Solution Set for Exercise Session No.3

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

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1 More on Representations

(1)

1. We first recall the multiplication table for D_3 given in Table 1.

	e	c_3	c_3^{-1}	σ_1	σ_2	σ_3
e	e	c_3	c_3^{-1}	σ_1	σ_2	σ_3
c_3	c_3	c_3^{-1}	e	σ_3	σ_1	σ_2
c_3^{-1}	c_3^{-1}	e	c_3	σ_2	σ_3	σ_1
σ_1	σ_1	σ_2	σ_3	e	c_3	c_3^{-1}
σ_2	σ_2	σ_3	σ_1	c_3^{-1}	e	c_3
σ_3	σ_3	σ_1	σ_2	c_3	c_3^{-1}	e

Table 1: Multiplication table for D_3

As carried out in the lecture, let us identify $e, c_3, c_3^{-1}, \sigma_1, \sigma_2, \sigma_3$ with $\mathbf{e}_1 = (1,0,0,0,0,0)^T$, $\mathbf{e}_2 = (0,1,0,0,0,0)^T$, $\cdots$, $\mathbf{e}_6 = (0,0,0,0,0,1)^T$. Then under the action of the regular representation $\rho^{(reg)}(g)$ with $g \in G$, $\mathbf{v}$ defined by

$$\begin{aligned} \mathbf{v} &= \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 + \alpha_4\mathbf{e}_4 + \alpha_5\mathbf{e}_5 + \alpha_6\mathbf{e}_6 \\ &= \alpha_1 e + \alpha_2 c_3 + \alpha_3 c_3^{-1} + \alpha_4\sigma_1 + \alpha_5\sigma_2 + \alpha_6\sigma_3 \end{aligned}$$

transforms as (as we have seen in the lecture, the action of $\rho^{(reg)}(g)$ with $g \in G$ is to multiply g from the left)

$$\begin{aligned} \rho^{(reg)}(e)\mathbf{v} &= e\mathbf{v} = \alpha_1 e + \alpha_2 c_3 + \alpha_3 c_3^{-1} + \alpha_4\sigma_1 + \alpha_5\sigma_2 + \alpha_6\sigma_3 \\ &= \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 + \alpha_4\mathbf{e}_4 + \alpha_5\mathbf{e}_5 + \alpha_6\mathbf{e}_6\,, \\ \rho^{(reg)}(c_3)\mathbf{v} &= c_3\mathbf{v} = \alpha_1 c_3 + \alpha_2 c_3^{-1} + \alpha_3 e + \alpha_4\sigma_3 + \alpha_5\sigma_1 + \alpha_6\sigma_2 \\ &= \alpha_1\mathbf{e}_2 + \alpha_2\mathbf{e}_3 + \alpha_3\mathbf{e}_1 + \alpha_4\mathbf{e}_6 + \alpha_5\mathbf{e}_4 + \alpha_6\mathbf{e}_5\,, \\ \rho^{(reg)}(c_3^{-1})\mathbf{v} &= c_3^{-1}\mathbf{v} = \alpha_1 c_3^{-1} + \alpha_2 e + \alpha_3 c_3 + \alpha_4\sigma_2 + \alpha_5\sigma_3 + \alpha_6\sigma_1 \\ &= \alpha_1\mathbf{e}_3 + \alpha_2\mathbf{e}_1 + \alpha_3\mathbf{e}_2 + \alpha_4\mathbf{e}_5 + \alpha_5\mathbf{e}_6 + \alpha_6\mathbf{e}_4\,, \\ \rho^{(reg)}(\sigma_1)\mathbf{v} &= \sigma_1\mathbf{v} = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 + \alpha_4 e + \alpha_5 c_3 + \alpha_6 c_3^{-1} \\ &= \alpha_1\mathbf{e}_4 + \alpha_2\mathbf{e}_5 + \alpha_3\mathbf{e}_6 + \alpha_4\mathbf{e}_1 + \alpha_5\mathbf{e}_2 + \alpha_6\mathbf{e}_3\,, \\ \rho^{(reg)}(\sigma_2)\mathbf{v} &= \sigma_2\mathbf{v} = \alpha_1\sigma_2 + \alpha_2\sigma_3 + \alpha_3\sigma_1 + \alpha_4 c_3^{-1} + \alpha_5 e + \alpha_6 c_3 \end{aligned}$$

$$
\begin{aligned}
&= \alpha_1\mathbf{e}_5 + \alpha_2\mathbf{e}_6 + \alpha_3\mathbf{e}_4 + \alpha_4\mathbf{e}_3 + \alpha_5\mathbf{e}_1 + \alpha_6\mathbf{e}_2\,, \\
\rho^{(reg)}(\sigma_3)\mathbf{v} &= \sigma_3\mathbf{v} = \alpha_1\sigma_3 + \alpha_2\sigma_1 + \alpha_3\sigma_2 + \alpha_4 c_3 + \alpha_5 c_3^{-1} + \alpha_6 e \\
&= \alpha_1\mathbf{e}_6 + \alpha_2\mathbf{e}_4 + \alpha_3\mathbf{e}_5 + \alpha_4\mathbf{e}_2 + \alpha_5\mathbf{e}_3 + \alpha_6\mathbf{e}_1\,.
\end{aligned}
$$

