Solution Set for Exercise Session No.5

Course: Mathematical Aspects of Symmetries in Physics,
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1 Some Basics on Manifolds

To see that S^2 and $\mathbb{R}P^2$ are Hausdorff spaces, please refer to some topology books.
(1)

1. To cover S^2, we introduce open sets $\{O_x^\pm, O_y^\pm, O_z^\pm\}$ where $O_x^\pm$ etc. are defined by

$$
\begin{aligned}
O_x^+ &= \{(x,y,z) \in S^2 | x > 0\}\,, \qquad O_x^- = \{(x,y,z) \in S^2 | x < 0\}\,,\\
O_y^+ &= \{(x,y,z) \in S^2 | y > 0\}\,, \qquad O_y^- = \{(x,y,z) \in S^2 | y < 0\}\,,\\
O_z^+ &= \{(x,y,z) \in S^2 | z > 0\}\,, \qquad O_z^- = \{(x,y,z) \in S^2 | z < 0\}\,.
\end{aligned}
$$

One can construct maps from $O_x^\pm$ etc. to a unit open disk in $\mathbb{R}^2$, $D^2 = \{(a,b) \in \mathbb{R}^2 | a^2 + b^2 < 1\}$ as

$$
\begin{aligned}
\varphi_x^+(x,y,z) = \varphi_x^-(x,y,z) = (y,z)\,,\\
\varphi_y^+(x,y,z) = \varphi_y^-(x,y,z) = (x,z)\,,\\
\varphi_z^+(x,y,z) = \varphi_z^-(x,y,z) = (x,y)\,.
\end{aligned}
$$

We check that this atlas is indeed an atlas of class C^∞ (an atlas where all the transition functions are C^∞ and bijective and their inverses are C^∞). It is obvious that the union of the open sets satisfies $O_x^+ \cup O_x^- \cup O_y^+ \cup O_y^- \cup O_z^+ \cup O_z^- = S^2$. Here we note that $\varphi_x^\pm : O_x^\pm \to D^2$, $\varphi_y^\pm : O_y^\pm \to D^2$ and $\varphi_z^\pm : O_z^\pm \to D^2$. Then the inverses of these maps are

$$
\begin{aligned}
(\varphi_x^\pm)^{-1}(a,b) = (\pm(1-a^2-b^2)^{1/2}, a, b)\,,\\
(\varphi_y^\pm)^{-1}(a,b) = (a, \pm(1-a^2-b^2)^{1/2}, b)\,,\\
(\varphi_z^\pm)^{-1}(a,b) = (a, b, \pm(1-a^2-b^2)^{1/2})\,.
\end{aligned}
$$

Obviously these $\varphi_x^\pm$ etc. are homeomorphism (bijective, continuous, and their inverses are also continuous).

Let us now consider the intersection of O_x^+ and O_y^+, $O_x^+ \cap O_y^+ = \{(x,y,z) \in S^2 | x > 0, y > 0\}$, which is not empty. Then the transition function from $\varphi_x^+(O_x^+ \cap O_y^+) = \{(c,d) \in D^2 | d > 0\}$ to $\varphi_y^+(O_x^+ \cap O_y^+) = \{(a,b) \in D^2 | a > 0\}$ is given by (for $(a,b) \in \varphi_x^+(O_x^+ \cap O_y^+)$)

$$
\varphi_y^+ \circ (\varphi_x^+)^{-1}(a,b) \;=\; \varphi_y^+((1-a^2-b^2)^{1/2}, a, b)
$$

$$= \quad ((1-a^2-b^2)^{1/2}, b) \,.$$

This is obviously infinite time differentiable and bijective. The inverse of transition function is (for $(c,d) \in \varphi_y^+(O_x^+ \cap O_y^+)$)

$$\begin{aligned}(\varphi_y^+ \circ (\varphi_x^+)^{-1})^{-1}(c,d) &= \varphi_x^+ \circ (\varphi_y^+)^{-1}(c,d) \\ &= \varphi_x^+(c, (1-c^2-d^2)^{1/2}, d) \\ &= ((1-c^2-d^2)^{1/2}, d) \,.\end{aligned}$$

This map is also infinite time differentiable. Therefore this transition function is C^∞ and bijective and its inverse is also C^∞. We can show the same statement for the other overlaps of the open sets in a similar way. Therefore S^2 is a two-dimensional differentiable manifold of class C^∞.

