 104, 184506 (2010) PHYSICAL REVIEW LETTERS week ending 7 MAY 2010

Turbulence in More than Two and Less than Three Dimensions

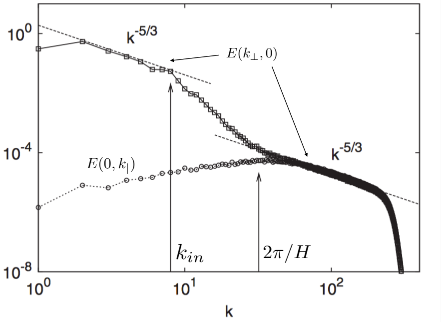
Antonio Celani,[1] Stefano Musacchio,[2,3] and Dario Vincenzi[3]

[1]*CNRS URA 2171, Institut Pasteur, 28 rue du docteur Roux, 75724 Paris Cedex 15, France*

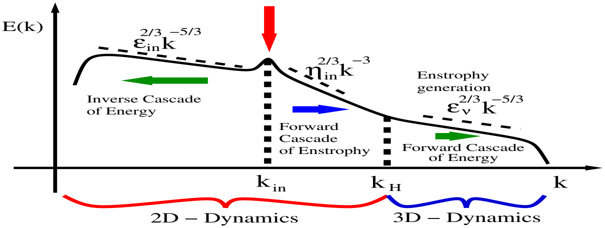
[2]*Dipartimento di Fisica Generale and INFN, Università di Torino, via P. Giuria 1, 10125 Torino, Italy*

[3]*CNRS UMR 6621, Laboratoire J. A. Dieudonné, Université de Nice Sophia Antipolis, Parc Valrose, 06108 Nice, France*

(Received 12 January 2010; published 7 May 2010)



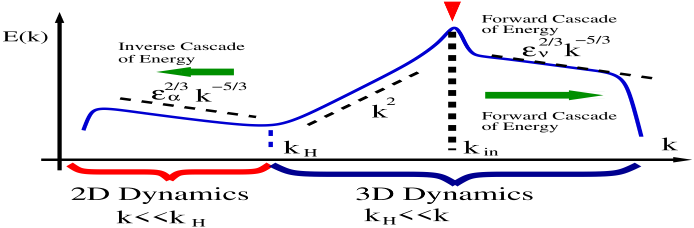
Case I: thin layer $H < \ell_{in}$, $(\lambda = \ell_{in}/H)$



- Enstrophy arrives at k_H at a rate $\eta_{in} = \epsilon_{in} k_{in}^2$
- Energy arrives at k_ν at a rate $\epsilon_\nu \propto \epsilon_{in} \left(\frac{k_{in}}{k_H}\right)^2$
- Energy that arrives at small scales is given by:

$$\epsilon_\nu = \epsilon_{in} \left(\frac{k_{in}}{k_H}\right)^2 = \epsilon_{in} \left(\frac{H}{\ell_{in}}\right)^2 = \epsilon_{in} \frac{1}{\lambda^2}$$

Boffetta, et al Phys. Rev. E 83, 066302 (2011)



$$E(k) \propto \epsilon_\alpha^{2/3} k^{-5/3} \qquad k \ll k_H$$

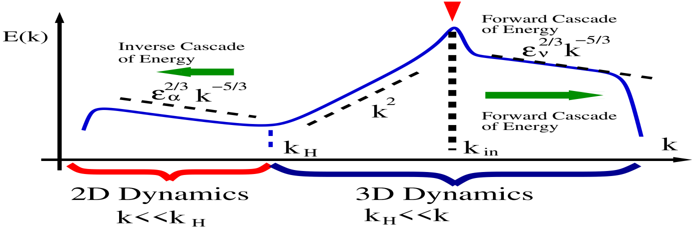
$$E(k) \propto (\epsilon_{in}^{2/3} k_{in}^{-11/3}) k^2 \qquad k_H \ll k \ll k_{in}$$