Thus we obtain the matrix representation $M^{(reg)}(g)$ of $\rho^{(reg)}(g)$ defined as

$$
\rho^{(reg)}(g) : \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \\ \alpha_4' \\ \alpha_5' \\ \alpha_6' \end{pmatrix} = M^{(reg)}(g) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix}
$$

(or, equivalently, $(\mathbf{e}_1', \cdots, \mathbf{e}_6') = (\mathbf{e}_1, \cdots, \mathbf{e}_6)M^{(reg)}(g)$) as follows:

$$
M^{(reg)}(e) = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \qquad
M^{(reg)}(c_3) = \begin{pmatrix} 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \end{pmatrix},
$$

$$
M^{(reg)}(c_3^{-1}) = \begin{pmatrix} 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \end{pmatrix}, \qquad
M^{(reg)}(\sigma_1) = \begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix},
$$

$$
M^{(reg)}(\sigma_2) = \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}, \qquad
M^{(reg)}(\sigma_3) = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}.
$$

—— memo ——

Since $\mathbf{v}$ can be written as

$$
\mathbf{v} = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_6) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix},
$$

which transforms under the action of $\rho^{(reg)}(g)$ to

$$\rho^{(reg)}(g)\mathbf{v} = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_6) M^{(reg)}(g) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} ,$$

one can regard this transformation as the transformation of α_i's, $(\alpha_1, \cdots, \alpha_6)^T \to M^{(reg)}(g)(\alpha_1, \cdots, \alpha_6)^T$. ——- On definition of regular representation ——
We first start with the left action of G on a element $g \in G$, $\hat{h}(g) = hg$ $(h \in G)$. Then from this, we have

$$\hat{h} \sum_{g \in G} \alpha(g) g = \sum_{g \in G} \alpha(g) hg = \sum_{g \in G} \alpha(h^{-1}g) g \,.$$

Then we also have

$$\begin{aligned} \hat{h}_1 \hat{h}_2 \sum_{g \in G} \alpha(g) g &= \hat{h}_1 \sum_{g \in G} \alpha(g) h_2 g = \sum_{g \in G} \alpha(g) h_1 h_2 g = \sum_{g \in G} \alpha(g)(h_1 h_2) g = \widehat{(h_1 h_2)} \sum_{g \in G} \alpha(g) g \,, \\ &= \sum_{g \in G} \alpha((h_1 h_2)^{-1} g) g \,. \end{aligned}$$

Thus, the action of $h \in G$ onto the function $\alpha(g)$ is given by $\hat{h}\alpha(g) = \alpha(h^{-1}g)$. In the above (as well as the lecture), we have used this definition of the regular representation.

2. Let us in general consider a finite group G and its representations $\rho^{(\alpha)}$ and $\rho^{(\beta)}$. We assume that these representations are n_α-dimensional and n_β-dimensional, respectively. We denote the matrix representation of the element $g \in G$ corresponding to these two representations by $M^{(\alpha)}(g)$ and $M^{(\beta)}(g)$, respectively. Then we consider the tensor product representation for the representations $\rho^{(\alpha)}$ and $\rho^{(\beta)}$. The matrix representation of $g \in G$ denoted by $M^{(\alpha \otimes \beta)}(g)$ is now a $(n_\alpha n_\beta) \times (n_\alpha n_\beta)$ matrix whose $(n_\beta(i-1)+a, n_\beta(j-1)+b)$ component is given by $M^{(\alpha)}_{ij}(g) M^{(\beta)}_{ab}(g)$.