2. We can easily see that $U^+ \cup U^- = S^2$. Let us next construct $f^\pm$ explicitly. The map f^+ is a stereographic projection from the north pole $(0,0,1)$ to (x,y)-plane (that is, $f^+ : U^+ \to \mathbb{R}^2$). Thus, under this map, we can identify where $(x,y,z) \in S^2 \backslash \{(0,0,1)\}$ is mapped to as follows: by solving

$$t((x,y,z)-(0,0,1)) + (0,0,1) = (X,Y,0)$$

with respect to t, we obtain $t = 1/(1-z)$. Thus f^+ is

$$f^+(x,y,z) = \left(\frac{x}{1-z}, \ \frac{y}{1-z} \right) .$$

Similarly we can construct $f^- : U^- \to \mathbb{R}^2$ as (we have only to replace $(0,0,1)$ by $(0,0,-1)$ in the above computation)

$$f^-(x,y,z) = \left(\frac{x}{1+z}, \ \frac{y}{1+z} \right) .$$

These $f^\pm$ are obviously bijective and continuous.

Now we calculate the inverse of f^+. Since $X = x/(1-z)$ and $Y = y/(1-z)$, we have

$$X^2 + Y^2 = \frac{x^2+y^2}{(1-z)^2} = \frac{1-z^2}{(1-z)^2} = \frac{1+z}{1-z} \,.$$

Here we have used $x^2+y^2+z^2 = 1$. Then by solving this, we obtain z as

$$z = \frac{X^2+Y^2-1}{X^2+Y^2+1} \,.$$

Then x and y are

$$x = \frac{2X}{X^2+Y^2+1} \,, \qquad y = \frac{2Y}{X^2+Y^2+1} \,.$$

To summarize, $(f^+)^{-1} : \mathbb{R}^2 \to U^+$ is (for $(X,Y) \in \mathbb{R}^2$)

$$(f^+)^{-1}(X,Y) = \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1} \right) .$$

In a similar way, we can get $(f^-)^{-1} : \mathbb{R}^2 \to U^-$ as

$$(f^-)^{-1}(X,Y) = \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{1-X^2-Y^2}{X^2+Y^2+1} \right) .$$

These inverse maps are obviously continuous. Thus $f^\pm$ are homeomorphism (bijective, continuous, and their inverse is also continuous).

Now we consider the transition function and its differentiability. We consider the intersection $U^+ \cap U^- = S^2 \backslash \{(0,0,\pm 1)\} (\neq \emptyset)$. Then the transition function is $f^+ \circ (f^-)^{-1} : f^-(U^+ \cap U^-) = \mathbb{R}^2 \backslash \{(0,0)\} \to f^+(U^+ \cap U^-) = \mathbb{R}^2 \backslash \{(0,0)\}$. This map is explicitly written as (for $(X,Y) \in f^-(U^+ \cap U^-)$)

$$\begin{aligned} f^+ \circ (f^-)^{-1}(X,Y) &= f^+ \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{1-X^2-Y^2}{X^2+Y^2+1} \right) \\ &= \left(\frac{X}{X^2+Y^2}, \frac{Y}{X^2+Y^2} \right) , \end{aligned}$$

and its inverse is (for $(X,Y) \in f^+(U^+ \cap U^-)$)

$$\begin{aligned} (f^+ \circ (f^-)^{-1})^{-1}(X,Y) &= f^- \circ (f^+)^{-1}(X,Y) \\ &= f^- \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1} \right) \\ &= \left(\frac{X}{X^2+Y^2}, \frac{Y}{X^2+Y^2} \right) . \end{aligned}$$

Obviously from these expressions, this transition function is bijective and infinite time differentiable, and its inverse is also infinite time differentiable. Thus this transition function is C^∞ and bijective and its inverse is also C^∞. Therefore, we have confirmed that S^2 is a two-dimensional differentiable manifold.

(2)

1. We first notice that $U_x \cup U_y \cup U_z = \mathbb{R}P^2$. It is obvious that $\varphi_x, \varphi_y, \varphi_z$ are continuous and bijective maps to $\mathbb{R}^2$. We also note that the inverses of $\varphi_x, \varphi_y, \varphi_z$ are

$$\begin{aligned} (\varphi_x)^{-1}(Y,Z) &= [1 : Y : Z] , \qquad \text{for } (Y,Z) \in \mathbb{R}^2 , \\ (\varphi_y)^{-1}(X,Z) &= [X : 1 : Z] , \qquad \text{for } (X,Z) \in \mathbb{R}^2 , \\ (\varphi_z)^{-1}(X,Y) &= [X : Y : 1] , \qquad \text{for } (X,Y) \in \mathbb{R}^2 . \end{aligned}$$

which are all continuous. Thus $\varphi_x, \varphi_y, \varphi_z$ are all homeomorphism.