$$E(k) \propto \epsilon_{in}^{2/3} k^{-5/3} \qquad k_{in} \ll k$$

Equating $E(k)$ at $k = k_H$ we obtain

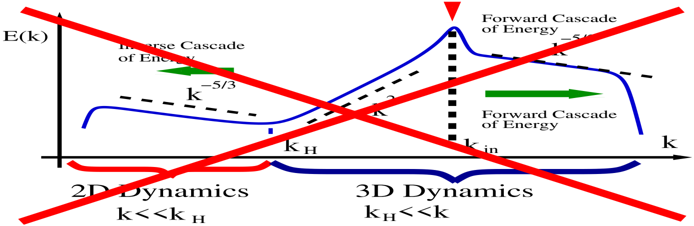
$$\boxed{\epsilon_\alpha = \epsilon_{in} \left(\frac{k_H}{k_{in}} \right)^{11/2} = \epsilon_{in} \lambda^{11/2}}$$

Case I: $H > \ell_{in}$, thick layer Turbulence $(\lambda = \ell_{in}/H)$



at $\ell \simeq H$:

Inverse Flux due to local 2D interactions		Forward flux due to a turbulent eddy viscosity
$\Pi_{INV} \propto u_H^3/H$		$\Pi_{FWD} \propto \nu_{eddy} \frac{U_H^2}{H^2}$
$\epsilon_{in} \left(\frac{k_H}{k_{in}}\right)^{11/2}$	$\ll$	$\epsilon_{in} \left(\frac{k_H}{k_{in}}\right)^{5}$



at $\ell \simeq H$:

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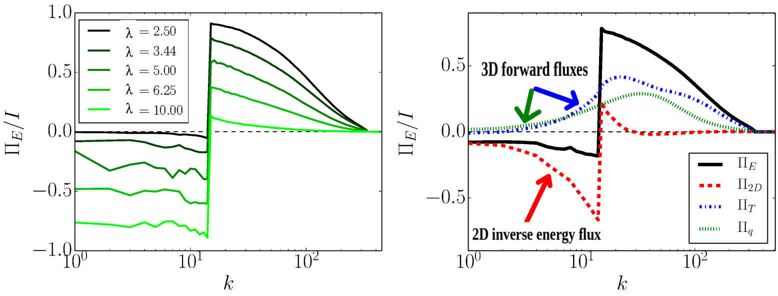
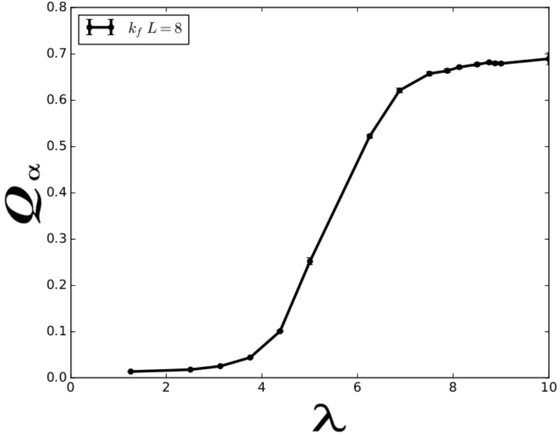
$\epsilon_{in} \left(\frac{k_H}{k_{in}}\right)^{11/2}$	$\ll$	$\epsilon_{in} \left(\frac{k_H}{k_{in}}\right)^{5}$

Case I: From 3D to a 2D cascade, $(\lambda = \ell_{in}/H)$



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

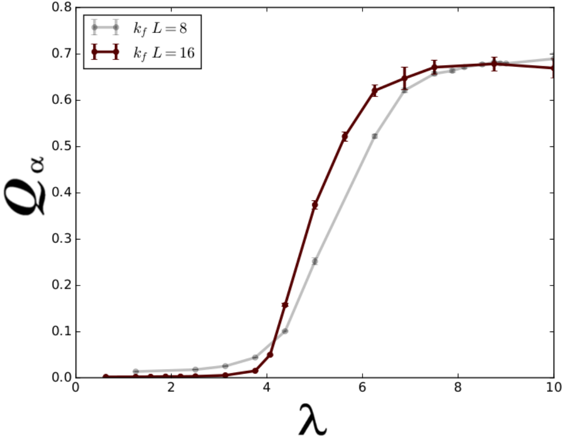
Inverse flux $(Q_\alpha = \epsilon_\alpha/\epsilon_{in})$ as a function of $(\lambda = \ell_{in}/H)$