   Now we recall that for ρ of the dihedral group D_3, we have

$$M(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad M(c_3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \qquad M(c_3^{-1}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ,$$

$$M(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \qquad M(\sigma_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \qquad M(\sigma_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Therefore, for the tensor product representation of two ρ's we obtain

$$M^{(\rho\otimes\rho)}(e) = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$M^{(\rho\otimes\rho)}(c_3) = \begin{pmatrix} 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0 \end{pmatrix},$$

$$M^{(\rho\otimes\rho)}(c_3^{-1}) = \begin{pmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$M^{(\rho\otimes\rho)}(\sigma_1) = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0 \end{pmatrix},$$

$$M^{(\rho\otimes\rho)}(\sigma_2) = \begin{pmatrix} 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$M^{(\rho\otimes\rho)}(\sigma_3) = \begin{pmatrix} 0&0&0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

(2)

1. For $g_1, g_2 \in G$, we confirm $\bar{\rho}(g_1)\bar{\rho}(g_2) = \bar{\rho}(g_1 g_2)$:

$$\begin{aligned} \bar{\rho}(g_1)\bar{\rho}(g_2) &= (\rho(g_1^{-1}))^T(\rho(g_2^{-1}))^T \\ &= (\rho(g_2^{-1})\rho(g_1^{-1}))^T \\ &= (\rho(g_2^{-1}g_1^{-1}))^T \\ &= (\rho((g_1g_2)^{-1}))^T \\ &= \bar{\rho}(g_1g_2)\,. \end{aligned}$$

2. For $g_1, g_2 \in G$, we confirm $\rho^*(g_1)\rho^*(g_2) = \rho^*(g_1 g_2)$:

$$\rho^*(g_1)\rho^*(g_2) = (\rho(g_1))^*(\rho(g_2))^* = (\rho(g_1)\rho(g_2))^* = (\rho(g_1g_2))^* = \rho^*(g_1g_2)\,.$$

2 Symmetry in Quantum Mechanics

1. Since

$$H|\psi\rangle = E|\psi\rangle\,, \qquad H|\phi\rangle = E'|\phi\rangle\,,$$

we have

$$\langle\psi|H|\phi\rangle = E\langle\psi|\phi\rangle = E'\langle\psi|\phi\rangle\,.$$

Then we obtain

$$(E - E')\langle\psi|\phi\rangle = 0\,,$$

which means $\langle\psi|\phi\rangle = 0$ for $E \neq E'$.

2. The existence of id is OK from the assumption (note that obviously $idH = Hid$). For A, B in this set, we have

$$ABH = AHB = HAB\,.$$

Since A in this set is invertible, there exists A^{-1} such that $AA^{-1} = A^{-1}A = E$. By multiplying A^{-1} both from the left and right of $AH = HA$, we have

$$(LHS) = A^{-1}AHA^{-1} = HA^{-1}\,, \qquad (RHS) = A^{-1}HAA^{-1} = A^{-1}H\,.$$

Thus we have $HA^{-1} = A^{-1}H$. The associativity of the elements in this set is obviously satisfied.

3. Since A commutes with H, we have

$$HA|\psi\rangle = AH|\psi\rangle = EA|\psi\rangle\,.$$

4. We first define a matrix M as (for representations of G labeled by α and β and an element $A \in G$)

$$P = \sum_{A\in G} M^{(\alpha)}(A^{-1})QM^{(\beta)}(A)\,.$$

Here Q is an arbitrary $n_\alpha \times n_\beta$ matrix which we will take to a specific matrix later. Then we have (for $B \in G$)

$$\begin{aligned}
M^{(\alpha)}(B)P &= \sum_{A\in G} M^{(\alpha)}(B)M^{(\alpha)}(A^{-1})QM^{(\beta)}(A) \\
&= \sum_{A\in G} M^{(\alpha)}(BA^{-1})QM^{(\beta)}(A) \\
&= \sum_{A'\in G} M^{(\alpha)}(A'^{-1})QM^{(\beta)}(A'B) \\
&= \sum_{A'\in G} M^{(\alpha)}(A'^{-1})QM^{(\beta)}(A')M^{(\beta)}(B) \\
&= PM^{(\beta)}(B)\,.
\end{aligned}$$

In the middle, we have defined $A'^{-1} = BA^{-1}$ and replaced $\sum_{A\in G}$ by $\sum_{A'\in G}$ since $\{A' = AB^{-1} | A \in G\} = G$.