Now we consider the intersection $U_x \cap U_y = \{[x : y : z] \in \mathbb{R}P^2 | x \neq 0, y \neq 0\} \neq \emptyset$. The transition function $\varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) = \{(X, Z) \in \mathbb{R}^2 | X \neq 0\} \to \varphi_y(U_x \cap U_y) = \{(Y, Z) \in \mathbb{R}^2 | Y \neq 0\}$ is given by (for $(Y, Z) \in \varphi_x(U_x \cap U_y)$ and $Y \neq 0$)

$$\varphi_y \circ \varphi_x^{-1}(Y, Z) = \varphi_y([1 : Y : Z]) = \left(\frac{1}{Y}, \frac{Z}{Y} \right),$$

and its inverse map is (for $(X, Z) \in \varphi_y(U_x \cap U_y)$ and $X \neq 0$)

$$\begin{aligned} (\varphi_y \circ \varphi_x^{-1})^{-1}(X, Z) &= \varphi_x \circ \varphi_y^{-1}(X, Z) \\ &= \varphi_x([X : 1 : Z]) \\ &= \left(\frac{1}{X}, \frac{Z}{X} \right). \end{aligned}$$

This transition function is obviously bijective and infinite time differentiable, and its inverse is also infinite time differerentiable. We can check these properties of the other transition functions in a similar way. Thus $\mathbb{R}P^2$ is a two-dimensional differentiable manifold of class C^∞.

2. We denote the equivalence class for $(x, y, z) \in S^2$ under the identification of the antipodal point as $[x, y, z] \in S^2/\sim$. Let us consider a map $f : \mathbb{R}^3 \backslash \{(0, 0, 0)\} \to S^2$ defined by

$$f(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right).$$

From this, one can construct a map from $\mathbb{R}P^2$ to $S^2/\sim$ as

$$\bar{f} : [x : y : z] \quad \mapsto \quad \left[\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right],$$

for $[x : y : z] \in \mathbb{R}P^2$. Let us now consider a map $g : S^2 \to \mathbb{R}^3 \backslash \{(0, 0, 0)\}$ defined by $g(x, y, z) = (x, y, z)$. This induces a map $\bar{g} : S^2/\sim \to \mathbb{R}P^2$ as $\bar{g}([x, y, z]) = [x : y : z]$. One can confirm that $\bar{g}$ is the inverse of $\bar{f}$ as follows: For $(x, y, z) \in S^2$, we have

$$(\bar{f} \circ \bar{g})([x, y, z]) = \bar{f}([x : y : z]) = \left[\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right] = [x, y, z].$$

Here we have used $x^2 + y^2 + z^2 = 1$ since $(x, y, z) \in S^2$. We can also show, for $(x, y, z) \in \mathbb{R}^3 \backslash \{(0, 0, 0)\}$,

$$\begin{aligned} (\bar{g} \circ \bar{f})([x : y : z]) &= \bar{g} \left(\left[\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right] \right) \\ &= \left[\frac{x}{\sqrt{x^2 + y^2 + z^2}} : \frac{y}{\sqrt{x^2 + y^2 + z^2}} : \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right] \\ &= [x : y : z]. \end{aligned}$$

Thus we have confirmed that $\bar{g}$ is the inverse of $\bar{f}$. It is obvious that $\bar{f}$ is bijective and continuous, and its inverse $\bar{g}$ is continuous. Thus we have confirmed that $S^2/\sim$ is homeomorphic (bijective, continuous, continuous inverse) to $\mathbb{R}P^2$).

2 Tangent Vector

We note that $p \in U^+$. Let us recall that, by using the stereographic projection from the north pole, we have introduced a local coordinate on the open set $S^2 \backslash \{(0,0,1)\}$ as

$$(X, Y) = f^+(x, y, z) = \left(\frac{x}{1-z}, \; \frac{y}{1-z} \right) .$$

Thus we have

$$(f^+ \circ c)(t) = \left(\frac{\sin\phi_0 \cos t}{1-\cos\phi_0}, \; \frac{\sin\phi_0 \sin t}{1-\cos\phi_0} \right) .$$

By using this, the tangent vector at p is given by

$$\begin{aligned} V_p &= \frac{d}{dt}\left(\frac{\sin\phi_0 \cos t}{1-\cos\phi_0} \right)_{t=0} (\partial_X)_p + \frac{d}{dt}\left(\frac{\sin\phi_0 \sin t}{1-\cos\phi_0} \right)_{t=0} (\partial_Y)_p \\ &= \frac{\sin\phi_0}{1-\cos\phi_0} (\partial_Y)_p \, . \end{aligned}$$