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Case I: From 3D to a Split cascade, $(\lambda = \ell_{in}/H)$

Inverse flux $(Q_\alpha = \epsilon_\alpha/\epsilon_{in})$ as a function of $(\lambda = \ell_{in}/H)$



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

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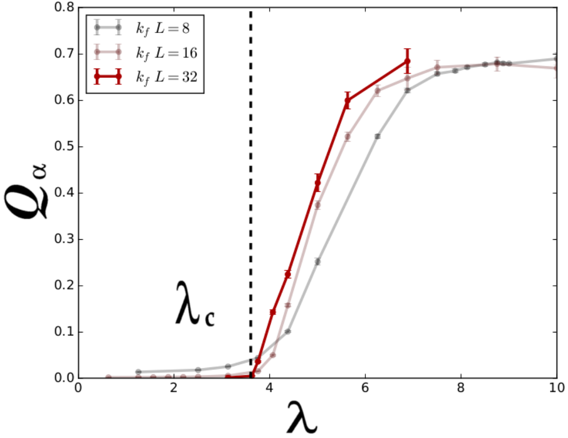

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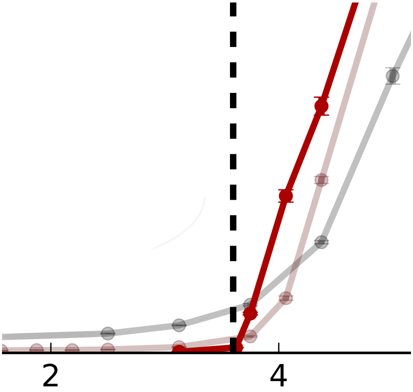
Inverse flux $(Q_\alpha = \epsilon_\alpha/\epsilon_{in})$ as a function of $(\lambda = \ell_{in}/H)$



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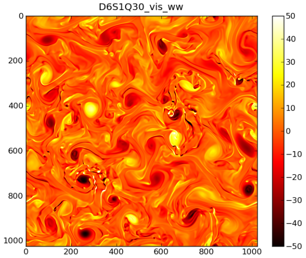
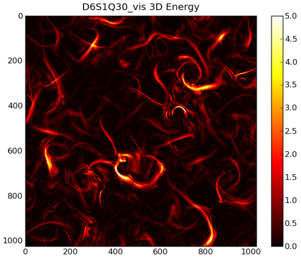
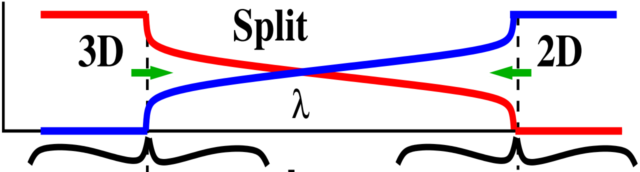
Case I: From 3D to a 2D cascade, $(\lambda = \ell_{in}/H)$

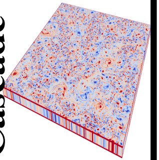
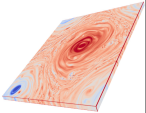




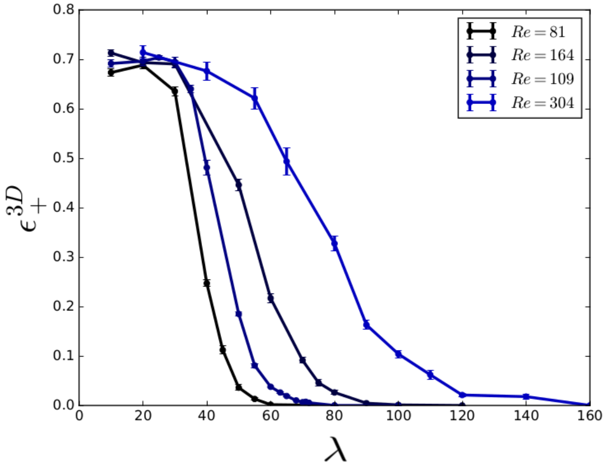
Predator-prey dynamics?