Thus from the Schur's lemma, if $\alpha \neq \beta$ we have $P = 0$. By taking N to be $Q_{ab} = 1$ and otherwise 0 (that is, (a, b)-component is 1 while the others are all zero), we can write $P = 0$ as

$$0 = \sum_{A\in G} M^{(\alpha)}_{ca}(A^{-1})M^{(\beta)}_{bd}(A) = \sum_{A\in G} M^{(\alpha)}_{ac}(A)^* M^{(\beta)}_{bd}(A)\,.$$

Here we have used the fact that $M^{(\alpha)}$ is unitary:

$$M^{(\alpha)}_{ca}(A^{-1}) = (M^{(\alpha)}(A)^{-1})_{ca} = (M^{(\alpha)}(A)^\dagger)_{ca} = (M^{(\alpha)}(A)^*)_{ac} = M^{(\alpha)}_{ac}(A)^*\,.$$

For $\alpha = \beta$, from the Schur's lemma, we have $P = C1_{n_\alpha}$ where C is a constant and 1_{n_α} is $n_\alpha \times n_\alpha$ unit matrix. Then, by taking Q as above, we obtain

$$C\delta_{cd} = \sum_{A\in G} M^{(\alpha)}_{ca}(A^{-1})M^{(\alpha)}_{bd}(A)\,.$$

By taking the trace with respect to (c, d), we obtain

$$\begin{aligned} Cn_\alpha &= \sum_{A\in G} M^{(\alpha)}_{ca}(A^{-1})M^{(\alpha)}_{bc}(A) \\ &= \sum_{A\in G} M^{(\alpha)}_{ba}(AA^{-1}) \\ &= \sum_{A\in G} M^{(\alpha)}_{ba}(id) \\ &= |G|\delta_{ab}\,. \end{aligned}$$

Here we have used the fact that $M^{(\alpha)}_{ba}(id) = \delta_{ab}$. From this, we have $C = |G|\delta_{ab}/n_\alpha$. Therefore we have obtained

$$\frac{|G|}{n_\alpha}\delta_{ab}\delta_{cd} = \sum_{A\in G} M^{(\alpha)}_{ca}(A^{-1})M^{(\alpha)}_{bc}(A) = \sum_{A\in G}(M^{(\alpha)}_{ac}(A))^* M^{(\alpha)}_{bd}(A)\,.$$

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Schur's lemma

Let us consider two complex irreducible representations $\rho^{(\alpha)} : G \to GL(n_\alpha, \mathbb{C})$ and $\rho^{(\beta)} : G \to GL(n_\beta, \mathbb{C})$ of a finite group G. We denote their matrix representation by $M^{(\alpha)}$ and $M^{(\beta)}$, respectively. We denote a matrix expression of an equivariant from $\mathbb{C}^{n_\alpha}$ to $\mathbb{C}^{n_\beta}$ by N which satisfies $NM^{(\alpha)}(g) = M^{(\beta)}(g)N$ for $g \in G$. Then

(a) if N is not isorporphic, then $N = 0$.

(b) if $n_\alpha = n_\beta$, then $N = \lambda 1_{n_\alpha}$ where $\lambda \in \mathbb{C}$.

Thus for $\alpha \neq \beta$, we have $N = 0$, while for $\alpha = \beta$ we have $N = \lambda 1_{n_\alpha}$.

5. (i) Proof by using the orthogonality relation of the representation

$$\begin{aligned} \langle\psi^{(\alpha)}_a|O|\psi^{(\beta)}_b\rangle &= \frac{1}{|G|}\sum_{A\in G}\langle\psi^{(\alpha)}_a|A^{-1}AO|\psi^{(\beta)}_b\rangle \\ &= \frac{1}{|G|}\sum_{A\in G}\langle\psi^{(\alpha)}_a|A^{-1}OA|\psi^{(\beta)}_b\rangle \\ &= \frac{1}{|G|}\sum_{A\in G}\sum_{\gamma}\sum_{c}\langle\psi^{(\alpha)}_a|A^{-1}|\psi^{(\gamma)}_c\rangle\langle\psi^{(\gamma)}_c|OA|\psi^{(\beta)}_b\rangle \\ &= \frac{1}{|G|}\sum_{A\in G}\sum_{\gamma}\sum_{c,d,e}\langle\psi^{(\alpha)}_a|\psi^{(\gamma)}_d\rangle M^{(\gamma)}_{dc}(A^{-1})\langle\psi^{(\gamma)}_c|O|\psi^{(\beta)}_e\rangle M^{(\beta)}_{eb}(A) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|G|}\sum_{A\in G}\sum_{\gamma}\sum_{c,d,e}\delta_{ad}\delta_{\alpha\gamma}M^{(\gamma)}_{dc}(A^{-1})\langle\psi^{(\gamma)}_c|O|\psi^{(\beta)}_e\rangle M^{(\beta)}_{eb}(A)\\
&= \frac{1}{|G|}\sum_{A\in G}\sum_{c,e}M^{(\alpha)}_{ac}(A^{-1})\langle\psi^{(\alpha)}_c|O|\psi^{(\beta)}_e\rangle M^{(\beta)}_{eb}(A)\\
&= \frac{1}{|G|}\sum_{A\in G}\sum_{c,e}M^{(\alpha)}_{ca}(A)^*\langle\psi^{(\alpha)}_c|O|\psi^{(\beta)}_e\rangle M^{(\beta)}_{eb}(A)\\
&= \frac{1}{|G|}\sum_{A\in G}\sum_{c=1}^{n_\alpha}\sum_{d=1}^{n_\beta}M^{(\alpha)}_{ca}(A)^*M^{(\beta)}_{db}(A)\langle\psi^{(\alpha)}_c|O|\psi^{(\beta)}_d\rangle\\
&= \frac{1}{n_\alpha}\delta_{\alpha\beta}\sum_{c=1}^{n_\alpha}\sum_{d=1}^{n_\beta}\delta_{ab}\delta_{cd}\langle\psi^{(\alpha)}_c|O|\psi^{(\beta)}_d\rangle\\
&= \frac{1}{n_\alpha}\delta_{\alpha\beta}\delta_{ab}\sum_{c=1}^{n_\alpha}\langle\psi^{(\alpha)}_c|O|\psi^{(\alpha)}_c\rangle\,.
\end{aligned}$$