Transitions and Critical points



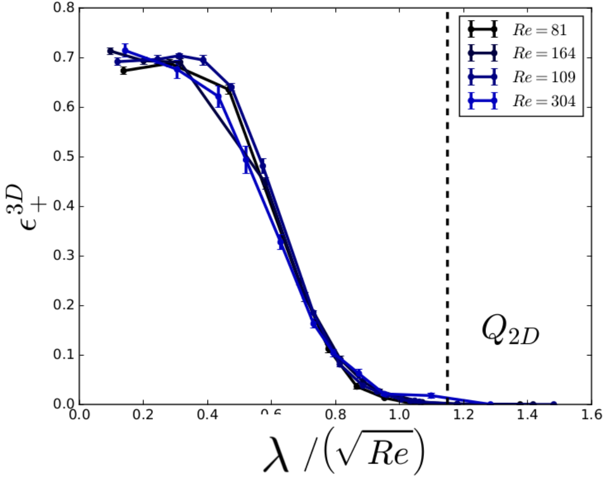
Cascade 	**Case I** From 3D turbulence to split cascade	**Case II** From 2D turbulence to split cascade
Condensate 	**Case III** From 3D turbulence to a condensate	**Case IV** From 2D condensate to split cascade

Case II: From 2D to a Split cascade, $(\lambda = \ell_{in}/H)$

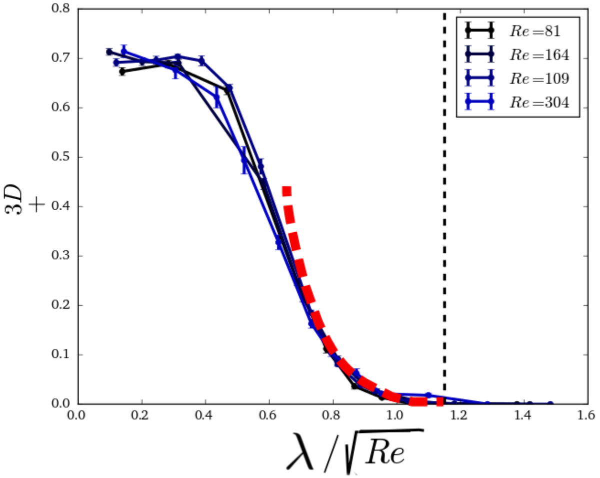


Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Case II: From 2D to a Split cascade, $(\lambda = \ell_{in}/H)$

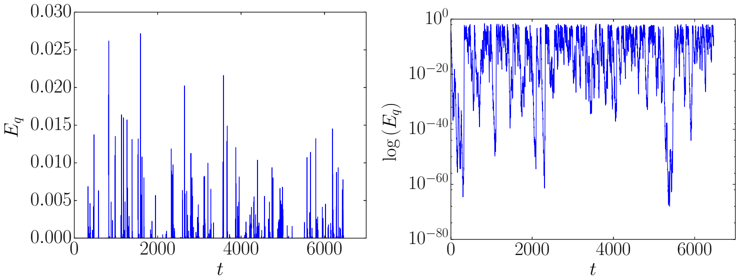


Results: Near Q_{2D}



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Intermittency



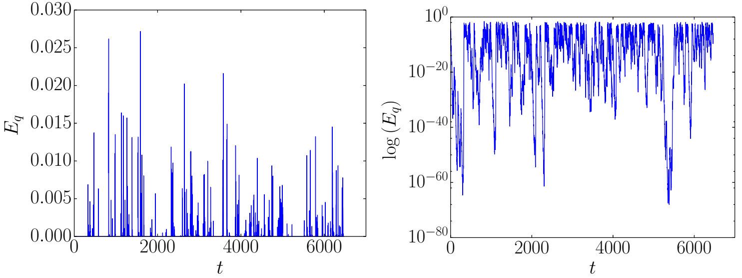
$$\mathbf{u} = \mathbf{u}_{2D} + \mathbf{v}_{3D}, \qquad \mathbf{v}_{3D} \ll \mathbf{u}_{2D}$$

$$\partial_t \mathbf{u}_{2D} + \mathbf{u}_{2D} \cdot \nabla \mathbf{u}_{2D} = -\overline{\mathbf{v}_{3D} \cdot \nabla \mathbf{v}_{3D}} - \overline{\nabla P} + \nu \Delta \mathbf{u}_{2D} + \eta \Delta^{-2} \mathbf{u}_{2D} + \mathbf{F}_{2D}$$

$$\partial_t \mathbf{v}_{3D} + \mathbf{u}_{2D} \cdot \nabla \mathbf{v}_{3D} + \mathbf{v}_{3D} \cdot \nabla \mathbf{u}_{2D} = -\mathbf{v}_{3D} \cdot \nabla \mathbf{v}_{3D} + \nabla p' + \nu \Delta \mathbf{v}_{3D}$$

Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

On-Off Intermittency



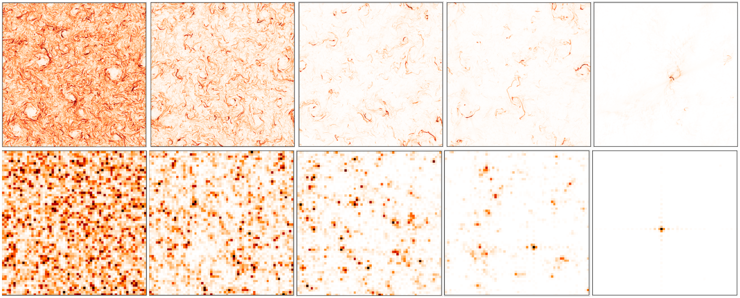
$$\frac{d}{dt}A = (\mu + \xi)A - A^3 \qquad \text{or} \qquad \frac{d}{dt}\log(A) = (\mu + \xi) - A^2$$

$\mu = \lambda_c - \lambda$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$, $\boxed{\langle A^n \rangle \propto \mu}$

H. Fujisaka et al, Prog. Theor. Phys. 76 1198 (1986).
N. Platt, et al, Phys. Rev. Lett. 70, 279 (1993)

Case II: From 2D to a Split cascade, $(\lambda = \ell_{in}/H)$

Extensive On-Off Intermittency



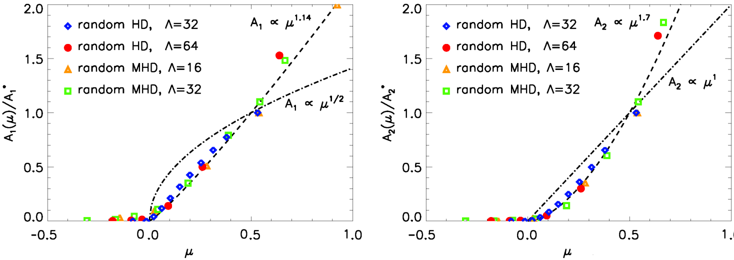
$\lambda/\lambda_c = 0.60,$ $\lambda/\lambda_c = 0.70,$ $\lambda/\lambda_c = 0.80,$ $\lambda/\lambda_c = 0.90,$ $\lambda/\lambda_c = 0.95,$