This is what we want to prove with $C^{(\alpha)} = (1/n_\alpha)\sum_c\langle\psi^{(\alpha)}_c|O|\psi^{(\alpha)}_c\rangle$. Here we have used the orthogonality relation of the irreducible representationa

$$\sum_{A\in G}(M^{(\alpha)}_{ac}(A))^*M^{(\beta)}_{bd}(A) = \frac{|G|}{n_\alpha}\delta_{\alpha\beta}\delta_{ab}\delta_{cd}\,.$$

(ii) Proof by directly using the Schur's lemma
We notice that

$$\begin{aligned}
\langle\psi^{(\alpha)}_a|A|\psi^{(\beta)}_b\rangle &= \langle\psi^{(\alpha)}_a|\sum_{c=1}^{n_\beta}|\psi^{(\beta)}_c\rangle M^{(\beta)}_{cb}(A)\\
&= \sum_{c=1}^{n_\beta}\delta_{ac}\delta_{\alpha\beta}M^{(\beta)}_{cb}(A)\\
&= \delta_{\alpha\beta}M^{(\beta)}_{ab}(A)\,.
\end{aligned}$$

Since $[O,A]=0$ for $A\in G$, we have by inserting the identity operator $id = \sum_\alpha\sum_a|\psi^{(\alpha)}_a\rangle\langle\psi^{(\alpha)}_a|$

$$\begin{aligned}
0 &= \langle\psi^{(\alpha)}_a|[O,A]|\psi^{(\beta)}_b\rangle\\
&= \langle\psi^{(\alpha)}_a|OA|\psi^{(\beta)}_b\rangle - \langle\psi^{(\alpha)}_a|AO|\psi^{(\beta)}_b\rangle\\
&= \langle\psi^{(\alpha)}_a|OA|\psi^{(\beta)}_b\rangle - \langle\psi^{(\alpha)}_a|AO|\psi^{(\beta)}_b\rangle\\
&= \sum_\gamma\sum_c\left(\langle\psi^{(\alpha)}_a|O|\psi^{(\gamma)}_c\rangle\langle\psi^{(\gamma)}_c|A|\psi^{(\beta)}_b\rangle - \langle\psi^{(\alpha)}_a|A|\psi^{(\gamma)}_c\rangle\langle\psi^{(\gamma)}_c|O|\psi^{(\beta)}_b\rangle\right)\\
&= \sum_c\left(\langle\psi^{(\alpha)}_a|O|\psi^{(\beta)}_c\rangle M^{(\beta)}_{cb}(A) - M^{(\alpha)}_{ac}(A)\langle\psi^{(\alpha)}_c|O|\psi^{(\beta)}_b\rangle\right)
\end{aligned}$$

Because of the Schur's lemma, we first notice that $\langle\psi_a^{(\alpha)}|O|\psi_c^{(\beta)}\rangle$ is zero unless $\alpha = \beta$. Moreover, from this lemma, for $\alpha = \beta$ we also have $\langle\psi_a^{(\alpha)}|O|\psi_c^{(\alpha)}\rangle$ is proportional to δ_{ac}. Thus we finally have

$$\langle\psi_a^{(\alpha)}|O|\psi_b^{(\beta)}\rangle = C^{(\alpha)}\delta_{ab}\delta_{\alpha\beta}\,,$$

where $C^{(\alpha)}$ is a constant independent of a. One can determine $C^{(\alpha)}$ by setting $\alpha = \beta$ and taking the trace with respect to (a, b). Then we reproduce $C^{(\alpha)}$ obtained by using the method (i).