$$\frac{d}{dt}A = (\mu + \xi)A - A^3 + \kappa\nabla^2 A$$

$\mu = \lambda_c - \lambda$ and
$\langle \xi(t)\xi(t')\rangle = \delta(t - t')$

Y. Tu, G. Grinstein, and M. A. Munoz (1997)
W. Genovese, M.A. Munoz, and J. M. Sancho, (1998)

Universality class of multiplicative noise



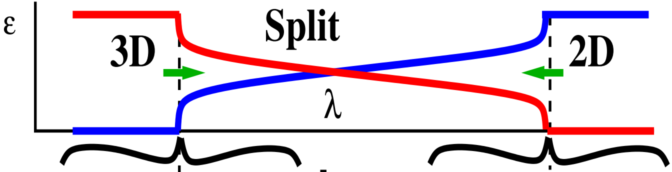
$$\mu = \lambda_c - \lambda$$

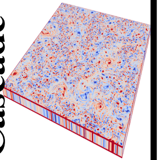
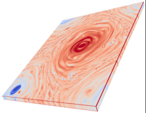
$$A_1 = \langle |\mathbf{v_{3D}}| \rangle, \qquad A_2 = \langle |\mathbf{v_{3D}}|^2 \rangle$$

$$\text{Model Predictions}: \quad \boxed{A_1 \propto \mu^{1.14...}}, \qquad A_2 \propto \boxed{\mu^{1.7...}}$$

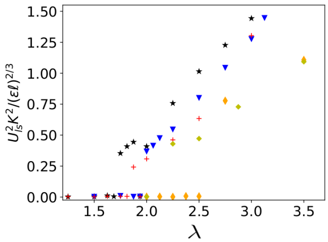
A. Alexakis, F. Petrelis, S.J. Benavides, K. Seshasayanan P.R.Fluids 6, 024605 (2021)

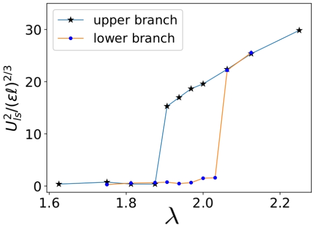
Transitions and Critical points



Cascade		**Case I** From 3D turbulence to split cascade	**Case II** From 2D turbulence to split cascade
Condensate		**Case III** From 3D turbulence to a condensate	**Case IV** From 2D condensate to split cascade

Case III: From 3D to a Condensate, $(\lambda = \ell_{in}/H)$

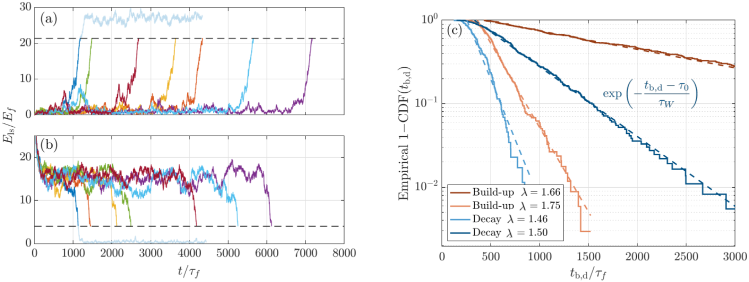




- Discontinuous transition to a condensate
- Hysterisis diagram

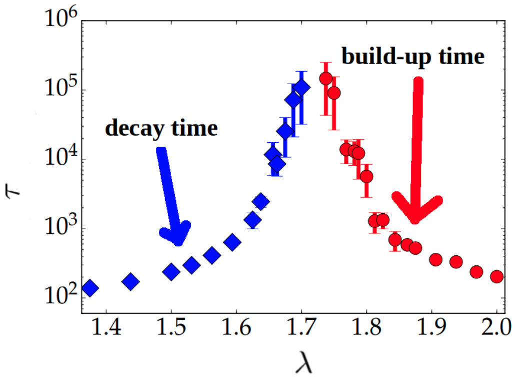
A. van Kan, A. Alexakis Journal of Fluid Mechanics 864, 490-518 (2019)

Case III: From 3D to a Condensate, $(\lambda = \ell_{in}/H)$



- Rare transitions from one state to the other
- Exponentially distributed transition times (memoryless process)
- Mean transition time $\tau(\lambda, \Lambda, Re)$

Case III: From 3D to a Condensate, $(\lambda = \ell_{in}/H)$



- Mean transition τ time diverges as criticality is approached

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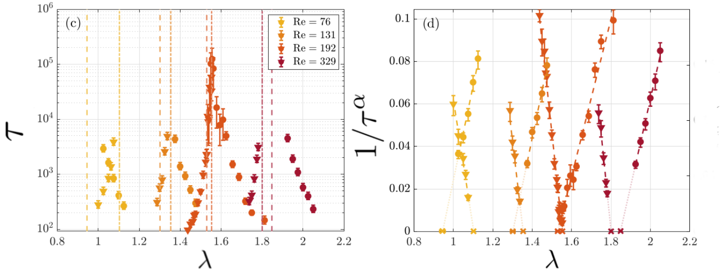
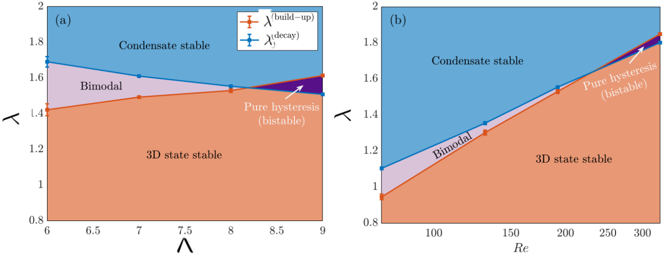
A. van Kan, T. Nemoto, A. Alexakis, Journal of Fluid Mechanics 878, 356-369 (2019)



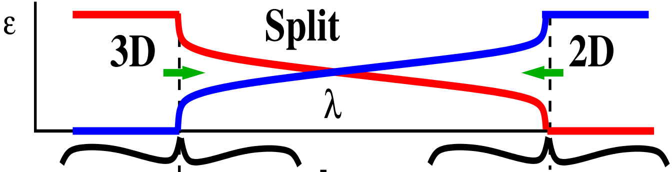
- Fit $\tau = \tau^*(\lambda - \lambda_c)^{-1/\alpha}$
- Decay: $\alpha = 1/2$, Build-up: $\alpha = 1/3$

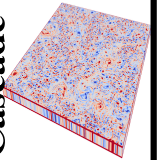
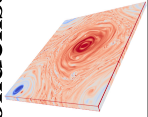
X.M. de Wit, A. van Kan, A. Alexakis J. Fluid Mech. 939, R2 (2022)



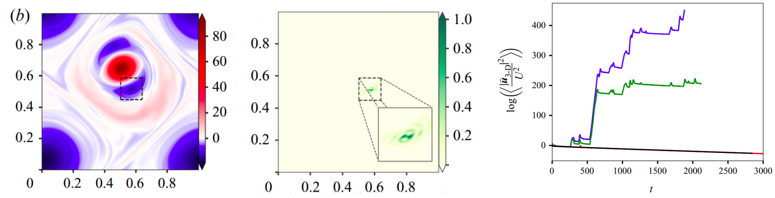
- Always in Condensate state
- Always in 3D turbulence state
- Bimodal: random switching from one state to the other
- True hysteresis: both states are attractors

Transitions and Critical points



Cascade 	**Case I** **From 3D turbulence to split cascade**	**Case II** **From 2D turbulence to split cascade**
Condensate 	**Case III** **From 3D turbulence to a condensate**	**Case IV** **From 2D condensate to split cascade**

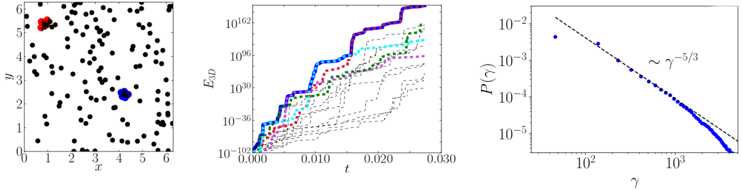
Case IV: From Condensate to 3D turbulence



- Localized unstable modes
- Long quiet periods $A \propto e^{-0.001t}$
- Short time of very fast growth $A \propto e^{-100.0t}$

K Seshasayanan, B Gallet Journal of Fluid Mechanics 901, R5 (2020)

A point vortex model



- Power-law distributed growth-rates of 3D mode amplitudes A
- A performs a Lévy walk in log-space

$$\frac{d}{dt}A = (\mu + \xi_L)A - A^3$$

where ξ_L is a Lévy noise

---

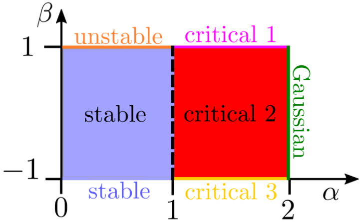
such that if $x = \int_0^T \xi_L \, dt, \quad P(x) = P_{\alpha,\beta}(x/T^{1/\alpha})$

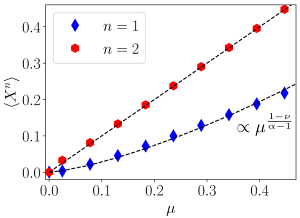
$$P_{\alpha,\beta}(|y| \to \infty) \propto (1 + \beta \mathrm{sign}(y))|y|^{-1-\alpha}$$

A. van Kan, A. Alexakis, M.E. Brachet Phys. Rev. E 103, 053102 (2021)

Case IV: From Condensate to 3D turbulence

A new type of intermitency

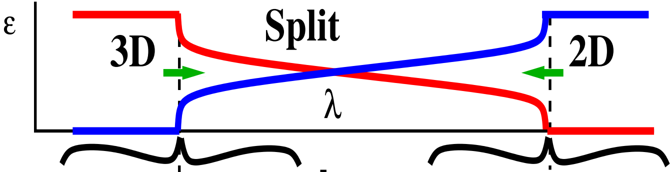


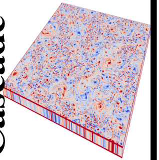
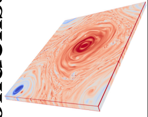


- α, β parameters of Lévy noise
- New exponents $\boxed{\langle A^n \rangle \propto \mu^{\gamma(\alpha,\beta,n)}}$

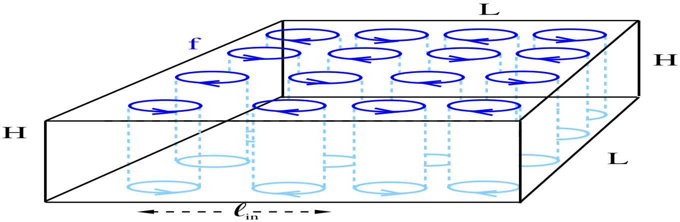
A Van Kan, F Petrelis Journal of Statistical Mechanics 013204 (2023)

Summary



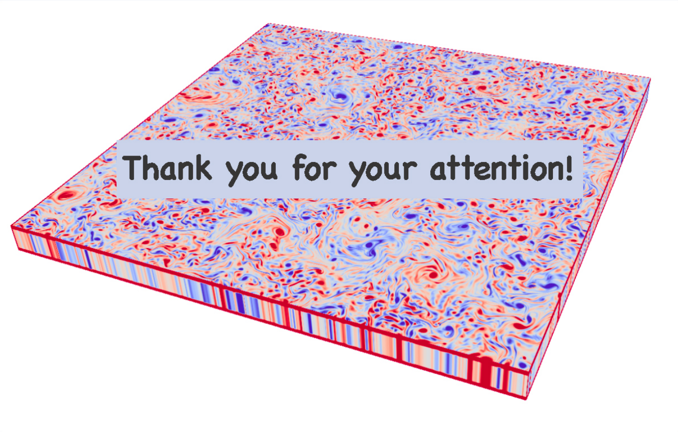
Cascade 	**Case I** Predator Prey Dynamics	**Case II** Extensive On–Off intermittency
Condensate 	**Case III** Hysteresis rare transitions	**Case IV** Levy On–Off intermittency

Phase transitions in turbulence have shown a far richer phenomenology than ever anticipated.



Take home message

- Take a system
- Simplify as much as possible
- Push it to its limits
- **Interesting Physics will come out!**


Thank you for your attention!